# On the Computation of Convex Hulls

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Joint work with:
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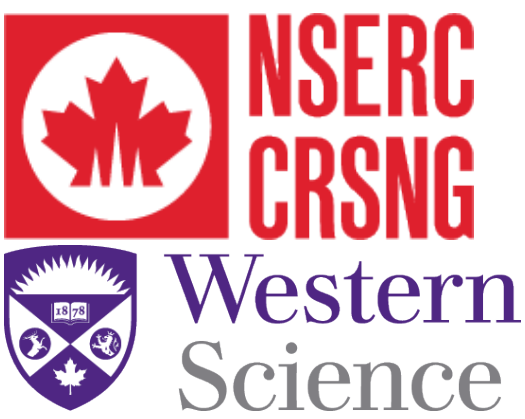
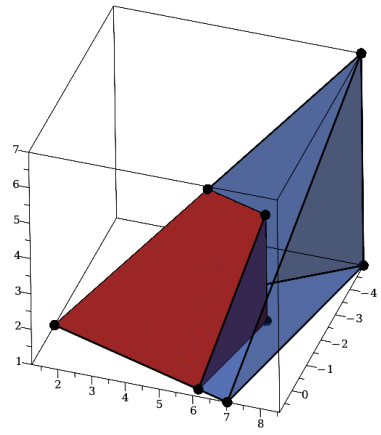
Sept 1, 2026




# Plan

1. Overview
2. The wrapping procedure
3. Computing ECH in the bounded case
4. Reduction to the bounded case

# A motivating example



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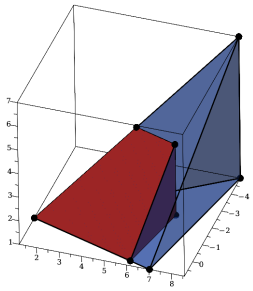

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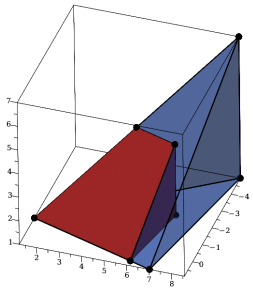

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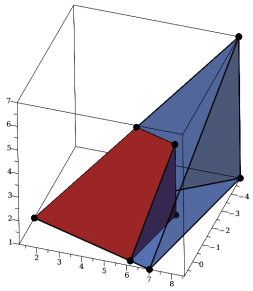

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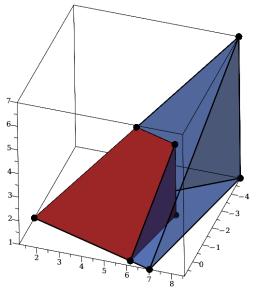

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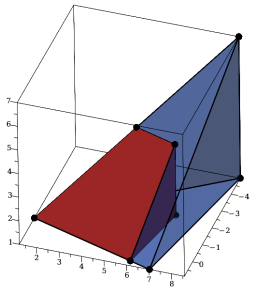

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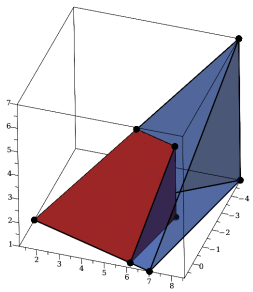

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5. Our ultimate goal is to decide whether or not the union of **finitely many $\mathbb{Z}$-polyhedral sets** is a $\mathbb{Z}$-polyhedra set, extending Verdoolaege's 2D constructions in [19].

# Computing convex hulls (1/2)

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3. But if $k$ input polyhedral sets $P_1, \ldots, P_k$ of $\mathbb{R}^d$ are given in H-representation rather than V-representation, reducing to the latter makes no sense since $|V| \in \mathcal{O}(m^{\lfloor d/2 \rfloor})$ where $m := |H|$, see [16].

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   a. the wrapping procedure of Chand and Kapur [7], and
   b. the reverse search enumeration developed by Avis and Fukuda [2–4].

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6. In fact, our plan is to reducing everything in polyhedral geometry to LP.

