On computer-assisted classification of coupled integrable equations.

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An integrable PDE

The very famous Korteweg–de Vries equation

$$u_t + 6\,u\,u_x + u_{xxx} = 0.$$

With the change of variable

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} = 2\frac{f f_x - {f_x}^2}{f^2}$$

we obtain in the 3D case

$$\begin{aligned} f &= 1 + \exp\theta_1 + \exp\theta_2 + \exp\theta_3 + a_{12}\exp(\theta_1+\theta_2) + \\ &= +a_{13}\exp(\theta_1+\theta_3) + a_{23}\exp(\theta_2+\theta_3) + b_{123}\exp(\theta_2+\theta_3+\theta_3) \end{aligned}$$

with

$$\theta_i = \exp(k_i\,x - k_i^3\,t + \delta_i)$$

where a_{ij}, b_{123} are functions of the k_i [Hereman & Nuseir, 1997].

Integrable (systems of) PDEs and solitons

- One of the most powerful tool for solving (systems of) PDEs is the *inverse scattering method.* See the paper by Richard S. Palais *the symmetries of solitons*, 1997.
- The PDEs solvable by this method posses *solitons.*
- Solitons are traveling waves that preserve their shape after a collision with other solitons.
- Solitons are used in several applications: Physics of plasmas, propagation along optic fibers.

Looking for solitons (1)

- No algorithms to construct PDEs solvable by the ISM.
- However ISM-solvable PDEs posses many other rare properties.
- Checking one or several of them is a powerful procedure to compute very good candidates.
- Since these properties are rare, one needs to work with families (of systems) of PDEs.

Looking for solitons (2)

A system of coupled integrable equations that posses solitons :

$$\left\{\begin{array}{lcl} u_t & = & u_{xxxxx} - 10uu_{xxx} + 30vv_{xxx} - 25u_x u_{xx} \\ & & +45v_x v_{xx} + 20u^2 u_x - 30v^2 u_x - 60uvv_x \\ v_t & = & -9v_{xxxxx} + 10vu_{xxx} + 30uv_{xxx} + 35v_x u_{xx} \\ & & +45u_x v_{xx} - 20uvu_x - 20u^2 v_x - 30v^2 v_x \end{array}\right. \qquad (1)$$

[Zhou, Jiang & Jiang, 1990]

Theorem 1 *System (1) is the only one in the following class (that does not belong to a lower class in the same hierarchy).*

$$
\left\{
\begin{array}{rcl}
u_t & = & a_1 u_{xxxxx} + a_2 v_{xxxxx} \\
& & + a_3 u u_{xxx} + a_4 v u_{xxx} + a_5 u v_{xxx} + a_6 v v_{xxx} \\
& & + a_7 u_x u_{xx} + a_8 v_x u_{xx} + a_9 u_x v_{xx} + a_{10} v_x v_{xx} \\
& & + a_{11} u^2 u_x + a_{12} u v u_x + a_{13} v^2 u_x \\
& & + a_{14} u^2 v_x + a_{15} u v v_x + a_{16} v^2 v_x \\
v_t & = & b_1 u_{xxxxx} + b_2 v_{xxxxx} \\
& & + b_3 u u_{xxx} + b_4 v u_{xxx} + b_5 u v_{xxx} + b_6 v v_{xxx} \\
& & + b_7 u_x u_{xx} + b_8 v_x u_{xx} + b_9 u_x v_{xx} + b_{10} v_x v_{xx} \\
& & + b_{11} u^2 u_x + b_{12} u v u_x + b_{13} v^2 u_x \\
& & + b_{14} u^2 v_x + b_{15} u v v_x + b_{16} v^2 v_x
\end{array}
\right.
$$

Higher order generalized symetries (1)

- One of the most popular method to investigate families (of systems) of PDEs.
  - $u_t = F(t, x, u, u_x, u_{xx})$ and other similar scalar equations [Mikhailov, Shabat and Sokolov, 1991].
  - Scalar equations in associative algebras [Olver and Sokolov, 1998].
  - Homogeneous **scalar** evolution equations (with additional conditions) [Sanders and Wang, 1998].
- This lead to the discovery of numerous new ISM-solvable PDEs.

