Computing with Constructible Sets

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Joint work with
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Outline

- Constructible set and removing redundancy
- Difference: $(A, B) \longmapsto A \setminus B$
- PairRefine: $(A, B) \longmapsto (A \setminus B,\ A \cap B, B \setminus A)$
- MPD: C (with redundancy) $\longmapsto$ C (without redundancy)
- SMPD:

  $C = \bigcup C_i \quad \longmapsto \quad C = \bigcup D_j$

  (with redundancy) (without redundancy and the $D_j's$ refine the $C_i's$)
- Complexity Analysis
- Experimental comparison (different algorithms for SMPD)

What is a Constructible Set?

Definition (Constructible Set)

A *constructible set* of $\overline{\mathbb{K}}^n$ is a finite union

$$(A_1 \setminus B_1) \cup \cdots \cup (A_e \setminus B_e)$$

where $A_i's$ and $B_i's$ are algebraic varieties in $\overline{\mathbb{K}}^n$.

Definition (Regular System)

A pair $[T, h]$ is a *regular system* if T is a regular chain, and $h \in \mathbb{K}[X]$ is regular with respect to $\mathrm{sat}(T)$. The zero set $Z(T, h)$ given by $[T, h]$ is $W(T) \setminus V(h)$.

Example (Regular Systems)

$$(Yes) \begin{cases} ax^2 + bx + c = 0 \\ a(b^2 - 4ac) \neq 0 \end{cases} \qquad (No) \begin{cases} x^2 - 2xy + t = 0 \\ y^2 - t = 0 \\ x - y \neq 0 \end{cases}$$

Representation of Constructible Sets

Example (Constructible Set)

For what value of a, b, c, does the equation

$$ax^2 + bx + c = 0$$

have solutions over $\mathbb{C}$?

- when a is not zero; $rs_1 = [a \neq 0]$
- when a is zero but b is not; $rs_2 = [a = 0, b \neq 0]$
- when a, b, c are all zero. $rs_3 = [a = 0, b = 0, c = 0]$

$cs = \{rs_1, rs_2, rs_3\}$ describes the answer.

Another Example

- **Example**: given two elliptic curves in the complex plane of coordinates (x, y): $g_1(x, y) = 0$ and $g_2(x, y) = 0$, where

$$\begin{aligned} g_1(x, y) &= x^3 + a_1 x - y^2 + 1, \\ g_2(x, y) &= x^3 + a_2 x - y^2 + 1 \end{aligned}$$

In invariant theory, a classical question is whether there exists a
linear fractional map from the first curve to the second one:

$$f : (x, y) \mapsto \left(\frac{A\,x + B\,y + C}{G\,x + H\,y + K}, \frac{D\,x + E\,y + F}{G\,x + H\,y + K} \right)$$

Another Example

- This problem can be turned into a parametric system:

$$g_1(x,y) - (Gx + Hy + K)^3 g_2(f(x,y)) = 0.$$

$$\begin{cases} 1 - K^3 & = 0 \\ -a_2 AK^2 + a_1 - 3GK^2 & = 0 \\ -3HK^2 - a_2 BK^2 & = 0 \\ GD^2 - a_2 G^2 A - A^3 - G^3 + 1 & = 0 \\ -3H^2K + E^2K - 1 - 2a_2 BHK & = 0 \\ -3G^2K - 2a_2 GAK + D^2K & = 0 \\ GE^2 - 2a_2 GBH - a_2 AH^2 - 3AB^2 - 3GH^2 + 2DEH & = 0 \\ E^2H - H^3 - a_2 BH^2 - B^3 & = 0 \\ D^2H - 3G^2H + 2GDE - 2a_2 GAH - 3A^2B - a_2 G^2B & = 0 \\ -3GHK - a_2 AHK - a_2 GBK + DEK & = 0 \end{cases}$$

- For which parameter values of a_1, a_2 does this system have solutions?

