

## Introduction

The Rosenfeld-Gröbner algorithm [1] is a central tool in constructive differential algebra [2,3]. It computes a *regular decomposition* of a radical differential ideal I, i.e., represents I as an intersection of ideals specified by regular systems [1] of differential polynomials. This decomposition allows to solve the membership problem for I.

No complexity estimates are known for the Rosenfeld-Gröbner algorithm. A first natural step towards estimating the complexity is obtaining a bound on the orders of derivatives in the polynomials computed by the Rosenfeld-Gröbner algorithm. For the particular cases of linear systems and systems with two polynomials in two differential indeterminates, relevant bounds are in the works of Jacobi, Ritt [2], Cohn [4], Lando, and Tomasovic. We prove a bound, which holds for all systems of ordinary differential polynomials in n differential indeterminates.

The performance of Rosenfeld-Gröbner significantly depends on the *ranking* on derivatives. Usually, *elimination rankings* yield more expensive computations than *orderly rankings*. So, the problem of efficient *transformation* of a regular decomposition of a radical differential ideal from one ranking to another arises.

For a prime differential ideal, the problem reduces to the transformation of its *characteristic set* [5,6,7]. We propose a different approach: **an algorithm that transforms the characteristic set algebraically**, taking advantage of the existing efficient techniques for the algebraic case. We have also generalized this approach to address the problem of algebraic transformation of a characteristic decomposition [8] of a radical differential ideal from one ranking to another (omitted in this poster).

