Exact Computation of the Real Solutions of Arbitrary Polynomial Systems

Presented by Marc Moreno Maza[1]
joint work with
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Plan

1. Computing the real solutions of polynomial systems symbolically
2. Triangular decomposition of semi-algebraic systems
3. Application to dynamical system analysis

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Solving polynomial systems? What does this mean?

The algebra text book says:

For $F \subset \mathbf{k}[x_1, \ldots, x_n]$ this is simply

- a *primary decomposition* of $\langle F \rangle$ or
- the *irreducible decomposition* of $V(F)$ (the zero set of F in $\overline{\mathbf{k}}^n$).

The computer algebra system does well:

For $F \subset \mathbf{k}[x_1, \ldots, x_n]$, with $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ or $\mathbf{k} = \mathbb{Q}$,

- computing a *Gröbner basis* of $\langle F \rangle$ or
- computing a *triangular decomposition* of $V(F)$.

But most scientists and engineers need:

- For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$, a useful description of the points of $V(F)$ whose coordinates are real.
- For $F \subset \mathbb{Q}[u_1, \ldots, u_d][x_1, \ldots, x_n]$, the real $(x_1, \ldots, x_n)$-solutions as a function of the real parameter $(u_1, \ldots, u_d)$.

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Solving for the real solutions: classical techniques

In dimension zero over $\mathbb{Q}$:

For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$, if $V(F)$ is finite, many standard and efficient techniques apply to identify the real solutions.

In (generic) dimension zero over $\mathbb{Q}[u_1, \ldots, u_d]$:

For $F \subset \mathbb{Q}[u_1, \ldots, u_d][x_1, \ldots, x_n]$ and an integer r one can determine "generic" conditions on $u_1, \ldots, u_d$ for F to admit exactly r real $(x_1, \ldots, x_n)$-solutions

For arbitrary systems:

For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$, one can partition $\mathbb{R}^n$ into *cylindrical cells* where the sign of each $f \in F$ does not change.

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Real root isolation for zero-dimensional systems

Applications Places System — moreno

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$with(RegularChains) : with(SemiAlgebraicSetTools) : with(ParametricSystemTools) :$ with(ParametricSystem

$R := PolynomialRing([x, y, z]);\ \ F := [x\text{^}2 + y + z - 1, x + y\text{^}2 + z - 1, x + y + z\text{^}2 - 1];$

$$polynomial\_ring$$

$$[x^2 + y + z - 1, y^2 + x + z - 1, x + y + z^2 - 1]$$

$dec := Triangularize(F, R);\ \ map(Display, dec, R);$

$$[regular\_chain, regular\_chain, regular\_chain, regular\_chain]$$

$$\left[\begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{cases}, \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0 \end{cases}\right]$$

$boxes := \left[seq\left(op\left(RealRootIsolate\left(rc, R, 'rerr' = \frac{1}{2^9}\right)\right), rc = dec\right)\right];\ map\left(Display, boxes, R\right.$

$$[box, box, box, box, box]$$

$$\left[\begin{cases} x = \left[\frac{3393}{8192}, \frac{6791}{16384}\right] \\ y = \left[\frac{3393}{8192}, \frac{6791}{16384}\right] \\ z = \left[\frac{217167}{524288}, \frac{868669}{2097152}\right] \end{cases}, \begin{cases} x = \left[-\frac{4947}{2048}, -\frac{2471}{1024}\right] \\ y = \left[-\frac{4947}{2048}, -\frac{2471}{1024}\right] \\ z = \left[-\frac{79109}{32768}, -\frac{316435}{131072}\right] \end{cases}, \begin{cases} x = 0 \\ y = 0 \\ z = 1 \end{cases}, \begin{cases} x = 0 \\ y = 1 \\ z = 0 \end{cases}, \begin{cases} x = 1 \\ y = 0 \\ z = 0 \end{cases}\right.$$

Real root classification: generically 0-dimensional systems

Applications Places System moreno

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with(*RegularChains*) : *with*(*SemiAlgebraicSetTools*) : *with*(*ParametricSystemTools*) : with(ParametricSystem

$R := PolynomialRing([x, y, z, \text{epsilon}]);\ \ F := [x\text{^}2 + y + z - 1, x + y\text{^}2 + z - 1, x + y + z\text{^}2 - 1 + \text{epsilon}];$

polynomial_ring

$$[x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1 + \varepsilon]$$

$dec := Triangularize(F, R);\ \ map(Equations, dec, R);$

[*regular_chain*, *regular_chain*]

$$[[2\,x + z^2 + \varepsilon - 1, 2\,y + z^2 + \varepsilon - 1, z^4 + (2\,\varepsilon - 4)\,z^2 + 4\,z - 4\,\varepsilon - 1 + \varepsilon^2], [x + y - 1, y^2 - y + z, z^2 + \varepsilon$$

For which values of epsilon does F have 2 solutions each of which has a positive x–coordinate?