Another Example

- The output produced by the command `ComprehensiveTriangularize` of the module `ParametricSystemTools` consists of 11 regular chains $[T_1, \ldots, T_{11}]$ and 3 constructible sets C_1, C_2 and C_3.

$$\begin{array}{lcl} C_1 & : & a_1^3 = a_2^3 = 9 \\ C_2 & : & a_1 = a_2 = 0 \\ C_3 & : & a_1^3 = a_2^3,\ a_2 \neq 0,\ a_2^3 \neq 9\ . \end{array}$$

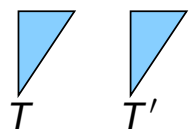
- The union of C_1, C_2, C_3 is the answer to our question: for which parameter values does the input system have solutions?

Redundancy in Computing with Constructible Sets

- Redundancy in a single constructible set
  - Two regular systems have a common part.
  - Remove redundancy: make regular systems pairwise disjoint (MPD)
- Redundancy in a list of constructible sets
  - Some zeroes appear in more than one constructible sets.
- Building block: compute the difference of two regular systems.

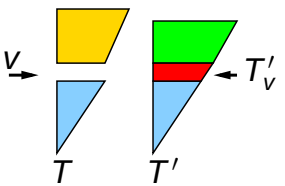
Sketch of Difference Algorithm to compute $V(T) \setminus V(T')$ by exploiting the triangular structure level by level.

Case 1:



$\langle T \rangle = \langle T' \rangle$, Easy!

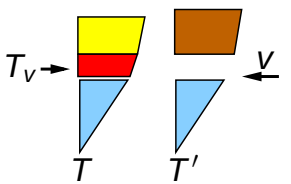
Case 2:



Output $[T, T'_v]$ and

$\texttt{Difference}(V(T_v) \cap V(T'_v), V(T'))$;

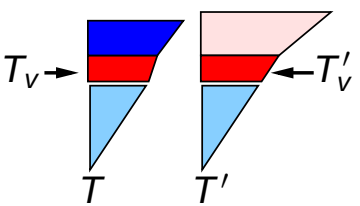
Case 3:



Output $\texttt{Difference}$

$(T, V(T_v) \cap V(T'))$;

Case 4:



- $g = \mathrm{GCD}(T_v, T'_v, T_{<v})$; • $g \in \mathbb{K} \Rightarrow$ Output $[T, 1]$;
- $\mathrm{mvar}(g) < v \Rightarrow$ Output $[T, g]$, $\texttt{Difference}(V(g) \cap V(T), T')$;
- Output $\texttt{Difference}(T_{<v} \cup \{g\} \cup T_{>v}, T')$;
- Output $\texttt{Difference}(T_{<v} \cup \{T_v/g\} \cup T_{>v}, T')$;

Algorithm 2 MPD

Input: a list L of regular systems

Output: a pairwise disjoint representation of L

1: $n \leftarrow |L|$
2: **if** $n < 2$ **then**
3: **return** L
4: **else**
5: $d \leftarrow L[n]$
6: $L^* \leftarrow \text{MPD}(L[1, \ldots, n-1])$
7: **for** $\ell' \in L^*$ **do**
8: $d \leftarrow \text{Difference}(d, \ell')$
9: **end for**
10: **return** $d \cup L^*$
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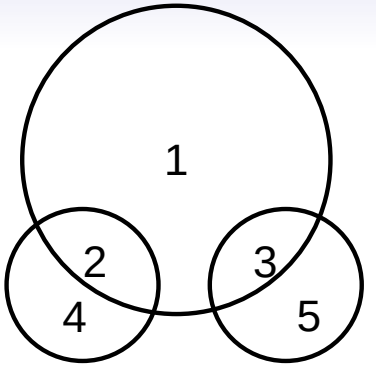

$A = \{1, 2, 3\}$,

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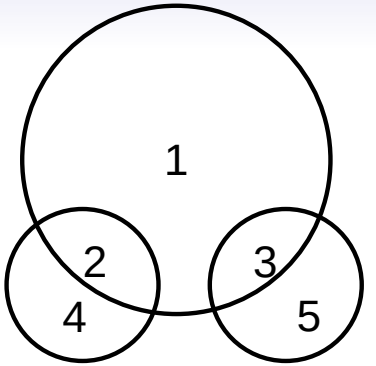

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$A = A \setminus B = \{1, 3\}$