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4. Recall that any ridge (that is, a facet of a facet) of a full-dimensional polytope belongs to exactly two facets.

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3. A facet $F$ of a full-dimensional $P_i$, for any $1 \leq i \leq k$ is **valid** for $P_0$ if this is a facet of $P_0$, that is, if all $P_j, 1 \leq j \leq k$ are on the same side of $F$.
4. Recall that any ridge (that is, a facet of a facet) of a full-dimensional polytope belongs to exactly two facets.
5. **Key observation 1:** if a ridge $R$ belongs to exactly one valid facet $F$, one can **rotate** its other facet $F'$ along $R$ into a supporting hyperpplane of a facet of $P_0$.

# Valid and non-valid facets

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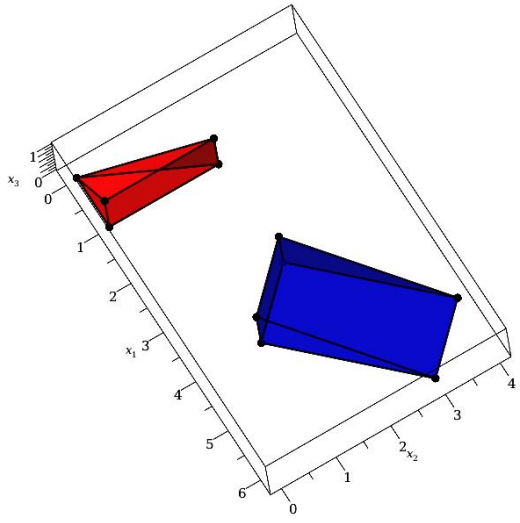
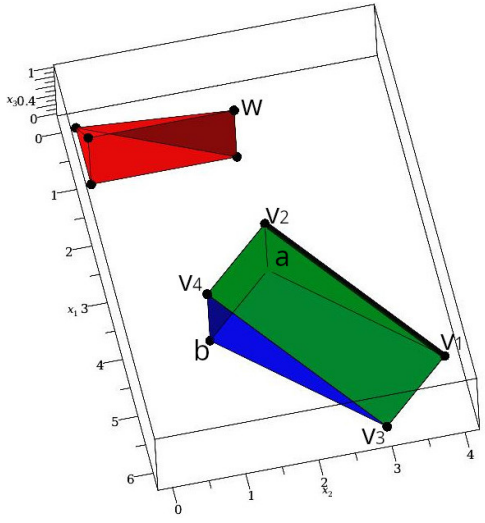
4. Recall that any ridge (that is, a facet of a facet) of a full-dimensional polytope belongs to exactly two facets.
5. **Key observation 1:** if a ridge $R$ belongs to exactly one valid facet $F$, one can **rotate** its other facet $F'$ along $R$ into a supporting hyperpplane of a facet of $P_0$.
6. **Key observation 2:** the rotation angle can be obtained by solving an **LP** problem.

