On the relationship between the Kaup–Kupershmidt and Sawada–Kotera equations

Mikhail V. Foursov * and Marc Moreno Maza †

Université de Lille-I, LIFL, 59655 Villeneuve d'Ascq Cedex, France

Abstract

In this paper we show a previously unknown relation between the Kaup–Kupershmidt and the Sawada–Kotera equations. Both equations are scalar reductions of a certain two-component system appearing in the Boussinesq hierarchy. Moreover, this relation may be the reason why it was more difficult to construct multi-soliton solutions for the Kaup–Kupershmidt equation then for the Sawada–Kotera equation.

*e-mail: foursov@lifl.fr

†e-mail: moreno@lifl.fr

1 Introduction

The Kaup–Kupershmidt (KK) equation [12]

$$u_t = u_{xxxxx} + 10uu_{xxx} + 10u_x u_{xx} + 20u^2 u_x \tag{1}$$

and the Sawada–Kotera (SK) equation [24] (sometimes called the Caudrey–Dodd–Gibbon equation [2])

$$u_t = u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x \tag{2}$$

are among the equations of the greatest mathematical and physical importance. Being the simplest integrable scalar evolution equations of order higher than 3, they helped to test and validate (or refute) several techniques of investigation of integrable evolution equations, such as the Wahlquist–Estabrook prolongation method [11] or the method of constructing Lax pairs from the Painlev analysis [16].

Since these two equations seem to be very similar, many authors tried to find relations between them (e.g. [5, 22]). Yet no simple and immediately usable relation was ever found.

In this paper, we propose a solution to this problem. We show that both the KK and SK equations appear as scalar reductions of a certain fifth-order two-component system from the Boussinesq hierarchy. In particular, the multi-soliton solutions can simply be found from the multi-soliton solutions of this coupled system. Moreover, the fact that the multi-soliton solutions of the KK equation were much harder to construct [20, 21] is explained since the KK equation is a scalar reduction of the "bad" Boussinesq equation.

2 Relations between the Kaup–Kupershmidt and Sawada–Kotera equations

Despite the striking similarities between the SK and KK equations, there is no point or contact transformation relating them. They are connected by a Bcklund transformation [4], but it does not allow to construct in a simple way the explicit solutions of one equation from the explicit solutions of the other one.

And yet, the properties of both equations are practically the same. Both of them possess a bi-Hamiltonian formulation [25]???, the Painlev property [???] and an auto-

Bcklund transformation [17]. They are associated with third-order scattering problems [5, 12] and one obtains Hnon-Heiles systems as scalar similarity reduction of both equations [10]. Moreover, they are both related to the Kupershmidt equation by a Miura-type map [5] (*en passant*, we think the Kupershmidt equation should rather be called the Fordy–Gibbons equation).

On the other hand, their differences are rather minor. For example, even though the SK equation can be written in a bilinear form in two different ways [23] (while there is only one way for the KK equation [20]), these two ways are equivalent.

However, in spite of all the appearances, the KK equation is still much harder to deal with. For example, the auto-Bcklund transformation for the SK equation was found back in 1977 [23], while it was only found in 1998 [17] for the KK equation. (Rogers and Carillo's earlier transformation [22] does not involve the crucial Bcklund parameter and thus cannot be used for constructing the solutions.) Furthermore, it is still not known whether the Hirota's method can provide a Bcklund transformation for the KK equation that can be linearized in the appropriate way.

Moreover, whereas the multi-soliton solutions for the SK equation are well-known [2, 23, 24], only implicit multi-soliton solutions for the KK equation were known until 2000. Using the inverse scattering scheme, Kaup [12] could only find a solitary wave in 1980. In 1994, Leble and Ustinov [13] did obtain a formula for the multi-soliton solutions using the Darboux transformation, but only implicitly. The explicit two- and three-soliton solutions were first found computationally by Hereman and Nuseir [9] in 1997. Finally, the explicit formulae for the N-soliton solutions were produced by Parker [20, 21] using a novel approach in 2000. The "anormal" structure of these solutions was explained by Musette and Verhoeven [18].

3 Some definitions

Let us consider a system of two equations

$$\begin{cases} u_t = F[u,v] \\ v_t = G[u,v] \end{cases} \tag{3}$$

where $F[u,v] = F(u, v, u_x, v_x, u_{xx}, \ldots)$ denotes a differential polynomial function of u and v, i.e. a function of u, v and their x-derivatives.

Definition: A second system of t-independent evolution equations

$$\begin{cases} u_t = Q_1[u,v] \\ v_t = Q_2[u,v] \end{cases} \tag{4}$$

is said to be a *generalized symmetry* of (3) if their flows formally commute

$$\mathbf{D_K}(\mathbf{Q}) - \mathbf{D_Q}(\mathbf{K}) = 0. \tag{5}$$

Here $\mathbf{Q} = (Q_1, Q_2)$, $\mathbf{K}[u,v] = (F[u,v], G[u,v])$ and $\mathbf{D_K}$ denotes the Fréchet derivative of $\mathbf{K}$ defined by $\mathbf{D_K}(\mathbf{Q}) = \frac{d}{d\varepsilon}\mathbf{K}[\mathbf{u} + \varepsilon\mathbf{Q}[\mathbf{u}]]\Big|_{\varepsilon=0}$. For a more detailed explanation of generalized symmetries, see Olver [19].