Algorithm 2 MPD

Input: a list L of monic square-free zero-dimensional regular chains

Output: a pairwise disjoint representation of L

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5: $d \leftarrow L[n]$
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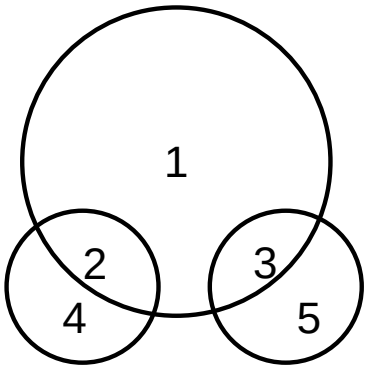

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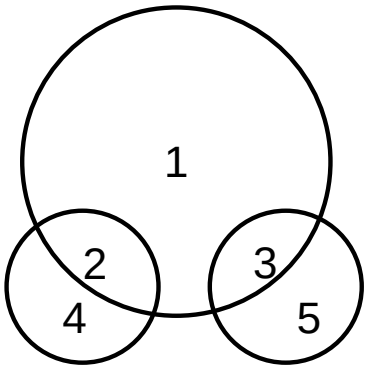

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$\text{MPD}(A, B, C) = \{1\}, \{2, 4\}, \{3, 5\}$

RCPairRefine

In dimension 0, we can do better.

- RCPairRefine performs Difference in a simple case.
- Compute two-side differences and intersection **in one pass**.

Example

$$\text{RCPairRefine}([x = 0, y(y+1) = 0], [x = 0, y(y+2) = 0])$$

outputs

$$\underbrace{[x = 0, y + 1 = 0]}_{\textit{difference}}, \quad \underbrace{[x = 0, y = 0]}_{\textit{intersection}}, \quad \underbrace{[x = 0, y + 2 = 0]}_{\textit{difference}}$$

RCPairRefine Algorithm

Algorithm 1 RCPairRefine

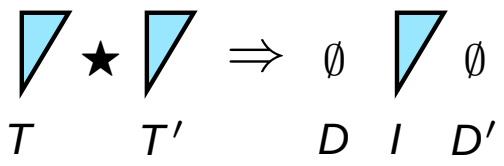
Input: two monic squarefree zero-dimensional regular chains T and T'

Output: three constructible sets D, I and D', such that $V(T) \setminus V(T') = Z(D)$, $V(T) \cap V(T') = Z(I)$ and $V(T') \setminus V(T) = Z(D')$

1: **if** $T = T'$ **then**
2: **return** $\emptyset,\ [T],\ \emptyset$
3: **else**
4: $D \leftarrow \emptyset;\ I \leftarrow \emptyset;\ D' \leftarrow \emptyset$
5: Let v be the largest variable s.t. $T_{<v} = T'_{<v}$
6: **for** $(g, G) \in \mathrm{GCD}(T_v, T'_v, T_{<v})$ **do**
7: **if** $g \in \mathbb{K}$ **or** $\mathrm{mvar}(g) < v$ **then**
8: $T_q \leftarrow G \cup \{T_v\} \cup T_{>v}$;
9: $T'_q \leftarrow G \cup \{T'_v\} \cup T'_{>v}$;
10: $D \leftarrow D \cup T_q$;
11: $D' \leftarrow D' \cup T'_q$
12: **else**
13: $q \leftarrow \mathrm{pquo}(T_v, g, G)$;
14: $q' \leftarrow \mathrm{pquo}(T'_v, g, G)$;
15: $T_g \leftarrow G \cup \{g\} \cup T_{>v}$;
16: $T'_g \leftarrow G \cup \{g\} \cup T'_{>v}$
17: **if** $\mathrm{mvar}(q) = v$ **then**
18: $T_q \leftarrow G \cup \{q\} \cup T_{>v}$;
19: $D \leftarrow D \cup T_q$
20: **end if**
21: **if** $\mathrm{mvar}(q') = v$ **then**
22: $T'_q \leftarrow G \cup \{q'\} \cup T'_{>v}$;
23: $D' \leftarrow D' \cup T'_q$
24: **end if**
25: $W, J, W' \leftarrow$ RCPairRefine(T_g, T'_g);
26: $D \leftarrow D \cup W$;
27: $I \leftarrow I \cup J$;
28: $D' \leftarrow D' \cup W'$
29: **end if**
30: **end for**
31: **return** D, I, D'
32: **end if**