△ <u>We focus now on systems of PDEs.</u>

The classification Problem

- Classification of general classes systems of PDEs requires solving systems of (very complicated) nonlinear PDEs ...
- The only attempts were done by [Mikhailov, Shabat and Sokolov, 1991] and [Svinolupov, 1989] who worked with Burgers-type and Schrödinger-type vector equations.
- Two-component systems of evolution equations come up in many physical applications and have often been considered in the literature [Gerdt and Zharkov, 1990] [Olver, 1993] [Olver and Sokolov, 1998] [Foursov, 1999-2000].
- It thus became necessary to consider narrower classes of equations to process these systems.

Restriction to homogeneous equations (1)

- Most important integrable equations are homogeneous as differential polynomials in a certain weighting scheme.
- The Kaup-Kupershmidt equation

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x \tag{2}$$

- The Sawada-Kotera equation

$$u_t = u_{xxxxx} + 10uu_{xxx} + 10u_x u_{xx} + 20u^2 u_x \tag{3}$$

Restriction to homogeneous equations (2)

- The Korteweg–de Vries equation:

$$u_t + 6u_{xxx} + uu_x = 0 \tag{4}$$

- A class of homogeneous equations that generalizes the KdV equation:

$$\left\{\begin{array}{lcl} u_t & = & a_1 u_{xxx} + a_2 v_{xxx} + a_3 uu_x + a_4 vu_x + a_5 uv_x + a_6 vv_x \\ v_t & = & b_1 u_{xxx} + b_2 v_{xxx} + b_3 uu_x + b_4 vu_x + b_5 uv_x + b_6 vv_x \end{array}\right. \tag{5}$$

Higher order generalized symetries (2)

Let us consider a system of two evolution equations

$$\begin{cases} u_t = F[u,v] \\ v_t = G[u,v] \end{cases} \tag{6}$$

A second system of t-independent evolution equations

$$\begin{cases} u_t &= & Q_1[u,v] \\ v_t &= & Q_2[u,v] \end{cases} \tag{7}$$

is a *generalized symmetry* of (6) if the *obstruction condition* holds

$$\mathbf{D_K}(\mathbf{Q}) - \mathbf{D_Q}(\mathbf{K}) = \mathbf{0}. \tag{8}$$

Here $\mathbf{Q} = (Q_1, Q_2)$, $\mathbf{K}[\mathbf{u}] = (F[\mathbf{u}], G[\mathbf{u}])$, $\mathbf{u} = (u,v)$ and $\mathbf{D_K}$ denotes the Fréchet derivative of $\mathbf{K}$ defined by $\mathbf{D_K}(\mathbf{Q}) = \frac{d}{d\varepsilon}\mathbf{K}[\mathbf{u} + \varepsilon\mathbf{Q}[\mathbf{u}]]\Big|_{\varepsilon=0}$.

Higher order generalized symmetries (3)

- Most important integrable equations are homogeneous as differential polynomials in a certain weighting scheme.
- Moreover their generalized symmetries can be split into homogeneous components, each of which is a symmetry itself.

⇒ In this case the *obstruction condition* becomes a system of algebraic equations.

- For a generic 2-component equation E_a of type

$$\begin{cases} u_t = \alpha u_{nx} + \beta v_{nx} + lower\,order\,terms \\ v_t = \gamma u_{nx} + \delta v_{nx} + lower\,order\,terms \end{cases} \tag{9}$$

of order a, together with a generalized symmetry S_b of the same type and order $b > a$, both homogeneous in the KdV weighting we consider now the cases $(a, b) = (3, 5), (3, 7), (5, 7)$.

The obstruction systems

- Each obstruction condition says that some differential polynomial is null leading to a system of algebraic equations, the *obstruction system.*

- Obstruction systems are quite large.