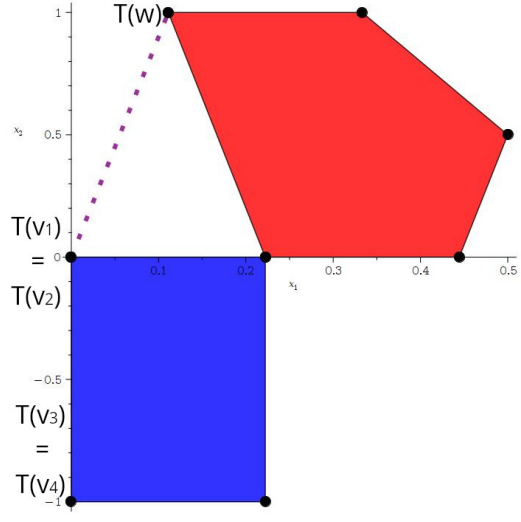
# The wrapping procedure: an example



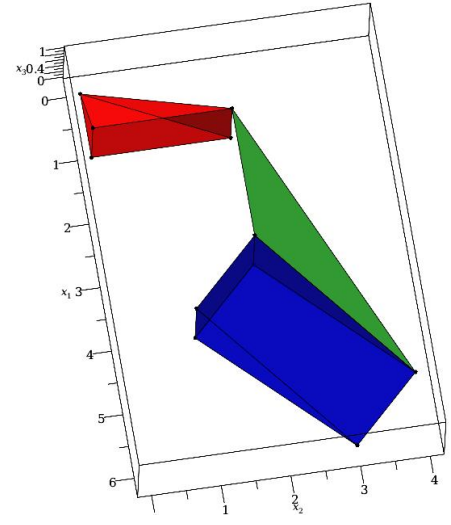
(a) Step 1




(b) Step 2




(c) Step 3



(d) Step 4

# The wrapping procedure: the formulas (1/2)

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5. Let $n_2$ be another vector satisfying $n_2 \perp n_1$, $n_2 \perp r_j$ for all $j$, and $n_2^t(v_d - v_1) < 0$. The vectors $r_1, \ldots, r_{d-2}, n_1, n_2$ a basis of $\mathbb{R}^d$, assembled into the $d \times d$ matrix $B := [r_1 \cdots r_{d-2}\ n_1\ n_2]$.

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6. The projection then maps a point $(x_1, \ldots, x_d)^t \in \mathbb{R}^d$ to $(y_1, y_2)^t \in \mathbb{R}^2$ via

$$y = T(x - v_1), \qquad T := \begin{bmatrix} 0 & \cdots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} B^{-1} \in \mathbb{R}^{2 \times d},$$

# The wrapping procedure: the formulas (2/2)

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$$x = \sum_{i=1}^{k} \lambda_i x^i, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i \geq 0, \quad x^i \in P_i.$$

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5. The wrapping procedure thus reduces to solving:

$$\begin{aligned}
\max \quad & y_2 \\
\text{s.t.} \quad & A^i \bar{x}^i \leq b^i \lambda_i, \quad i = 1, \ldots, k, \\
& \lambda_i \geq 0, \quad i = 1, \ldots, k, \\
\text{null} \quad & x = \sum_{i=1}^{k} \bar{x}^i, \\
& y = T\left(x - v_1 \sum_{i=1}^{k} \lambda_i\right), \\
& y_1 = 1.
\end{aligned} \tag{LP}$$

# Plan

1. Overview
2. The wrapping procedure
3. Computing ECH in the bounded case
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# The bounded case (1/3)

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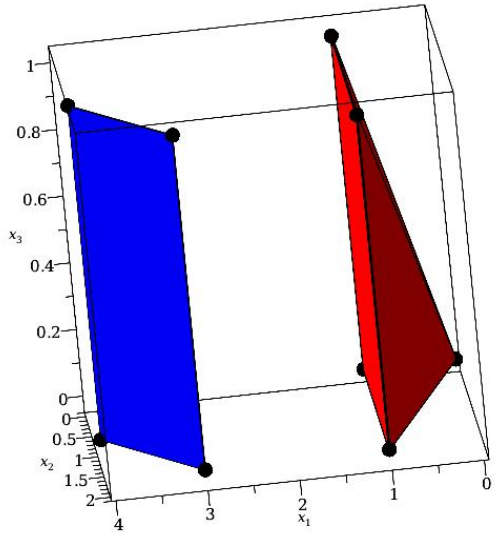
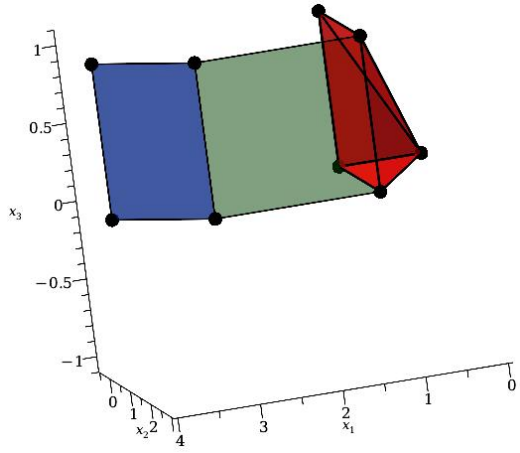
**(4)** return $\mathcal{F}$.