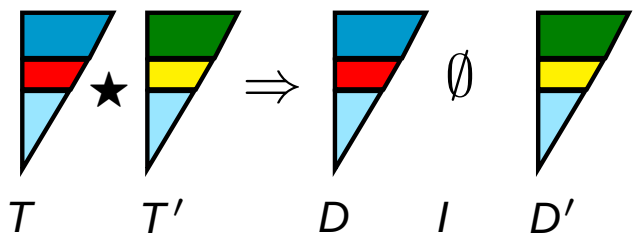
RCPairRefine Algorithm

Case 1: $T = T'$



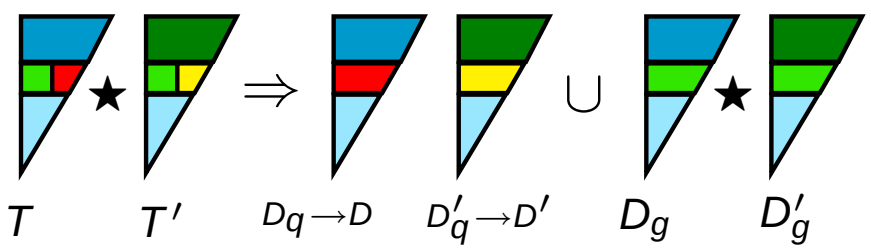
Case 2: $g = GCD(T_v, T'_v, T_{<v})$

$g \in \mathbb{K}$ or $\mathrm{mvar}(g) < v$



Case 3:

$g = GCD(T_v, T'_v, T_{<v})$; $\mathrm{mvar}(g) = v$



$$
\begin{aligned}
D &= T \setminus T' \\
I &= T \cap T' \\
D' &= T' \setminus T
\end{aligned}
$$

Irredundant Representation of a Family of Constructible Sets by *Symmetrically Make Pairwise Disjoint* (SMPD)

- Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a set of constructible sets.
- An intersection-free basis of $\mathcal{C}$: $\mathcal{D} = \{D_1, \ldots, D_n\}$ satisfies
  (1) $D_i \cap D_j = \emptyset$ for $1 \leq i \neq j \leq n$,
  (2) each C_i can be uniquely written as a union of some D_j's.

- CSPairRefine $= \mathrm{SMPD}(C_1, C_2) = C_1 \backslash C_2,\ C_1 \cap C_2,\ C_2 \backslash C_1$

- Based on RCPairRefine, CSPairRefine can be deduced naturally.

Example

$\mathrm{SMPD}(\{[x = 0]\},\ \{[y = 0]\})$ outputs:

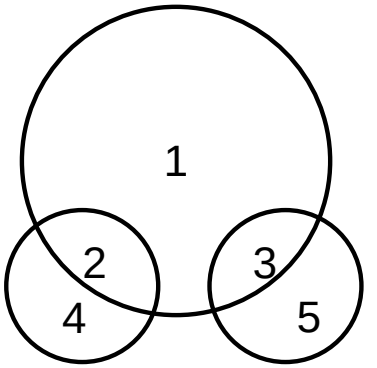
$\{[x = 0,\ y \neq 0]\},\ \{[x = 0, y = 0]\},\ \{[x \neq 0,\ y = 0]\}$

Algorithm 3 BachSMPD

Input: a list L of constructible sets
with each consisting of a
family of monic squarefree
zero-dimensional regular chains

Output: an intersection-free basis of L

1: $n \leftarrow |L|$
2: **if** $n < 2$ **then**
3: **return** L
4: **else**
5: $I \leftarrow \emptyset;\ D' \leftarrow \emptyset;\ d \leftarrow L[n]$
6: $L^* \leftarrow \text{BachSMPD}(L[1, \ldots, n-1])$
7: **for** $l' \in L^*$ **do**
8: $d, i, d' \leftarrow \text{CSPairRefine}(d, l')$
9: $I \leftarrow I \cup i;\ D' \leftarrow D' \cup d'$
10: **end for**
11: **return** $\{d\} \cup I \cup D'$
12: **end if**