(a,b)	(3,5)	(3,7)	(5,7)
(#eqs,#vars)	(110,44)	(255,92)	(543,112)
$(\overline{\#vars}, \overline{\#terms})$	(14,9)	(17,12)	(29,23)

- Interesting solutions are exceptional and there are infinitely many undesirable solutions (systems that can be decoupled).

$\Rightarrow$ We limit their number by adding inequalities, but the dimension of the consistent obstruction systems remain < 0.

Solving the obstruction systems

(1) Equations E_a and S_b split into canonical forms, splitting each obstruction system into 17 cases.

$\Rightarrow$ Does not reduce #eqs and #vars significantly but (together with the inequalities) renders efficient the following for most systems.

(2) For a system S to be solved <u>compute</u>

- a new ordering of the variables,
- a triangular set T (w.r.t. this new ordering)
- and a polynomial set R

such that:

- $\mathbf{V}(S) = \mathbf{W}(T) \cap \mathbf{V}(R)$,
- For every $p \in T$, $\mathsf{mvar}(p) >$ every variable in R,
- For every $\zeta \in \mathbf{V}(R)$ the set T specializes to a regular chain.

Solving the obstruction systems (2)

(3) We split the obstruction system into subsystems according to the total degree of the monomials in the obstruction polynomial.

(4) We use an incremental and parallel method [Moreno Maza, 2000]. The current implementation (in ALDOR) simulates parallelism.

(5) There are several other tricks like *time re-scaling* to raise new relations.

The results

Theorem 2 *A two-component system of evolution equations, homogeneous of order 3 in the KdV weighting, that possesses a symmetry of order 5, 7 or 9, can be reduced by a linear change of dependent variables either to a decoupled form, or to one of the following:*

$$\begin{cases} u_t = u_{xxx} + uu_x + vv_x \\ v_t = -2v_{xxx} - uv_x \end{cases} \tag{10}$$

$$\begin{cases} u_t = u_{xxx} + 3uu_x + 3vv_x \\ v_t = vu_x + uv_x \end{cases} \tag{11}$$

$$\begin{cases} u_t = u_{xxx} + v_{xxx} + 2vu_x + 2uv_x \\ v_t = v_{xxx} - 9uu_x + 6vu_x + 3uv_x + 2vv_x \end{cases} \tag{12}$$

$$\begin{cases} u_t &= u_{xxx} + 2vu_x + uv_x \\ v_t &= uu_x \end{cases} \tag{13}$$

$$\begin{cases} u_t &= 4u_{xxx} + 3v_{xxx} + 4uu_x + vu_x + 2uv_x \\ v_t &= 3u_{xxx} + v_{xxx} - 4vu_x - 2uv_x - 2vv_x \end{cases} \tag{14}$$

(Hirota-Satsuma, Ito, Drinfeld-Sokolov, and 2 <u>new systems</u>).

Consider the Boussinesq equation as a system of evolution equations

$$\left\{ \begin{array}{lcl} u_t & = & -3v_{xx} \\ v_t & = & u_{xx} + 4u^2 \end{array} \right. \tag{15}$$

Theorem 3 *An two-component system of evolution equations, homogeneous of order 5 in the KdV weighting, that possesses a symmetry of order 7, can be reduced by a linear change of dependent variables either to a decoupled form, to a symmetry of a lower-order equation ((15)-(14)), or else to*

$$\left\{ \begin{array}{lcl} u_t & = & u_{xxxxx} - 10uu_{xxx} + 30vv_{xxx} - 25u_xu_{xx} \\ & & +45v_xv_{xx} + 20u^2u_x - 30v^2u_x - 60uvv_x \\ v_t & = & -9v_{xxxxx} + 10vu_{xxx} + 30uv_{xxx} + 35v_xu_{xx} \\ & & +45u_xv_{xx} - 20uvu_x - 20u^2v_x - 30v^2v_x \end{array} \right. \tag{16}$$

(Zhou, Jiang & Jiang).