Definition: Let us call the system (3) *integrable* if it possesses infinitely many generalized symmetries [3].

4 Relation of the Kaup-Kupershmidt and Sawada-Kotera equations to the Boussinesq hierarchy

The *Boussinesq equation*

$$u_{tt} = \pm 3(u_{xx} + 4u^2)_{xx} \tag{6}$$

arises in the study of bi-directional propagation of long waves in shallow water [1]. It also describes one-dimensional nonlinear lattice-waves, vibrations in a nonlinear string and ion sound waves in a plasma. Zakharov [26] was first to notice that one can rewrite it as an equivalent integrable evolutionary system

$$\begin{cases} u_t &= \pm 3w_x \\ w_t &= u_{xxx} + 8uu_x \end{cases} \tag{7}$$

possessing infinitely many commuting conserved densities. Later the *Boussinesq system* (7) was shown to be solvable by the inverse scattering [27] and its bi-Hamiltonian formulation was found [19].

However, the properties of the equation and of its solutions depend on the sign on the right-hand side of the equation. The geometric properties of two versions are, of course, exactly the same, since nothing forbids us from making an imaginary change of the time variable $t \to i\,t$. The analytic properties are though different. Following the terminology of McKean [14], we call the version

$$u_{tt} = -3u_{xxxx} - 12(u^2)_{xx} \tag{8}$$

the *good Boussinesq equation*, and the other version

$$u_{tt} = 3u_{xxxx} + 12(u^2)_{xx} \tag{9}$$

the *bad Boussinesq equation.* The reason for calling them the "good" and "bad" versions lies in the analytic properties of its solutions. In particular, it is a much harder problem to construct solutions to the bad Boussinesq equation (9), see [14].

We remark that the Boussinesq system (7) is not the only way to write the Boussinesq equation as a integrable system of evolution equations. Beside the system equivalent to (7) by an invertible contact change of variables [6, 8], there exists another less-known system found by Mikhailov, Shabat & Yamilov [15], which we will write in the form

$$\left\{\begin{array}{lcc} u_t & = & \pm 3v_{xx} \\ v_t & = & u_{xx} + 4u^2 \end{array}\right. \tag{10}$$

We will call the system (10) the *nonstandard Boussinesq system.* Mikhailov, Shabat & Yamilov showed that it possesses infinitely many generalized symmetries and conservation laws.

However, we were unable to find any further investigation of (10). We do not know whether it possesses a Lax pair and whether can be directly solved by the inverse scattering. One does not need though a special inverse scattering scheme to construct solutions (and in particular multi-soliton solutions) to (10). Taking a solution (u, w) to the standard Boussinesq system (7), one immediately obtains a solution (u, v) of the nonstandard Boussinesq system (10) by taking $(u, v) = (u, \int w dx)$. This fact is a simple consequence of the relation $w = v_x$ obtained from (7) and (10). This was probably the main reason not to study the nonstandard Boussinesq system (10) in more detail.

And yet, this is the equation we are interested in. Let us now look at its fifth-order symmetry

$$\left\{\begin{array}{lcc} u_t & = & u_{xxxxx} + 10uu_{xxx} + 25u_xu_{xx} \pm 15v_xv_{xx} + 20u^2u_x \\ v_t & = & v_{xxxxx} + 10uv_{xxx} + 5v_xu_{xx} + 5u_xv_{xx} + 20u^2v_x \end{array}\right. \tag{11}$$

The substitution $v = 0$ reduces both versions to the KK equation, while the substitution $v = \pm u$ reduced the "good" version to the SK equation. To reduce the other version to the SK equation, we would have to make the complex substitution $v = iu$. However, such a substitution is equivalent to an imaginary change of the time variable, and we know that this would change the analytic properties of the solutions.

As we remarked before, the solutions of the good Boussinesq equation (8) were much simpler to construct than the solutions of the bad Boussinesq equation (9) [14]. In our opinion, the fact that the KK equation is a reduction of equations from both the good and the bad Boussinesq hierarchies, whereas the SK equation comes only from the good Boussinesq hierarchy, explains why it was so difficult to find the multi-soliton solutions of the KK equation, compared to the SK equation.

5 Conclusion

In this article, we showed that the KK and SK equations are reductions of the fifth-order symmetry of a nonstandard form of the Boussinesq equation. We hope that it will help to shed some light on the differences and similarities of the KK and SK equations. It would also be interesting to study how the multi-soliton solutions of the nonstandard Boussinesq system (10) reduce to the multi-soliton solutions of the KK and SK equations.

We obtained the main result of this paper while investigating the results of our classification of coupled integrable equations of the KK and SK type [7]. It shows that implementing classifications of integrable evolution equations may lead not only to finding new integrables equations, but also to more theoretical results.

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