$rrc := RealRootClassification(F, [\ ], [x], [\ ], 1, 2, R);\ Display(rrc[1][1], R);\ Display(rrc[2], R)$

[[*regular_semi_algebraic_set*], *border_polynomial*]

$$\begin{cases} \varepsilon < 0 \textbf{ and } 16\,\varepsilon < -1 \textbf{ and } 5\,\varepsilon - 1 \neq 0 \textbf{ and } 16\,\varepsilon^2 - 71\,\varepsilon + 2 \neq 0 \\ \textbf{or } \varepsilon > 0 \textbf{ and } 16\,\varepsilon + 1 \neq 0 \textbf{ and } 5\,\varepsilon < 1 \textbf{ and } 16\,\varepsilon^2 - 71\,\varepsilon + 2 > 0 \end{cases}$$

$$\left[\varepsilon, \varepsilon - \frac{1}{5}, \varepsilon + \frac{1}{16}, \varepsilon^2 - \frac{71}{16}\,\varepsilon + \frac{1}{8}\right]$$

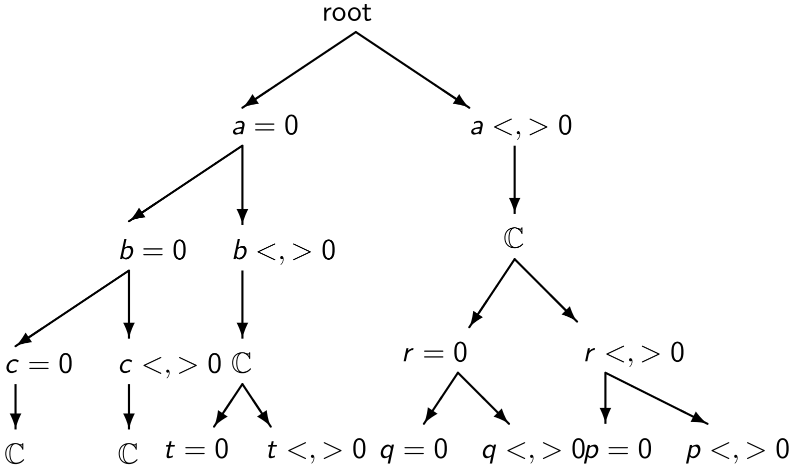
For which values of epsilon does F have 5 solutions ?

$rrc := RealRootClassification(F, [\ ], [\ ], [\ ], 1, 5, R);\ \ Display(rrc[2], R)$

[[ ], *border_polynomial*]

$$\left[\varepsilon, \varepsilon + \frac{1}{16}, \varepsilon^2 - \frac{71}{16}\,\varepsilon + \frac{1}{8}\right]$$

Cylindrical algebraic decomposition of $\{ax^2 + bx + c\}$



The cylindrical algebraic decomposition of $\{ax^2 + bx + c\}$ is given by the tree above, where $t = bx + c$, $q = 2ax + b$, and $r = 4ac - b^2$. This is the best possible output for that method, leading to **27 cells**!

Can a computer program be as good as a high-school student?

For the equation $ax^2 + bx + c = 0$, can a computer program produce?

$$\begin{cases} ax^2 + bx + c = 0 \\ a \neq 0 \wedge b^2 - 4ac > 0 \end{cases} \qquad \begin{cases} 2ax + b = 0 \\ 4ac - b^2 = 0 \\ a \neq 0 \end{cases}$$

$$\begin{cases} bx + c = 0 \\ a = 0 \\ b \neq 0 \end{cases} \qquad \begin{cases} c = 0 \\ b = 0 \\ a = 0 \end{cases}$$

Yes, our new algorithm RealTriangularize can do that!