$$
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A &= \{1,2,3\},\\
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C &= \{3,5\}.\\
A \bigstar B &= \{1,3\},\{2\},\{4\}\\
d \bigstar C &= \{1\},\{3\},\{5\}
\end{aligned}
$$

$$
\text{SMPD}(A,B,C) = \{1\},\{2\},\{4\},\{3\},\{5\}
$$

Algorithm 4 DCSMPD

1: $n \leftarrow |L|$
2: **if** $n < 2$ **then**
3: **return** L
4: **else**
5: $z \leftarrow \lfloor n/2 \rfloor$
6: $L_1 \leftarrow \text{DCSMPD}(L[1, \ldots, z])$
7: $L_2 \leftarrow \text{DCSMPD}(L[z+1, \ldots, n])$
8: **for** j in $1..|L_1|$ **do**
9: **for** k in $1..|L_2|$ **do**
10: $L_1[j], i, L_2[k] \leftarrow$ CSPairRefine$(L_1[j], L_2[k])$
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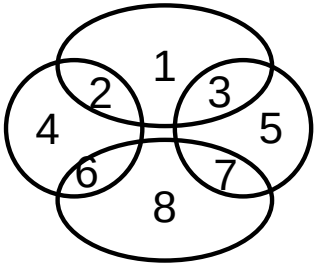
Merge: Line 8-line 13



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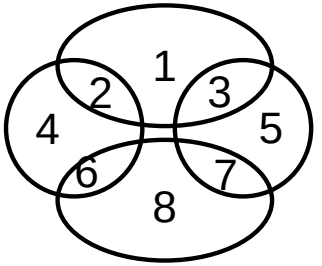

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$A \bigstar C = \{4, 6\}, \boxed{\{2\}}, \{1, 3\}$

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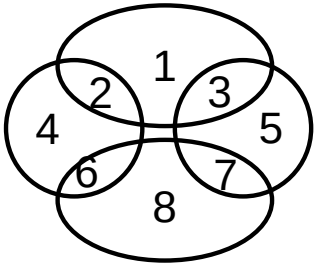

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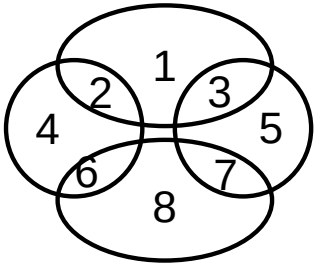

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$B \bigstar C = \{5, 7\}, \boxed{\{3\}}, \boxed{\{1\}}$

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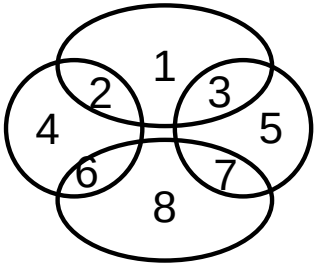
15: **end if**

Merge: Line 8-line 13



$A = \{2, 4, 6\}, B = \{3, 5, 7\},$
$C = \{1, 2, 3\}, D = \{6, 7, 8\}.$

$A \bigstar C = \{4, 6\}, \boxed{\{2\}}, \{1, 3\}$

$A \bigstar D = \boxed{\{4\}}, \boxed{\{6\}}, \{7, 8\}$

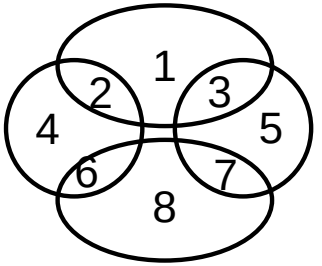
$B \bigstar C = \{5, 7\}, \boxed{\{3\}}, \boxed{\{1\}}$

$B \bigstar D = \boxed{\{5\}}, \boxed{\{7\}}, \boxed{\{8\}}$

Algorithm 4 DCSMPD

1: $n \leftarrow |L|$
2: **if** $n < 2$ **then**
3: **return** L
4: **else**
5: $z \leftarrow \lfloor n/2 \rfloor$
6: $L_1 \leftarrow \text{DCSMPD}(L[1, \ldots, z])$
7: $L_2 \leftarrow \text{DCSMPD}(L[z+1, \ldots, n])$
8: **for** j in $1..|L_1|$ **do**
9: **for** k in $1..|L_2|$ **do**
10: $L_1[j], i, L_2[k] \leftarrow$ CSPairRefine$(L_1[j], L_2[k])$
11: $I \leftarrow I \cup i$
12: **end for**
13: **end for**
14: **return** $L_1 \cup I \cup L_2$
15: **end if**