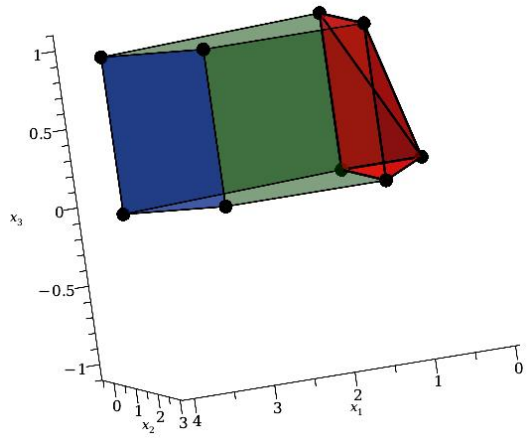
# The bounded case (2/3)



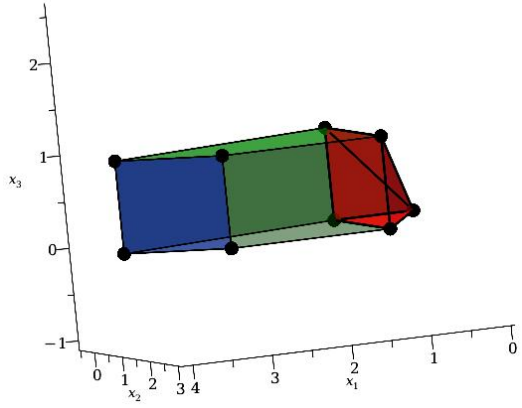

(a) Input



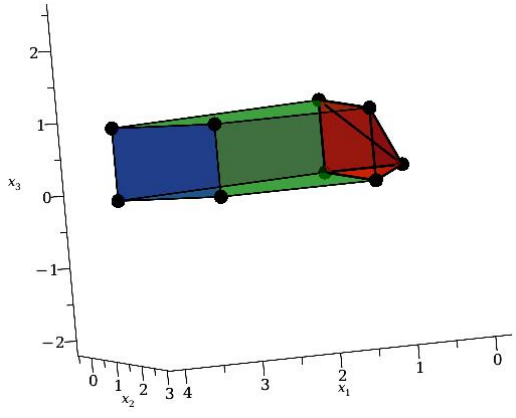

(b) Step 1




(c) Step 2

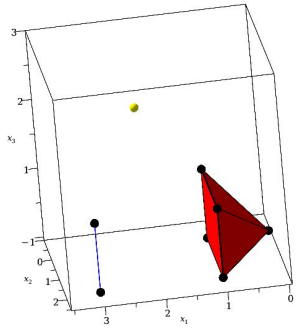



(d) Step 3




(e) Step 4

# The bounded case (3/3)



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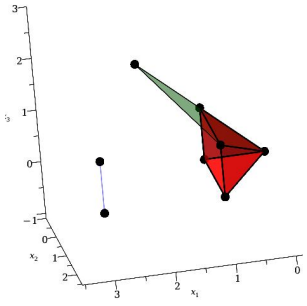
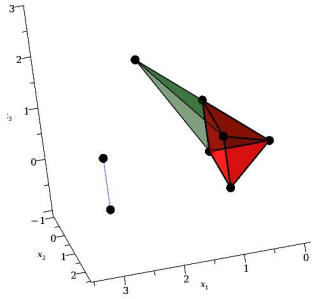
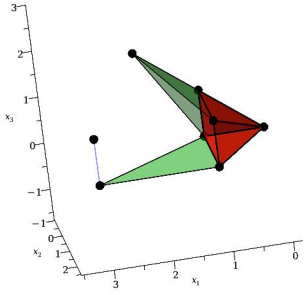


