Triangular Decompositions of Polynomial Systems: From Theory to Practice

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Why a tutorial on triangular decompositions?

- The theory is mature:
  - the objects are well understood,
  - the interactions with other theories also,
  - notions and terminologies are unifying.
- The algorithms are evolving very quickly:
  - modular algorithms are available now,
  - complexity estimates also,
  - fast polynomial and matrix arithmetic start to be used.
- The implementation effort is growing
  - triangular decompositions are available in major computer algebra systems,
  - implementation techniques are a priority.

Where are triangular decompositions used?

- Books and Papers, for instance:
  - differential algebra **(Ritt, 1932), (Kolchin, 1973), (Boulier, Lazard, Ollivier & Petitot, 1995), (Kondratieva, Levin, Mikhalev & Pankratiev, 1999) (Hubert, 2003) (Sit, 2002) (Golubisky, 2004) (Ovchinnikov, 2004)**
  - difference polynomial systems **(Gao & Luo, 2004)**
  - polynomial systems **(Chen & M^3 , 2011)**
  - automatic theorem proving **(Wu, 1984), (Chou, 1988)**
  - geometric computation **(Chen & Wang, 2004)**
  - primary decomposition **(Shimoyama & Yokoyama, 1994)**
  - isolating real roots **(Rioboo, 1992), (Aubry, Rouillier & Safey El Din, 2001), (Boulier, Chen, Lemaire & M^3 , 2009)**
  - structured polynomial systems **(Boulier, Lemaire & M^3 , 2001), (Dahan, Jin, M^3 & Schost, 2006)**

- cryptology (**Schost & Gaudry, 2003**)
- algebraic geometry **(Alvandi, Chen, Marcus, $\mathbf{M}^3$, Schost & Vrbik, 2012-2014)**
- real algebraic geometry **(Chen, Davenport, $\mathbf{M}^3$, Xia & Xiao, 2010)**
- symbolic-numeric computations **($\mathbf{M}^3$, Reid, Scott & Wu, 2005)**
- theoretical physics **(Foursov & $\mathbf{M}^3$, 2001)**
- classification problems in geometry **(Kogan & $\mathbf{M}^3$, 2002)**.
- …

- Software, for instance:
  - *Diffalg* by Boulier and Hubert in MAPLE
  - *Dynamic Evaluation* by Duval and Gómez Díaz in AXIOM
  - *RealClosure* by Rioboo in AXIOM
  - *RAG'lib* by Safey El Din in MAPLE

- *Epsilon* by Wang in MAPLE
- *Discoverer* by Xia in MAPLE
- for primary decomposition in MAGMA and SINGULAR
- `RegularChains` by Alvandi, Chen, Lemaire, M^3 and Xie in MAPLE see also `www.regularchains.org`
- `RegularChains` in AXIOM and ALDOR by M^3
- *Elimino* parallel implementation by Wu, Liao, Lin, and Wang in C
- *Basic Polynomial Algebra Subprograms* by Chen, Covanov, M^3 Xie & Xie in `CilkPlus`.

• Related concepts

- resultants
- Gröbner bases
- geometric resolutions
- comprehensive Gröbner bases.

- …

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- Xavier Dahan (Kyushu Univ., Japan)
- James Davenport (Univ. of Bath, UK)
- Jürgen Gerhard and John May (Maplesoft)
- Wenyuan Wu (CIGIT, Chinese Academy of Science, China)
- Bican Xia (Peking Univ., China)

An overview of this tutorial

- **Main objective:** an introduction for non-experts.
- **Prerequisites:** some familiarity with Gröbner bases would be useful, but not necessary.
- **Outline:**

***Day 1*:** the case of polynomial systems with finitely many solutions

***Day 2*:** the general case: triangular sets, characteristic sets, Wu's method, regular chains, reduction to dimension zero

***Day 3*:** the `RegularChains` library in MAPLE and an overview of its solvers

***Day 4*:** Applications to real algebraic geometry

***Day 5*:** Applications to the study of dynamical systems

How triangular decompositions look like?

For the following input polynomial system:

$$F : \begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases}$$

One possible triangular decompositions of the solution set of F is:

$$\begin{cases} z = 0 \\ y = 1 \\ x = 0 \end{cases} \bigcup \begin{cases} z = 0 \\ y = 0 \\ x = 1 \end{cases} \bigcup \begin{cases} z = 1 \\ y = 0 \\ x = 0 \end{cases} \bigcup \begin{cases} z^2 + 2z - 1 = 0 \\ y = z \\ x = z \end{cases}$$

Another one is:

$$\begin{cases} z = 0 \\ y^2 - y = 0 \\ x + y = 1 \end{cases} \bigcup \begin{cases} z^3 + z^2 - 3z = -1 \\ 2y + z^2 = 1 \\ 2x + z^2 = 1 \end{cases}$$