Applications Places System moreno

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with(RegularChains) : with(SemiAlgebraicSetTools) : with(ParametricSystemTools) : with(ParametricSystem

$R := PolynomialRing([x, c, b, a]);\ F := [a \cdot x^2 + b \cdot x + c];$

$$polynomial\_ring$$

$$[a\,x^2 + b\,x + c]$$

Solving for the real solutions:

RealTriangularize(F, R, output = record);

$$\left\{\begin{array}{c} a\,x^2 + b\,x + c = 0 \\ -4\,c\,a + b^2 > 0 \\ a \neq 0 \end{array}\right., \left\{\begin{array}{c} b\,x + c = 0 \\ b \neq 0 \\ a = 0 \end{array}\right., \left\{\begin{array}{c} c = 0 \\ b = 0 \\ a = 0 \end{array}\right., \left\{\begin{array}{c} 2\,a\,x + b = 0 \\ 4\,a\,c - b^2 = 0 \\ a \neq 0 \end{array}\right.$$

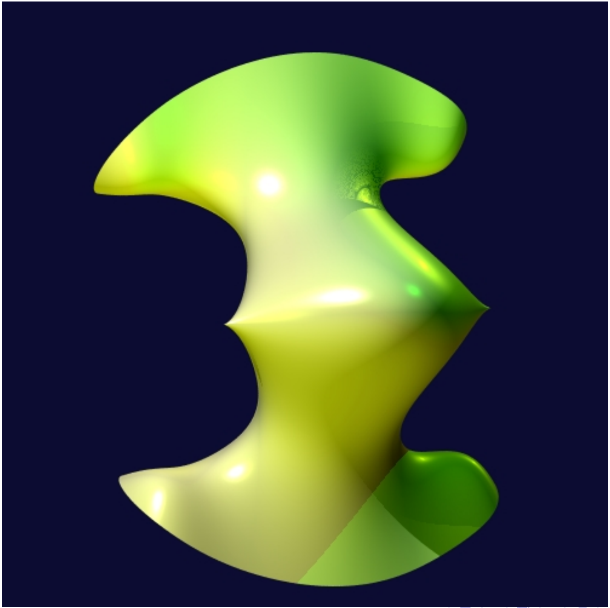
Solving **for** *the complex solutions*

dec := Triangularize(F, R, output = lazard); map(Display, dec, R);

$$[regular\_chain, regular\_chain, regular\_chain]$$

$$\left[\left\{\begin{array}{c} a\,x^2 + b\,x + c = 0 \\ a \neq 0 \end{array}\right., \left\{\begin{array}{c} b\,x + c = 0 \\ a = 0 \\ b \neq 0 \end{array}\right., \left\{\begin{array}{c} c = 0 \\ b = 0 \\ a = 0 \end{array}\right.\right]$$

RealTriangularize applied to the *Eve* surface (1/2)



RealTriangularize applied to the *Eve* surface (2/2)

Applications Places System moreno

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$R := PolynomialRing([x, y, z]);\ F := [5*x^\wedge 2 + 2*x*z^\wedge 2 + 5*y^\wedge 6 + 15*y^\wedge 4 + 5*z^\wedge 2 - 15*y^\wedge 5 - 5*y^\wedge$

$$\mathit{polynomial\_ring}$$

$$[5x^2 + 2xz^2 + 5y^6 + 15y^4 + 5z^2 - 15y^5 - 5y^3]$$

$RealTriangularize(F, R, output = record);$

$$\left\{\begin{array}{l} 5x^2 + 2z^2x + 5y^6 + 15y^4 - 5y^3 - 15y^5 + 5z^2 = 0 \\ 25y^6 - 75y^5 + 75y^4 - z^4 - 25y^3 + 25z^2 < 0 \end{array}\right. ,$$

$$\left\{\begin{array}{c} 5x + z^2 = 0 \\ 25y^6 - 75y^5 + 75y^4 - 25y^3 - z^4 + 25z^2 = 0 \\ 64z^4 - 1600z^2 + 25 > 0 \\ z \neq 0 \\ z - 5 \neq 0 \\ z + 5 \neq 0 \end{array}\right. , \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right. , \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z = 0 \end{array}\right. , \left\{\begin{array}{c} x + 5 = 0 \\ y - 1 = 0 \\ z - 5 = 0 \end{array}\right. ,$$