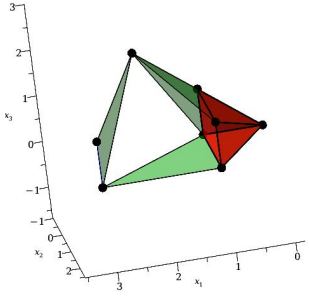
(b) Step 1



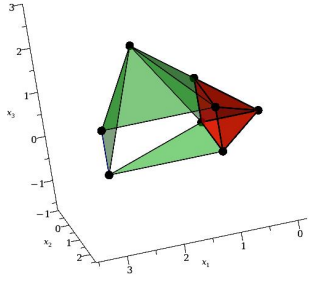
(c) Step 2



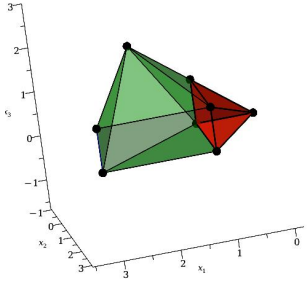
(d) Step 3



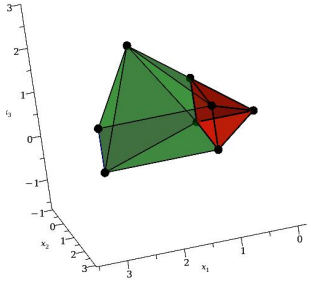
(e) Step 4



(f) Step 5



(g) Step 6



(h) Step 7

# Plan

1. Overview
2. The wrapping procedure
3. Computing ECH in the bounded case
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3. Assume that the homogeneization cone of $P$

$$T = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax - \lambda b \leq 0, \lambda \geq 0 \right\}.$$

is pointed, that is, it does not contain any line.

# Mapping an unbounded to a bounded polyhedron (1/3)

1. Let $P = \{x \in \mathbb{R}^d \mid Ax \leq b\}$ be an unbounded polyhedron.
2. Let $\{v_1, \ldots, v_t\} \subseteq \mathbb{R}^d$ be the vertices of $P$ and let $\{r_1, \ldots, r_s\} \subseteq \mathbb{R}^d$ be its extreme rays.
3. Assume that the homogeneization cone of $P$

$$T = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax - \lambda b \leq 0, \lambda \geq 0 \right\}.$$

   is pointed, that is, it does not contain any line.
4. We transform $P$ into a bounded polyhedron $Q$ given in V-representation, using an invertible transformation matrix $S$.

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4. We transform $P$ into a bounded polyhedron $Q$ given in V-representation, using an invertible transformation matrix $S$.
5. We then show that the H-representation of $P$ can easily be recovered from the H-representation of $Q$.

# Mapping an unbounded to a bounded polyhedron (2/3)

1. Homogenization$(P)$ is the conical hull of the columns of the matrix

$$M = \begin{pmatrix} V & R \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{(d+1)\times(t+s)},$$

where the columns of $V$ and $R$ are of the vertices and extreme rays of $P$.

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6. Removing that last row, we obtain a matrix, the columns of which contain the vertices of the desired polytope $Q$.

# Mapping an unbounded to a bounded polyhedron (3/3)

1. If $Bx \leq c$ is an H-representation of $Q$, then, by definition, we have

$$\text{Homogenization}(Q) = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \;\middle|\; \begin{pmatrix} B & -c \\ \mathbf{0} & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq \mathbf{0} \right\}. \tag{4.1}$$

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2. In order to relate $Q$ to $P$, we define another polyhedral cone in $\mathbb{R}^{d+1}$ by

$$H := \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \middle| \begin{pmatrix} A & -b \\ \mathbf{0} & -1 \end{pmatrix} S^{-1} \begin{pmatrix} x \\ t \end{pmatrix} \leq \mathbf{0} \right\}, \tag{4.2}$$

where $Ax \leq b$ is an H-representation of $P$.