Merge: Line 8-line 13



$A = \{2, 4, 6\}, B = \{3, 5, 7\},$
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$B \bigstar C = \{5, 7\}, \boxed{\{3\}}, \boxed{\{1\}}$

$B \bigstar D = \boxed{\{5\}}, \boxed{\{7\}}, \boxed{\{8\}}$

$\text{SMPD}(A, B, C) = \{1\}, \{2\},$
$\{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}$

Complexity Analysis

- The main cost comes from GCD computation modulo regular chains.
- By means of evaluation/interpolation techniques, these GCDs can be performed modulo 0-dim regular chains.
- Let $T = [T_1, \ldots, T_n]$ be a squarefree regular chain in $\mathbb{K}[X_1 < \cdots < X_n]$.
- $\mathbb{K}(T) := \mathbb{K}[X_1, \ldots, X_n]/\langle T \rangle$ is a direct product of fields (DPF).
- Let $\deg_i T := \deg(T_i, Xi)$, for $1 \leq i \leq n$, and $\deg_T$ be their product.

Complexity Analysis

- For practical purpose, we rely on classical i.e. quadratic algorithms.
- We adapt the extended Euclidean algorithm for polynomials with coefficients in $\mathbb{K}$ to $\mathbb{K}(T)$.
- Recall an extended GCD of f_1 and f_2 in $\mathbb{K}[x]$ with degrees $d_1 \geq d_2$, can be computed in $O(d_1 d_2)$ operations in $\mathbb{K}$.

Complexity Analysis

- What is different in $\mathbb{K}(T)$ for EEA?

  In a Euclidean division step i, to do

  $$r_{i-1} \; rem \; r_i \; mod \; T,$$

  we need first to get the inverse u_i of $lc(r_i)$ s.t.

  $$u_i \times lc(r_i) = 1 \; mod \; T,$$

  which is called a quasi-inverse.

  Let X_k be the main variable of $lc(r_i)$. We need to compute

  $$GCD(lc(r_i), T_{x_k}) \; mod \; T_{<x_k}$$

Complexity Analysis

- In case of a zero-divisor, T splits into a family of $T^1, \cdots, T^m$. Then computation splits into m branches.
- All the elements then need to project to each branch. To perform this projection efficiently, each family of

$$T^1_{x_j}, \cdots, T^m_{x_j} \text{ for } 1 \leq j \leq m$$

should be pair-wise coprime, which is called a *non-critical* triangular decomposition.

- Thus GCD-free basis computation modulo regular chains is needed. We extend Bach's augment refinement method to $\mathbb{K}(T)$ for this.

Complexity Analysis

- Following the inductive process applied in “On the complexity of D5 principle”, we obtain an arithmetic time for regular chains based on classical algorithms.

- Recall that an arithmetic time $T \mapsto \mathsf{A}_n(\deg_1 T, \ldots, \deg_n T)$ is an asymptotic upper bound for the cost of basic polynomial arithmetic operations in $\mathbb{K}(T)$ (counted in $\mathbb{K}$), i.e. $+, *$, quasi-inverse, projection and computation of non-critical decompositions.

An Arithmetic Time for Regular Chain based on Classical Algorithms

Theorem (An arithmetic time for regular chain)

There exists a constant C such that, writing

$$\mathrm{A}_n(d_1, \ldots, d_n) = C^n (d_1 \times \cdots \times d_n)^2,$$

the function $T \mapsto \mathrm{A}_n(\deg_1 T, \ldots, \deg_n T)$ is an arithmetic time for regular chain T in n variables, for all n.

Basic Complexity Result

- An extended GCD of f_1 and f_2 in $\mathbb{K}(T)[y]$ with degrees $d_1 \geq d_2$ can be computed in $O(d_1 d_2 \mathsf{A}_n(T))$ operations in $\mathbb{K}$.