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Regular chain

Example

$$T := \begin{cases} f_3 = x_4 x_5^2 + 2x_5 + 1 \\ f_2 = (x_1 + x_2) x_3^2 + x_3 + 1 \\ f_1 = x_1^2 - 1. \end{cases}$$

Under the order $x_5 > x_4 > x_3 > x_2 > x_1$,

- $\mathrm{mvar}(f_2) = x_3$ and $\mathrm{mvar}(f_1) = x_1$
- $\mathrm{init}(f_2) = x_1 + x_2$ and $\mathrm{init}(f_1) = 1$
- T is a regular chain. Indeed:
  - $\mathrm{init}(f_2)$ is regular (neither zero nor zero-divisor) modulo $\langle f_1 \rangle$.
  - $\mathrm{init}(f_3)$ is regular modulo $\langle f_1, f_2 \rangle : \mathrm{init}(f_2)^\infty$.

Proposition

Let $T = \{t_1, \ldots, t_s\}$ be a regular chain of $\mathbf{k}[\mathbf{x}]$. Then the saturated ideal $\mathrm{sat}(T) := \langle T \rangle : (\prod_{i=1}^{s} \mathrm{init}(t_i))^\infty$ is a proper equi-dimensional ideal of $\mathbf{k}[\mathbf{x}]$.

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Regular semi-algebraic system

Notation

- Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ be a regular chain with $\mathbf{y} := \{\mathrm{mvar}(t) \mid t \in T\}$ and $\mathbf{u} := \mathbf{x} \setminus \mathbf{y} = u_1, \ldots, u_d$.
- Let P be a finite set of polynomials, s.t. every $f \in P$ is regular modulo $\mathrm{sat}(T)$.
- Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{u}]$.

Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(*i*) $\mathcal{Q}$ defines a non-empty open semi-algebra ic set S in $\mathbb{R}^d$,

(*ii*) the regular system $[T, P]$ specializes well at every point u of S

(*iii*) at each point u of S, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

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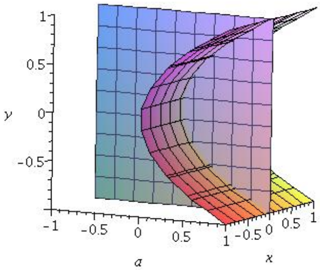
Example

The system $[\mathcal{Q}, T, P_>]$, where

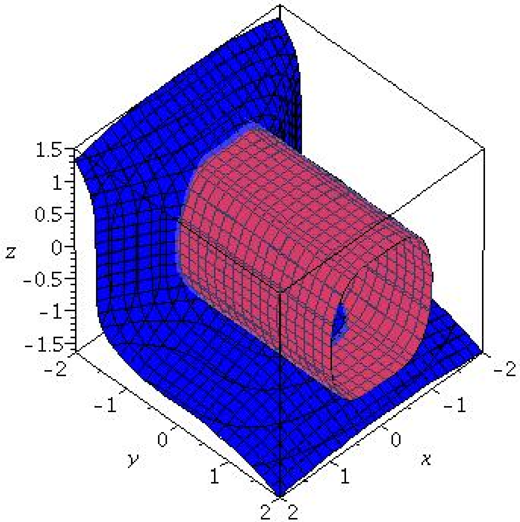
$$\mathcal{Q} := a > 0, \;\; T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases}, \;\; P_> := \{y > 0\}$$

is a regular semi-algebraic system.



RealTriangularize applied to *sofa* and *cylinder* (1/2)

$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$



RealTriangularize applied to *sofa* and *cylinder* (2/2)

Applications Places System — Mon 18 Jul, 3:0

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$$\mathit{true}$$

> $R := \mathit{PolynomialRing}([z, y, x]);\ F := [x\^{}2 + y\^{}3 + z\^{}5,\ x\^{}4 + z\^{}2 - 1];\ \mathit{RealTriangularize}(F, R, \mathit{output} =$

$$R := \mathit{polynomial\_ring}$$

$$F := [x^2 + y^3 + z^5,\ x^4 + z^2 - 1]$$

$$\left[\left\{\begin{array}{c} (-2x^4 + x^8 + 1)\,z + y^3 + x^2 = 0 \\ y^6 + 2x^2y^3 + 10x^{12} - 10x^8 + x^{20} - 5x^{16} + 6x^4 - 1 = 0 \\ x < 1 \\ x + 1 > 0 \\ x^{12} - 4x^8 + 5x^4 - 1 \neq 0 \end{array}\right.,\ \left\{\begin{array}{c} (-2x^4 + x^8 + 1)\,z - x^2 = 0 \\ y^3 + 2x^2 = 0 \\ x^{12} - 4x^8 + 5x^4 - 1 = 0 \end{array}\right.,\right.$$

