Fraction-free Comprehensive LU Decomposition and Variants

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- This paper is an experimental comparison of different strategies for doing PLU.

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This is the most common method, where the L matrix ends up with $1's$ on the diagonal.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Recursive PLU Steps

If the first column of A contains a non-zero entry a, we write

$$P_1 A = \left[\begin{array}{c|c} a & w^T \\ \hline v & A' \end{array}\right]$$

and set $c = 1/a$, otherwise we set $c = 0$.

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Let $v' = P'v$, thus

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P' \end{array}\right] P_1 A = \left[\begin{array}{c|c} 1 & 0 \\ \hline cv' & L' \end{array}\right] \left[\begin{array}{c|c} a & w^T \\ \hline 0 & U' \end{array}\right].$$

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5X5 Kac Murdock Szegö Matrix [2]

$$A5 = \begin{bmatrix} 1 & -p & 0 & 0 & 0 \\ -p & p^2+1 & -p & 0 & 0 \\ 0 & -p & p^2+1 & -p & 0 \\ 0 & 0 & -p & p^2+1 & -p \\ 0 & 0 & 0 & -p & 1 \end{bmatrix}$$

Comprehensive LU Decomposition: Example

Given an input matrix $\mathbf{A}$ and an initially empty constructible set cs:

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Input: $\mathbf{A}$ be an $\mathbf{m} \times \mathbf{n}$-matrix over $\mathbb{K}[X_1, \ldots, X_v]$ and W be a constructible set given by polynomials of $\mathbb{K}[X_1, \ldots, X_v]$.

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2. $\mathbf{A} = \mathbf{P}_i \mathbf{L}_i \mathbf{U}_i \mathbf{Q}_i$ holds at every point of W_i.

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In a nutshell, a *constructible set* is the solution set of a disjunction of conjunctions of polynomial equations and inequations.

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- For every constructible set C one can compute regular systems $S_1, \ldots, S_e$ so that $C = Z(S_1) \cup \cdots \cup Z(S_e)$.
- Given two constructible sets C_1, C_2 represented by regular systems, one can deduce a regular system representation for the sets $C_1 \setminus C_2$ and $C_1 \cap C_2$ and $C_1 \cup C_2$.

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  - Should we use these simplifications in conjunction with fraction-free techniques or not?

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Review of Bareiss Algorithm: pseudo-code

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Algorithm 3 Compute leading principal minors in-place

Input: A — an $n \times n$ matrix with all leading principal minors $[A]_{k,k} \neq 0$

$A_{0,0} \leftarrow 1$

for $k \leftarrow 1$ to $n-1$ **do**

  for $i \leftarrow k+1$ to n **do**

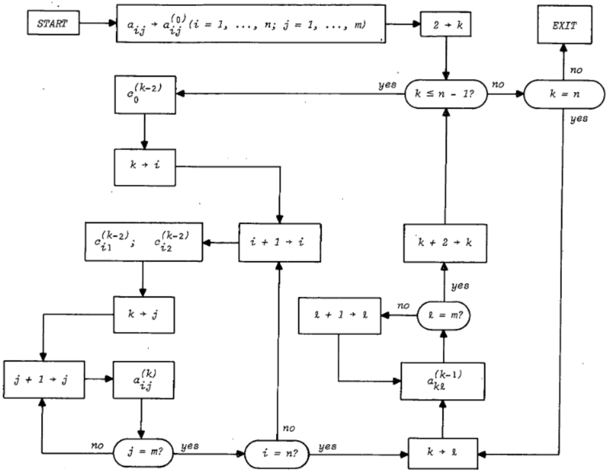
    for $j \leftarrow k+1$ to n **do**

      $A_{i,j} \leftarrow \dfrac{A_{i,j}A_{k,k} - A_{i,k}A_{k,j}}{A_{k-1,k-1}}$ ▷ exact division

    $A_{i,k} \leftarrow 0$

Output: Modified A with $A_{k,k} = [A]_{k,k}$, and $A_{n,n} = \det(A)$

Review of Bareiss Algorithm: Bareiss flow-chart



Fraction-Free LU Factorization

The fraction-free LU form is given by:

$$PA = LD^{-1}U, \quad \text{where}$$

1. the matrices L, D, U are constructed from the determinants of the submatrices $A_{i,j}^{(k)}$ generated during Bareiss algorithm [5], and
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$$L = \begin{bmatrix} A_{1,1}^{(0)} & & & \\ A_{2,1}^{(0)} & A_{2,2}^{(1)} & & \\ \vdots & \vdots & \ddots & \\ A_{n,1}^{(0)} & A_{n,2}^{(1)} & \cdots & A_{n,n}^{(n-1)} \end{bmatrix}, \quad U = \begin{bmatrix} A_{1,1}^{(0)} & A_{1,2}^{(0)} & \cdots & A_{1,m}^{(0)} \\ & A_{2,2}^{(1)} & \cdots & A_{2,m}^{(1)} \\ & & \ddots & \vdots \\ & & & A_{n,m}^{(n-1)} \end{bmatrix}$$

$$D = \begin{bmatrix} A_{1,1}^{(0)} & 0 & \cdots & 0 \\ 0 & A_{1,1}^{(0)}A_{2,2}^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{n-1,n-1}^{(n-2)}A_{n,n}^{(n-1)} \end{bmatrix}$$