- For a family of monic squarefree polynomials $F = \{f_1, \ldots, f_m\}$ in $\mathbb{K}(T)[y]$ with degrees $d_1, \ldots, d_m$, we can extend the augment refinement method to compute a GCD-free basis of F modulo T in $O(\sum_{1 \leq i < j \leq m}(d_i d_j)\mathsf{A}_n(T))$ operations in $\mathbb{K}$.

- With $\deg(T) = d$ and $\deg(T') = d'$, RCPairRefine or `Difference`(T, T') costs $O(C^{n-1}dd')$ operations in $\mathbb{K}$.

Main Complexity Result

Assume all regular chains are monic and squarefree in dimension zero.

Theorem (Complexity of MPD)

Let $L = \{U_1, \ldots, U_m\}$ be a set of regular chains and the degree of U_i be d_i for $1 \leq i \leq m$. Then a pairwise disjoint representation of L can be computed in $O(C^{n-1} \sum_{1 \leq i < j \leq m} d_i d_j)$ operations in $\mathbb{K}$.

Remark

Consider the case when $d_i = d_j = d$, then MPD can be computed in $O(C^{n-1}(md)^2)$ operations.

Main Complexity Result

Theorem (Complexity of SMPD)

Given a set $L = \{C_1, \ldots, C_m\}$ of constructible sets, each of which is given by some pairwise disjoint regular chains. Let D_i be the number of points in C_i for $1 \leq i \leq m$. An intersection-free basis of L can be computed in $O(C^{n-1} \sum_{1 \leq i < j \leq m} D_i D_j)$ operations in $\mathbb{K}$.

Remark

A special case is when $D_i = D_j = D$, then the number of operations needed to compute an SMPD is bounded by $O(C^{n-1}(mD)^2)$.

An Experimental Comparison

- OldSMPD:
  - the defining regular systems of each constructible set are made (symmetrically) pairwise disjoint;
  - the set theoretical differences and the intersections are computed separately.
- BachSMPD
- DCSMPD: combining a divide-and-conquer approach with the augment refinement method for the operation SMPD.

Timing(s) of 15 Examples Computed by 3 SMPD Algorithms

Sys	OldSMPD	BachSMPD	DCSMPD
1	12.302	3.494	0.786
2	0.303	0.103	0.062
3	1.123	0.259	0.271
4	2.407	1.184	0.703
5	0.574	0.091	0.159
6	0.548	0.293	0.300
7	0.733	0.444	0.211
8	3.430	0.584	0.633
9	25.413	8.292	9.530
10	1097.291	82.468	122.575
11	11.828	0.930	0.985
12	54.197	1.934	1.778
13	0.530	0.047	0.064
14	27.180	13.705	4.626
15	–	1838.927	592.554

An Experimental Comparison

- Improved OldSMPD: extending the RCPairRefine algorithm to positive dimension and manages to compute the difference and the intersection in one pass

- BachSMPD + MPD and DCSMPD + MPD: cleaning each constructible set after SMPD.

Timing(s) of 15 Examples Computed by 3 SMPD Algorithms

Sys	OldSMPD(improved)	BachSMPD(+MPD)	DCSMPD+MPD
1	3.949	3.766	0.914
2	0.118	0.103	0.062
3	0.271	0.259	0.271
4	1.442	1.449	0.927
5	0.116	0.100	0.173
6	0.257	0.300	0.290
7	0.460	0.444	0.211
8	0.607	0.584	0.633
9	9.291	8.347	9.592
10	95.690	82.795	125.286
11	0.912	0.930	1.784
12	12.330	1.934	2.900
13	0.065	0.047	0.065
14	16.792	16.280	6.323
15	2272.550	1876.061	624.679

Conclusions

- We provide different algorithms for removing the redundancies and distinguished two cases:

MPD: *removal* of redundant components within a constructible set

SMPD: *removal with refinement* among a family of constructible sets.

- We rely on quadratic arithmetic motivated by practical concerns.

- We give a complexity analysis of these algorithms in dimension zero:
  - following the work of Bach, Driscoll and Shallit, we have obtained *essentially quadratic time complexity* based on classical (=quadratic) polynomial arithmetic.
  - the GCD of polynomials modulo a zero-dimensional regular chain can be done in *essentially quadratic time complexity* based on quadratic arithmetic.

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