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>

The RegularChains library in MAPLE (1/2)

Design goals

- Solving polynomial systems over $\mathbb{Q}$ and $\mathbb{F}_p$, including **parametric** systems and **semi-algebraic** systems.
- Offering tools to manipulate their solutions.
- Organized around the concept of a **regular chain**, accommodating all types of solving and providing space-and-time efficiency.

Features

- Use of types for algebraic structures: `polynomial_ring`, `regular_chain`, `constructible_set`, `quantifier_free_formula`, `regular_semi_algebraic_system`.
- Top level commands: `PolynomialRing`, `Triangularize`, `RealTriangularize` `SamplePoints`, ...
- Tool kits: `ConstructibleSetTools`, `ParametricSystemTools`, `FastArithmeticTools`, `SemiAlgebraicSetTools`, ...

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The RegularChains library in MAPLE (2/2)

Classical tools

- Isolating the real solutions of zero-dimensional polynomial systems: `SemiAlgebraicSetTools:-RealRootIsolate`
- Real root classification of parametric polynomial systems: `ParametricSystemTools:-RealRootClassification`
- Cylindrical algebraic decomposition of polynomial systems: `SemiAlgebraicSetTools:-CylindricalAlgebraicDecompose`

New tools

- Triangular decomposition of semi-algebraic systems: `RealTriangularize`
- Sampling all connected components of a semi-algebraic system: `SamplePoints`
- Set-theoretical operations on semi-algebraic sets: `SemiAlgebraicSetTools:-Difference`

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A biochemical network: setting (1/3)

The generic kinetic scheme of prion diseases is illustrated as follows:

$$\begin{array}{l} \downarrow 1 \\ PrP^C \quad \xrightarrow{3} PrP^{Sc} \xrightarrow{4} \text{Aggregates.} \\ \downarrow 2 \end{array}$$

where

- $\left[PrP^C\right]$ is the concentration of PrP^C (harmless form)
- $\left[PrP^{Sc}\right]$ is the concentration of PrP^{Sc} (infectious form) which catalyses the transformation from the normal form to itself,
- Step 1: synthesis of native PrP^C
- Step 2, 4: normal degradation.

A biochemical network: setting (2/3)

Let ν_i be the rate of Step i for $i = 1, \ldots, 4$.

$$
\begin{array}{l}
\downarrow 1 \\
PrP^C \quad \xrightarrow{3} PrP^{Sc} \xrightarrow{4} \text{Aggregates.} \\
\downarrow 2
\end{array}
$$

- Step 1: zero-order kinetic process, that is $\nu_1 = k_1$,
- Step 2, 4: first-order rate equations: $\nu_2 = k_2 \left[PrP^C\right]$, $\nu_4 = k_4 \left[PrP^{Sc}\right]$.
- Step 3: a nonlinear process

$$\nu_3 = \left[PrP^C\right] \frac{a\left(1 + b\left[PrP^{Sc}\right]^n\right)}{1 + c\left[PrP^{Sc}\right]^n}.$$

A biochemical network: setting (3/3)

$$
\begin{array}{l}
\quad\downarrow 1 \\
PrP^{C} \xrightarrow{3} PrP^{Sc} \xrightarrow{4} \text{Aggregates.} \\
\quad\downarrow 2
\end{array}
$$

We also have:

$$
\begin{aligned}
\frac{\mathrm{d}\left[PrP^{C}\right]}{\mathrm{d}t} &= \nu_1 - \nu_2 - \nu_3 \\
\frac{\mathrm{d}\left[PrP^{Sc}\right]}{\mathrm{d}t} &= \nu_3 - \nu_4
\end{aligned}
$$

Letting $x = \left[PrP^{C}\right]$ and $y = \left[PrP^{Sc}\right]$. we obtain the dynamical system:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= k_1 - k_2 x - ax\frac{(1 + by^n)}{1 + cy^n} \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= ax\frac{(1 + by^n)}{1 + cy^n} - k_4 y
\end{aligned}
$$