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Two strategies for simplifying our matrix entries w.r.t. constructible sets.

$$A \in \mathscr{M}_{m\times n}(\mathbb{K}[\underline{x}]),\ \ W \subseteq \mathbb{K}^n$$

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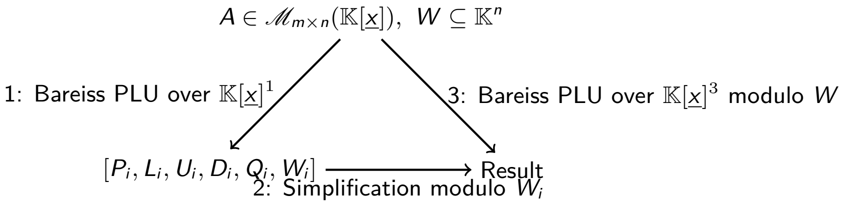

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- Strategy 2:
  3: The entries of the intermediate L_i, D_i, U_i are simplified w.r.t. W_i (after reach row reduction and after each exact division).

Details of the simplification

When we set the `simplified` flag to `true`, the following occur at each branch of the computation:

1. We "deconstruct" the constructible set Wi into a list of regular systems $[[T1, H1], ..., [Tk, Hk]]$.

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3. We compute the normal form of each matrix L_i, D_i, U_i w.r.t. each T_j.

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2. We make sure that each regular chain T_j is normalized (i.e, a Gröbner basis) which may split the computations.
3. We compute the normal form of each matrix L_i, D_i, U_i w.r.t. each T_j.

Details of the simplification

When we set the `simplified` flag to `true`, the following occur at each branch of the computation:

1. We "deconstruct" the constructible set Wi into a list of regular systems $[[T1, H1], ..., [Tk, Hk]]$.
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3. We compute the normal form of each matrix L_i, D_i, U_i w.r.t. each T_j.
4. To perform an exact division $\frac{a}{b} \mod T_j$ we compute the inverse of b w.r.t. T_j, which may split the computations.

We implemented the algorithms in MAPLE The algorithms output a list of cases along with their corresponding constraints. To illustrate these methods, we selected examples from various papers, and ran these examples over $\mathbb{K}[X_1, \ldots, X_v]$ with initial with initial constraints in MAPLE 2024 using an `HP` Pavilion x360 PC with `Windows` 11.

Examples: same as those from our SYNASC 2024 paper plus:

- Sylvester matrices with input constraints hence representative of sub-resultant chain computations (E_{14}, E_{17})
- Vandermonde matrices, hence representative of evaluation-interpolation problems.(E_{20}, E_{24})
- Toeplitz matrices, since LU decomposition gives a quick method for computing the determinant of such matrices (E_{23})

Algorithmic Variants Compared:

Variant	Fraction-Free	Simplification
CLU	No	No
SCLU	No	Yes
FFLU	Yes	No
SFFLU	Yes	Yes

Performance comparison: Time, number of cases(branches), output size

Example	CLU	SCLU	FFLU	SFFLU	Best Performer
E_5	**0.38s** 7 2406	0.46s 12 3525	0.52s 9 3532	0.46s 9 2846	**CLU**
E_{14}	**0.23**s 3 1382	0.25s 7 2801	0.25s 7 3595	0.37s 7 3007	**CLU**
E_{17}	10.94s 3 2500	**5.04s** 18 13281	14.08s 15 18253	11.91s 18 19156	**SCLU**
E_{20}	18.59s 4 4637	**0.29s** 4 2037	12.37s 4 16118	1.66s 4 2112	**SCLU**
E_{23}	10.69s 12 9957	**5.03s** 20 9647	11.64s 22 22584	6.42s 20 10269	**SCLU**
E_{24}	20.46s 2 3386	**0.44s** 2 2810	886.11s 2 62673	7.01s 2 2949	**SCLU**
E_4	0.39s 9 2121	0.39s 11 2393	0.41s 11 3013	**0.35s** 9 2163	**SFFLU**

Conclusions

- No single variant dominates across all test cases.
- For most simple test cases (less than a 1 sec for all methods) CLU (that is, no simplification, no fraction-free) works best.
- For most harder test cases, SCLU (simplification, but no fraction-free) works best (time and size).
- For some of these harder test cases, CLU alone does better than FFCLU (fraction-free, but no simplification)
- For all harder test cases, SFFLU (simplification + fraction-free) comes second.
- Our results illustrate the fact that
  1. using fraction-free methods is not always helpful (think of subresultant chain with no defective subresultants),
  2. it is hard to build examples where systematic simplification is a very bad idea.

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