An example in positive dimension

- Every prime ideal $\mathcal{P} = \langle F \rangle$ in a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ may be **represented** by a **triangular set** T encoding the **generic zeros** of $\mathcal{P}$.

$$F = \begin{cases} ax + by - c \\ dx + ey - f \\ gx + hy - i \end{cases} \simeq T = \begin{cases} gx + hy - i \\ (hd - eg)\, y - id + fg \\ (ie - fh)\, a + (ch - ib)\, d + (fb - ce)\, g \end{cases}$$

- **All the common zeros** of every polynomial system can be decomposed into **finitely many** triangular sets.

$$\begin{array}{rcl} \mathbf{V}(\mathcal{P}) & = & \mathbf{W}(T) \cup \mathbf{W} \begin{cases} dx + ey - f \\ hy - i \\ (ie - fh)\, a + (-ib + ch)\, d \\ g \end{cases} \cup \mathbf{W} \begin{cases} gx + hy - i \\ (ha - bg)\, y - ia + cg \\ hd - eg \\ ie - fh \end{cases} \\ & & \cup \mathbf{W} \begin{cases} x \\ (hd - eg)\, y - id + fg \\ fb - ce \\ ie - fh \end{cases} \cup \mathbf{W} \begin{cases} ax + by - c \\ hy - i \\ d \\ g \\ ie - fh \end{cases} \cup \cdots \end{array}$$

where $\mathbf{W}(T)$ denotes the generic zeros of T. We have : $\mathbf{W}(T) \subseteq \mathbf{V}(T)$.

Structured examples: implicitization, ranking conversions

- For $\mathcal{R} = x > y > z > s > t$ and $\overline{\mathcal{R}} = t > s > z > y > x$ we have:

$$\mathsf{convert}(\begin{cases} x - t^3 \\ y - s^2 - 1 \\ z - s\,t \end{cases}, \mathcal{R}, \overline{\mathcal{R}}) \;=\; \begin{cases} s\,t - z \\ (x\,y + x)s - z^3 \\ z^6 - x^2y^3 - 3x^2y^2 - 3x^2y - x^2 \end{cases}$$

- For $\mathcal{R} = \cdots > v_{xx} > v_{xy} > \cdots > u_{xy} > u_{yy} > v_x > v_y > u_x > u_y > v > u$ and $\overline{\mathcal{R}} = \cdots u_x > u_y > u > \cdots > v_{xx} > v_{xy} > v_{yy} > v_x > v_y > v$ we have:

$$\mathsf{convert}(\begin{cases} v_{xx} - u_x \\ 4\,u\,v_y - (u_x\,u_y + u_x\,u_y\,u) \\ u_x^2 - 4\,u \\ u_y^2 - 2\,u \end{cases} \mathcal{R}, \overline{\mathcal{R}}) \;=\; \begin{cases} u - v_{yy}^2 \\ v_{xx} - 2\,v_{yy} \\ v_y\,v_{xy} - v_{yy}^3 + v_{yy} \\ v_{yy}^4 - 2\,v_{yy}^2 - 2\,v_y^2 + 1 \end{cases}$$

How to compute triangular decompositions?

• Consider again solving the system F for $x > y > z$:

$$F : \begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{cases}$$

• Eliminating x leads to $\begin{cases} y^2 + (-1 + 2z^2)y - 2z^2 + z + z^4 = 0 \\ y^2 + z - y - z^2 = 0 \end{cases}$

• Eliminating y^2 and then y we can arrive to $r(z) = 0$ with $r(z) = z^8 - 4z^6 + 4z^5 - z^4$.

• Factorizing $r(z)$ leads to $z^4(z^2 + 2z - 1)(z - 1)^2 = 0$ and thus to $z = 0$, $z = 1$ or $z^2 + 2z = 1$. In each case, it is easy to conclude either by substitution, or by GCD computation in $(\mathbb{Q}[z]/\langle z^2 + 2z - 1\rangle)[y]$.

• Alternatively, one can directly perform GCD computation in $(\mathbb{Q}[z]/\langle r(z)\rangle)[y]$. But this is unusual since $\mathbb{Q}[z]/\langle r(z)\rangle$ is not a field! Let us see this now.

Computing a polynomial GCD over a ring with zero-divisors (I)

- Let us consider again the polynomials

$$\begin{cases} f_1 = y^2 + (2z^2 - 1)y - 2z^2 + z + z^4 \\ f_2 = y^2 + z - y - z^2 \end{cases}$$

- Let us compute their GCD in $\mathbb{L}[y]$ with $\mathbb{L} = \mathbb{Q}[z]/\langle s(z) \rangle$ where $s(z) = z(z^2 + 2z - 1)(z - 1)$ is the squarefree part of $r(z)$. (Replacing $r(z)$ with $s(z)$ makes the story simpler.)