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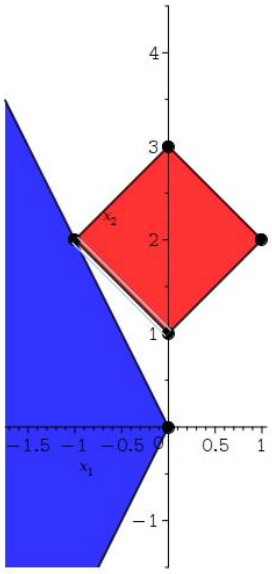
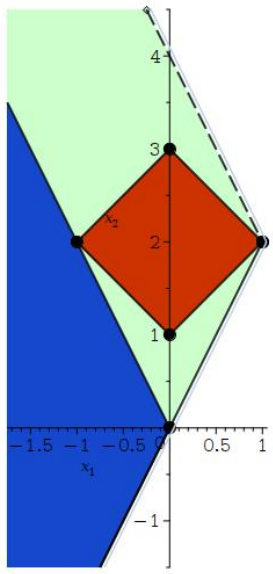
where $Ax \leq b$ is an H-representation of $P$.
3. **Theorem 1:** We have $\text{Homogenization}(Q) = H$.

# The convex hull of two closed sets may not be closed




(a) Input




(b) Output

Bounded polyhedral set:

$$\text{null} \begin{cases} -x_1 - x_2 <= -1 \\ -x_1 + x_2 <= 3 \\ x_1 - x_2 <= -1 \\ x_1 + x_2 <= 3 \end{cases}$$

Unbounded polyhedral set:

$$\begin{cases} x_1 - \frac{x_2}{2} <= 0 \\ x_1 + \frac{x_2}{2} <= 0 \end{cases}$$

The green region is contained in the convex hull of the blue and the red polyhedra. But the dotted line is no.

**Theorem 2:** $\text{ConvexHull}(V) + \text{ConicalHull}(R) = \overline{P_0}$.

# Reduction to the bounded case: algorithm (1/2)

1. Let $t_i$ and $s_i$ be the numbers of vertices and the extreme rays of $P_i$. We define the following matrix

$$M = \begin{pmatrix} V_1 & R_1 & \cdots & V_k & R_k \\ \mathbf{1} & \mathbf{0} & \cdots & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

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3. Removing the last row, the first $t_1 + s_1$ columns correspond to a bounded polyhedron $Q_1$, the next $t_2 + s_2$ columns correspond to a bounded polyhedron $Q_2$, etc.

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5. Let $K_0$ be the homogenization cone of $Q_0$ and $C_0$ be the result of inverting the map $x \longmapsto SxD$ on $K_0$.

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4. Next, we compute $Q_0 :=$ ConvexHull$(Q_1, \ldots, Q_k)$.
5. Let $K_0$ be the homogenization cone of $Q_0$ and $C_0$ be the result of inverting the map $x \longmapsto SxD$ on $K_0$.
6. Finally, dehomogenizing $C_0$, we obtain $\overline{P_0}$,