A biochemical network: semi-algebraic systems to solve

Dynamical system to study

M. Laurent (Biochem. J., 1996) suggests to set $b = 2$, $c = 1/20$, $n = 4$, $a = 1/10$, $k_4 = 50$ and $k_1 = 800$, leading to:

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & f_1 \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & f_2 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{lcl} f_1 & = & \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4} \\ f_2 & = & \frac{2(x+2xy^4-500y-25y^5)}{20+y^4} \end{array} \right. .$$

Semi-algebraic systems to solve

By Routh-Hurwitz criterion, an equilibrium (x, y) is asymptotically stable if

$$\Delta_1 := -(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}) > 0 \ \text{ and } \ a_2 := \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial x} > 0.$$

Letting p_1, p_2 the above polynomials, we obtain two semi-algebraic systems:

$$\mathcal{S}_1 : \{p_1 = p_2 = 0, k_2 > 0\} \text{ and } \mathcal{S}_2 : \{p_1 = p_2 = 0, k_2 > 0, \Delta_1 > 0, a_2 > 0\}$$

A biochemical network: solving $\mathcal{S}_1$

The real solutions of $\mathcal{S}_1$ are described by the following **triangular decomposition** into **regular semi-algebraic systems**.

$$A_1 := \left\{ \begin{array}{ccc} (2y^4+1)x - 25y^5 - 500y & = & 0 \\ (k_2+4)y^5 - 64y^4 + (2+20k_2)y - 32 & = & 0 \\ k_2 & > & 0 \\ r & \neq & 0 \end{array} \right. , \quad A_2 := \left\{ \begin{array}{ccc} t_x & = & 0 \\ t_y & = & 0 \\ r & = & 0 \\ k_2 & > & 0 \end{array} \right. .$$

where $t_y(k_2, y)$ has degree 4 in y and r is given by

$$\begin{array}{rcl} r & := & 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 \\ & - & 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056. \end{array}$$

The polynomial r has four real roots, two are positive: $\alpha_1 < \alpha_2$.

A biochemical network: solving $\mathcal{S}_1$

Regarding k_2 as a parameter, one can compute a **real comprehensive triangular decomposition** which gives:

$$\begin{cases} \{\ \} & k_2 \leq 0 \\ \{A_1\} & 0 < k_2 < \alpha_1 \\ \{A_2\} & k_2 = \alpha_1 \\ \{A_1\} & \alpha_1 < k_2 < \alpha_2 \\ \{A_2\} & k_2 = \alpha_2 \\ \{A_1\} & k_2 > \alpha_2 \end{cases}.$$

From where we deduce the number of real solutions:

$$\begin{cases} 0 & k_2 \leq 0 \\ 1 & k_2 > 0 \text{ and } r > 0 \\ 2 & k_2 > 0 \text{ and } r = 0 \\ 3 & k_2 > 0 \text{ and } r < 0 \end{cases}$$

A biochemical network: conclusion

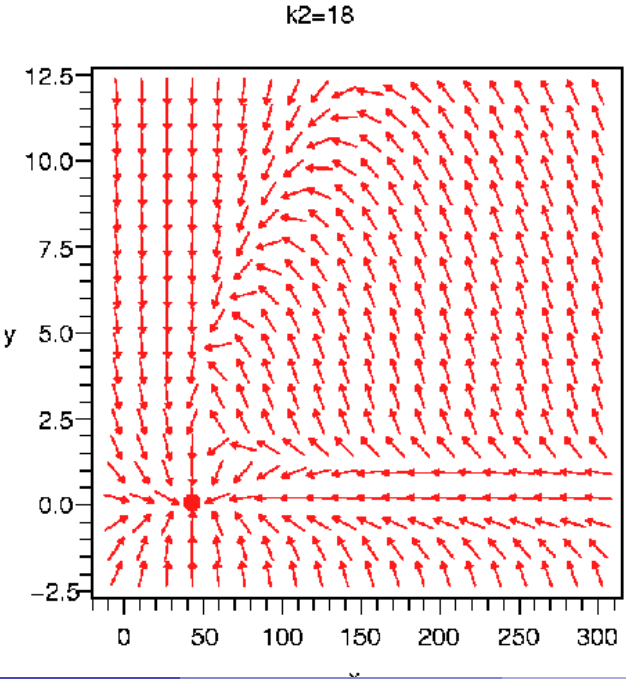
Theorem

Assume that $k_2 > 0$. Then we have: if $r > 0$, then the dynamical system has 1 *equilibrium; if $r = 0$, then it has* 2 *equilibria; if $r < 0$, it has* 3 *equilibria.*