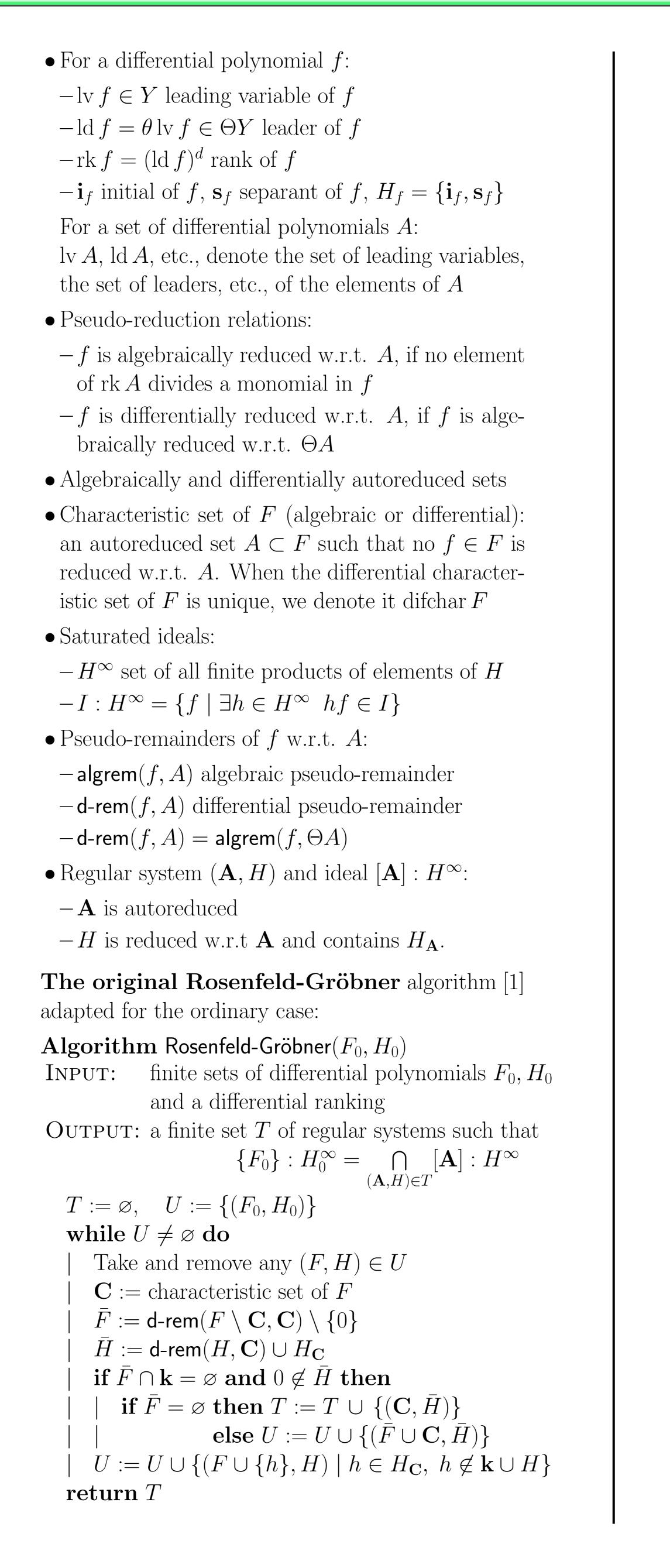
Basic concepts

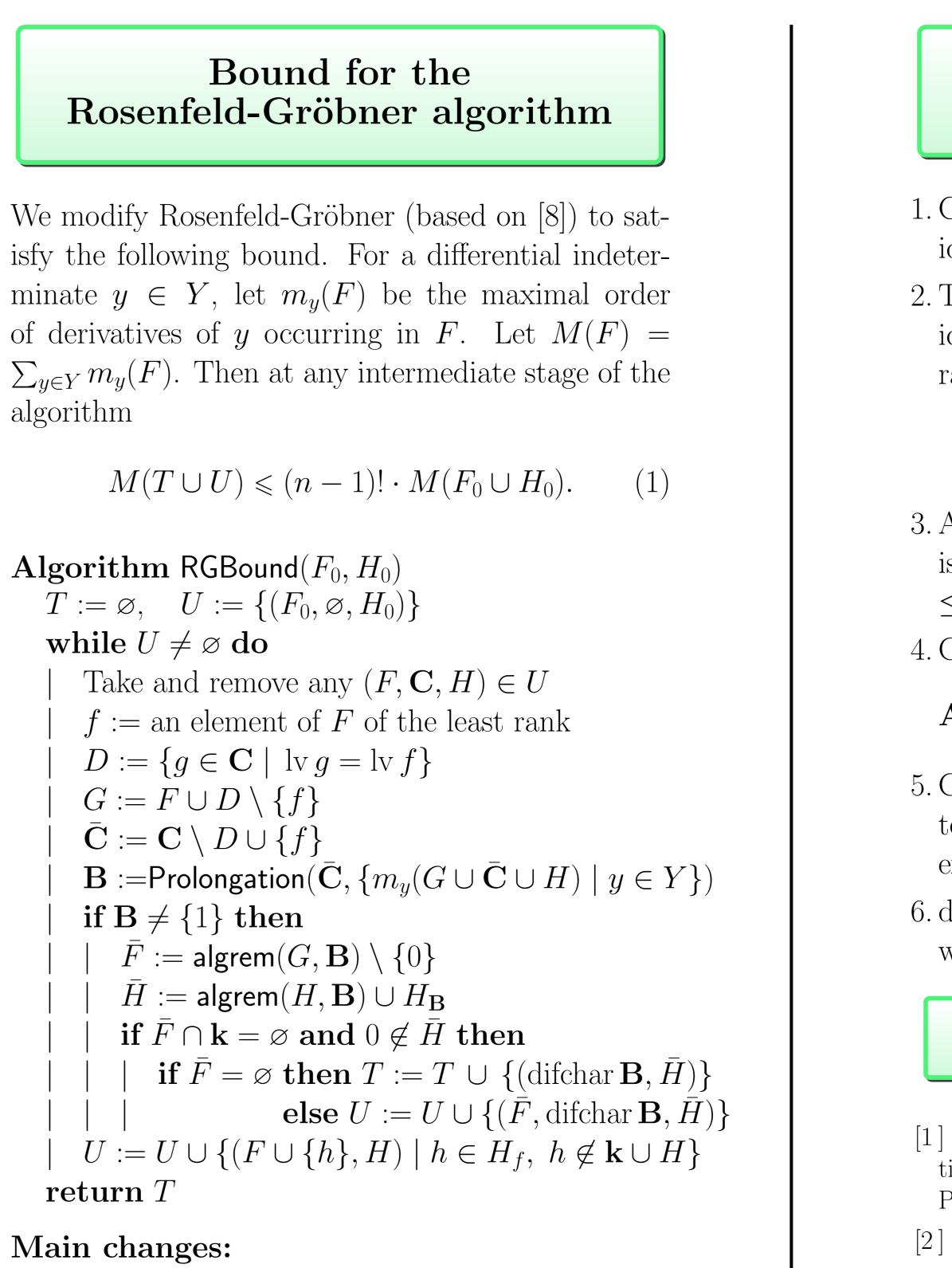
We treat the **ordinary case**, i.e.:

- **k** differential field of *characteristic zero* with derivation  $\delta : \mathbf{k} \to \mathbf{k}$
- $Y = \{y_1, \ldots, y_n\}$  set of differential indeterminates
- $\Theta Y = \{\delta^i y_j \mid i \ge 0, \ 1 \le j \le n\}$  set of derivatives
- $\mathbf{k}{Y} = \mathbf{k}[\Theta Y]$  the ring of differential polynomials over  $\mathbf{k}$  in *n* differential indeterminates
- Ranking  $\leq$  on derivatives  $\Theta Y$

## Bounds and algebraic algorithms for ordinary differential characteristic sets

Oleg Golubitsky<sup>a</sup>, Marina Kondratieva<sup>b</sup>, Marc Moreno Maza<sup>c</sup>, and Alexey Ovchinnikov<sup>d</sup> a: School of Computing, Queen's University, Kingston, Ontario, Canada, K7L 3N6, oleg.golubitsky@gmail.com b: Department of Mechanics and Mathematics, Moscow State University, Leninskie Gory, Moscow, Russia, 119992, kondra\_m@shade.msu.ru c: ORCCA, University of Western Ontario, London, Ontario, Canada, N6A 5B7, moreno@orcca.on.ca d: Department of Mathematics, North Carolina State University, Raleigh, NC, USA, 27695-8205, aiovchin@ncsu.edu ISSAC, July, 2006.





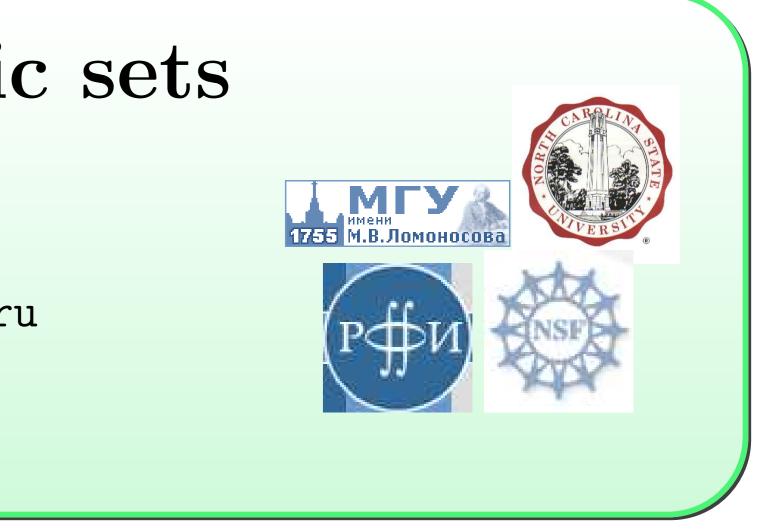
- In the original algorithm, set  $\mathbf{C}$  was autoreduced; in Algorithm **RGBound**, we can only claim that the leading variables of its elements are distinct. As a result, when new pseudo-remainders are replacing old elements of  $\mathbf{C}$ , no indeterminates disappear from lv  $\mathbf{C}$ , i.e., lv  $\overline{\mathbf{C}} \supseteq$  lv  $\mathbf{C}$ .
- d-rem $(F \setminus C, C)$  is replaced by algrem(G, B). Here *G* plays the role of  $F \setminus C$ , while **B** is a *differentiated* and *autoreduced* set **C** computed by Algorithm Prolongation in such a way that:

-it can be used in algrem as C in d-rem

-the orders of derivatives of non-leading variables  $y \in Y \setminus \operatorname{lv} \overline{\mathbf{C}}$  occurring in **B** satisfy

$$m_y(\mathbf{B}) \le m_y(\bar{\mathbf{C}}) + \sum_{z \in \text{lv}\,\bar{\mathbf{C}}} [m_z(G \cup \bar{\mathbf{C}} \cup H) - m_z(\text{ld}\,\bar{\mathbf{C}})].$$

This inequality is the key for proving bound (1).



## Algebraic transformation of characteristic sets

1. Given a characteristic set  $\mathbf{C}$  of a prime differential ideal I w.r.t. a ranking  $\leq$ .

2. The orders of derivatives occurring in the canonical characteristic set [9]  $\mathbf{D}$  of I w.r.t. any other ranking  $\leq'$  do not exceed the bound

 $M := |\mathbf{C}| \cdot \max_{f \in \mathbf{C}} \operatorname{ord} f.$ 

3. Algebraic ideal  $\overline{I} = I \cap \mathbf{k}\{Y\}_M$ , where  $\mathbf{k}\{Y\}_M$ is the subring of differential polynomials of order  $\leq M$  w.r.t. lv **C**, contains **D**.

4. Compute an algebraic characteristic set of  $\overline{I}$  w.r.t.  $\leq$ :

 $\mathbf{A} := \{ \operatorname{\mathsf{algrem}}(f, \Theta \mathbf{C} \setminus \{f\}) \mid f \in \Theta \mathbf{C}, \text{ ord } \operatorname{ld} f \leq M \}.$ 

5. Given **A**, compute the canonical algebraic characteristic set **B** of  $\overline{I}$  w.r.t.  $\leq'$ , applying one of the existing efficient algebraic algorithms, e.g. [10,11].

6. difchar **B** is the canonical characteristic set of I w.r.t.  $\leq'$ .

## References

[1] F. Boulier, D. Lazard, F. Ollivier, M. Petitot, Representation for the radical of a finitely generated differential ideal, Proc. ISSAC 1995.

[2] J.F. Ritt, *Differential Algebra*, Dover, 1950.

- [3] E.R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, 1973.
- [4] R. Cohn, The Greenspan bound for the order of differential systems, Proc. AMS, 79(4), 1980.
- [5] F. Boulier, Efficient computation of regular differential systems by change of rankings using Kähler differentials, TR Université Lille, 1999.
- [6] F. Boulier, F. Lemaire, M. Moreno Maza, PARDI! TR LIFL 2001.
- [7] O. Golubitsky, Gröbner walk for characteristic sets of prime differential ideals, Proc. CASC 2004.
- [8] E. Hubert, Notes on triangular sets and triangulationdecomposition algorithms II: Differential Systems, Symbolic and Numerical Scientific Computing 2001.
- [9] O. Golubitsky, M. Kondratieva, A. Ovchinnikov, Canonical characteristic sets of characterizable differential ideals, Preprint, 2006.
- [10] X. Dahan, X. Jin, M. Moreno Maza, É Schost, Change of ordering for regular chains in positive dimension, Proc. Maple Conf. 2006 (to appear).
- [11] F. Lemaire, M. Moreno Maza, Y. Xie, The RegularChains library, Proc. Maple Conf. 2005.