# Reduction to the bounded case: algorithm (2/2)

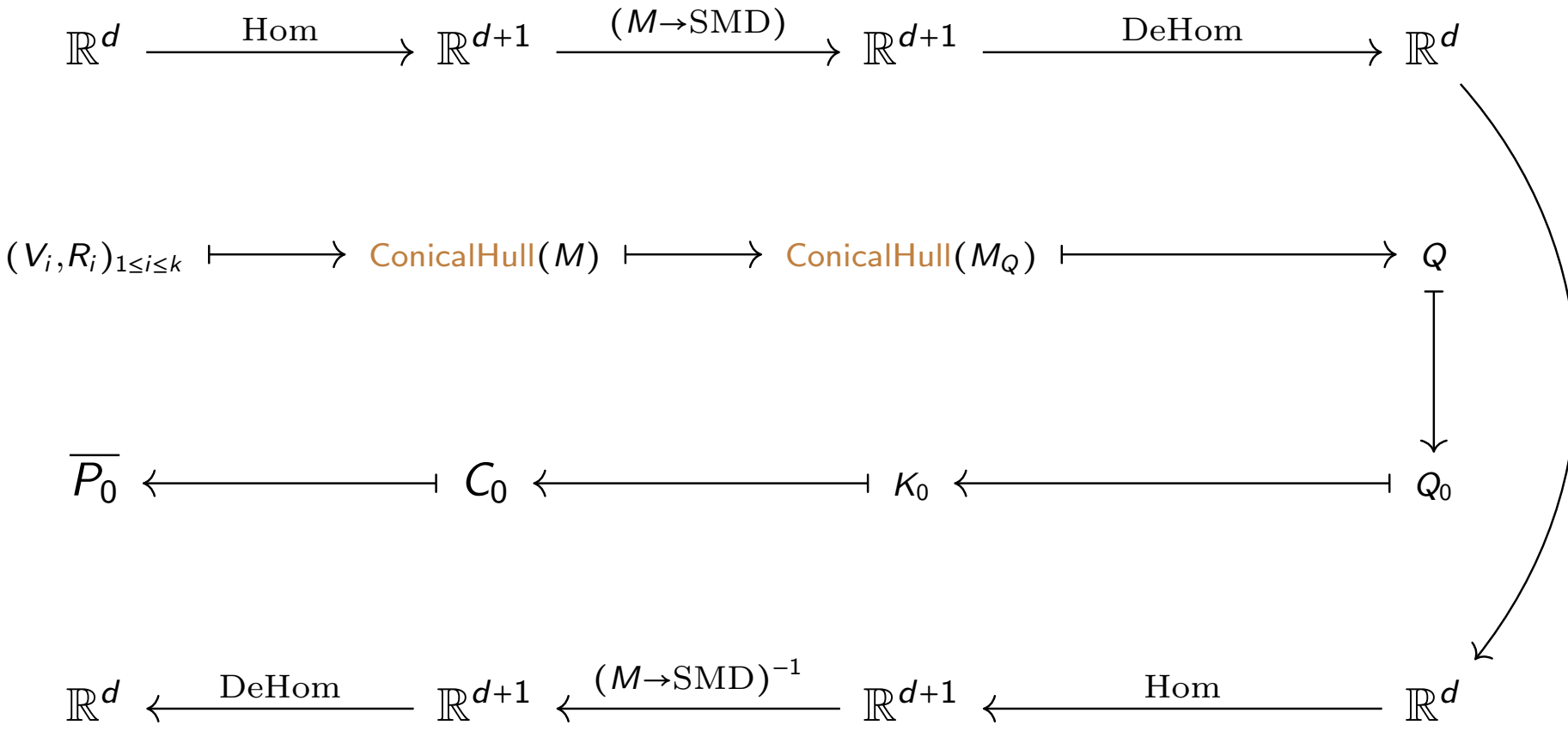
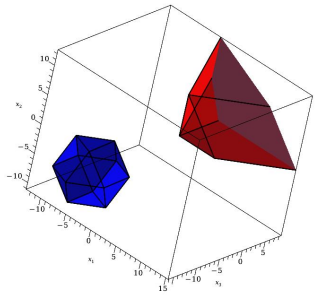



Figure: Reduction to the bounded case in 7 steps.

# Reduction to the bounded case: example (1/2)



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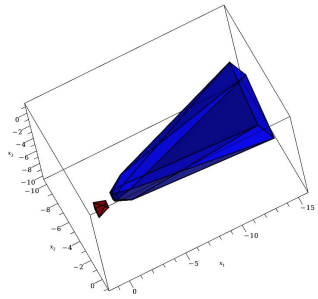
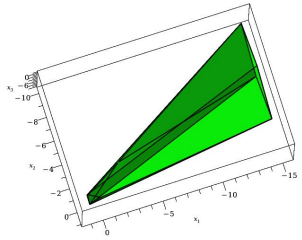
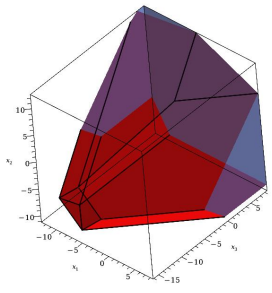


(b) Step 1



(c) Step 2



(d) Step 3

Octahedron:

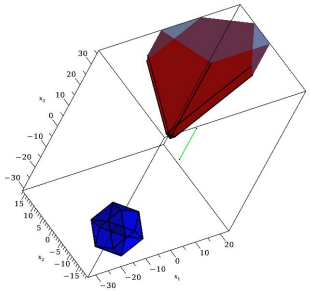
$$\text{null}\begin{cases} -x_3 \le 15 \\ x_3 \le -7 \\ -x_2 \le 10 \\ x_2 \le -2 \\ -x_1 - x_2 - x_3 \le 31 \\ -x_1 - x_2 + x_3 \le 9 \\ -x_1 \le 10 \\ -x_1 + x_2 - x_3 \le 19 \\ -x_1 + x_2 + x_3 \le -3 \\ x_1 - x_2 - x_3 \le 19 \\ -x_2 + x_3 + x_1 \le -3 \\ x_1 \le -2 \\ x_1 + x_2 - x_3 \le 7 \\ x_2 + x_3 + x_1 \le -15 \end{cases}$$

Unbounded polyhedral set:

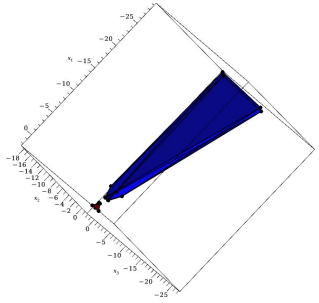
$$\begin{cases} -x_3 \le 0 \\ -x_2 \le 0 \\ -x_1 - \frac{x_2}{2} \le 0 \\ -x_1 - \frac{x_3}{3} \le 0 \\ x_1 + x_2 - x_3 \le 7 \\ x_1 + 2x_2 - x_3 \le 14 \end{cases}$$

Figure: Computation of the convex hull for a truncated octahedron and an unbounded polyhedral set.

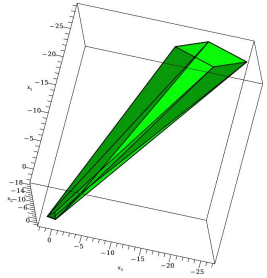
# Reduction to the bounded case: example (2/2)



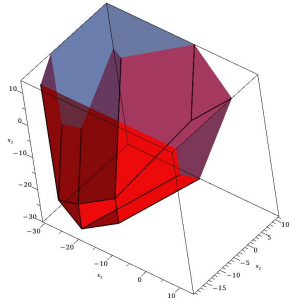
(a) Input



(b) Step 1



(c) Step 2



(d) Step 3

Octahedron:

$$\begin{cases} -x_3 \leq 28 \\ x_3 \leq -16 \\ -x_2 \leq 18 \\ x_2 \leq -6 \\ -x_1 - x_2 - x_3 \leq 66 \\ -x_1 - x_2 + x_3 \leq 22 \\ -x_1 \leq 26 \\ -x_1 + x_2 - x_3 \leq 42 \\ -x_1 + x_2 + x_3 \leq -2 \\ x_1 - x_2 - x_3 \leq 26 \\ -x_2 + x_3 + x_1 \leq -18 \\ x_1 \leq -14 \\ x_1 + x_2 - x_3 \leq 2 \\ x_2 + x_3 + x_1 \leq -42 \end{cases}$$

null

Unbounded polyhedral set:

$$\begin{cases} -x_3 \leq 0 \\ -x_2 \leq 0 \\ -x_1 - \frac{x_2}{2} \leq 0 \\ -x_1 - \frac{x_3}{3} \leq 0 \\ x_1 + x_2 - x_3 \leq 2 \\ x_1 + 2x_2 - x_3 \leq 4 \end{cases}$$

Line segment:

$$\begin{cases} -x_3 \leq 0 \\ x_3 \leq 15 \\ x_2 = -8 \\ x_1 = -2 \end{cases}$$

Figure: Computation of the convex hull for a truncated octahedron, an unbounded polyhedral set and a lower dimensional polytope which is a line segment.

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