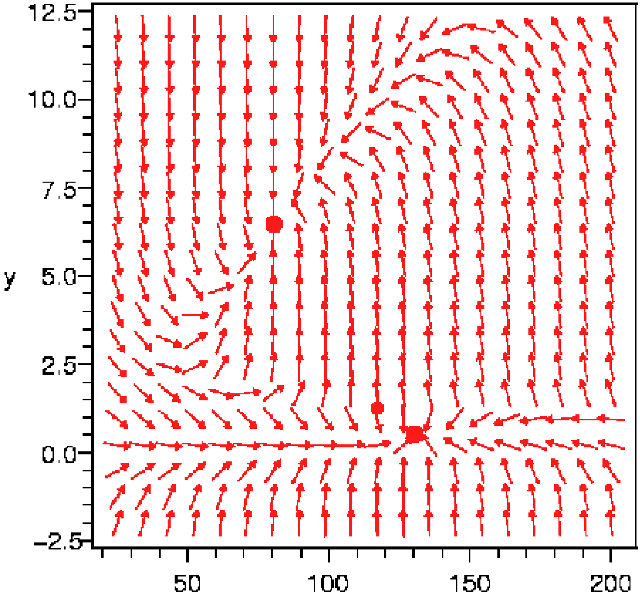
Theorem

Assume that $k_2 > 0$. Then we have: if $r > 0$, then the system has one *hyperbolic equilibrium, which is asymptotically stable; if $r < 0$ and $r_2 \neq 0$, then the system has* three *hyperbolic equilibria, two of which are asymptotically stable and the other one is unstable; if $r = 0$ or $r_2 = 0$, the system experiences a* bifurcation *where*

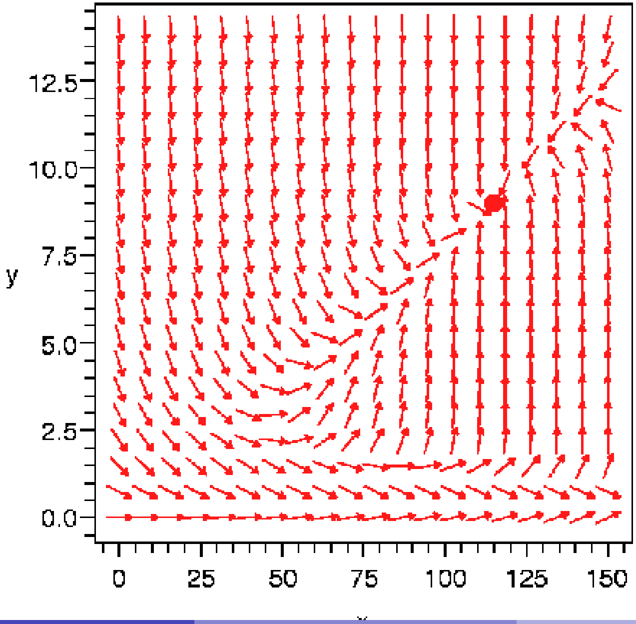
$$
\begin{aligned}
r_2 = {} & 10004737927168 k_2^9 + 624166300700672 k_2^8 + 7000539052537600 k_2^7 \\
& + 45135589467012800 k_2^6 - 840351411856453750 k_2^5 - 5009800435224846875 k_2^4 \\
& - 27388168989455000000 k_2^3 - 86752092666960000000 k_2^2 \\
& + 102960917356800000000 k_2 + 593254606410240000000.
\end{aligned}
$$


k2=18
12.5
10.0
7.5
5.0
2.5
0.0
−2.5
y
0
50
100
150
200
250
300
x

k2 = 8



k2 = 3



Summary and notes

- Solving for the **real roots** of (parametric or not) polynomial systems is a fundamental problem with many applications.
- Most of the time, this requires **exact (thus symbolic) computation**.
- Computer algebra systems used to have limited capabilities for that, especially for the parametric case.
- Recent work (Changbo Chen, James H. Davenport, M^3 , Bican Xia & Rong Xiao, ISSAC 2010-2011) is changing that.
- **RealTriangularize** is available MAPLE 15, as part of the MAPLE **RegularChains** library.