- We proceed **as if $\mathbb{L}$ were a field** and run the **Euclidean Algorithm in $\mathbb{L}[y]$**. Of course, before dividing by an element of $\mathbb{L}$ we check whether it is a zero-divisor. We pretend we are not aware of the factorization of $s(z)$.

- Dividing f_1 by f_2 is no problem since f_2 is monic. We obtain: $\begin{array}{c|c} f_1 & f_2 \\ \cline{2-2} f_3 & 1 \end{array}$ with $f_3 = 2z^2y - z^2 + 2z^2 - z$.

Computing a polynomial GCD over a ring with zero-divisors (II)

• In order to divide f_2 by f_3, we need to check whether $2z^2$ divides zero in $\mathbb{L}$. This is done by computing $\gcd(s(z), 2z^2)$ in $\mathbb{Q}[z]$, which is z.

• Hence $s(z)$ writes $z(z^3 + z^2 - 3z + 1)$ and we split the computations into two cases: $z = 0$ and $z^3 + z^2 - 3z = 1$.

• $\boxed{\text{Case } z = 0.}$ Then $f_3 = 0$ and $f_2 = y^2 - y$ is the GCD.

• $\boxed{\text{Case } z^3 + z^2 - 3z = -1.}$ Since $S(z)$ is square-free, $2z^2$ has an inverse in this case, namely $i(z) = -(3/2)z^2 - 2z + 4$.

• Thus, the polynomial $\tilde{f_3} = i(z)f_3 = y + (1/2)z^2 - (1/2)$ is monic. So, we can compute

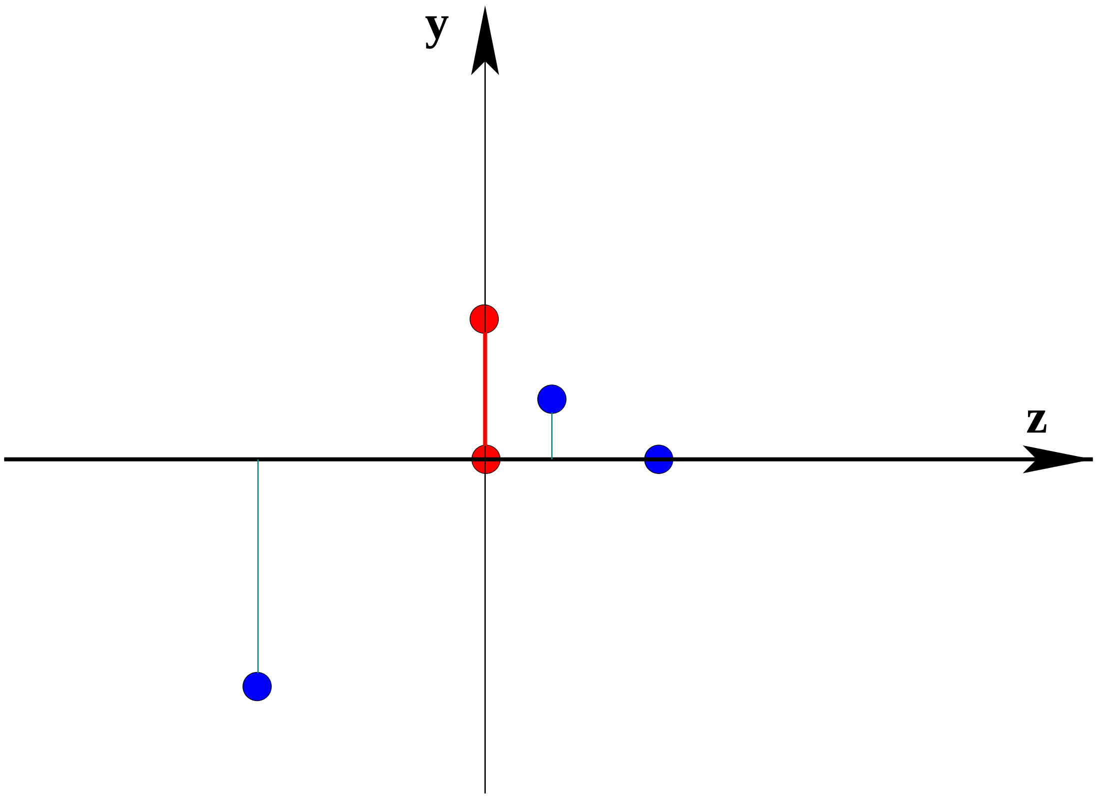
$$\begin{array}{c|c} f_2 & \tilde{f_3} \\ \hline 0 & y - (1/2)z^2 - (1/2) \end{array}.$$

• Finally $\gcd(f_1, f_2, \mathbb{L}[y]) = \begin{cases} y^2 - y & \text{if} \quad z = 0 \\ 2y + z^2 - 1 & \text{if} \quad z^3 + z^2 - 3z = -1 \end{cases}$

How those triangular sets look like? (I)

• Let us consider again the system $\left\{ \begin{array}{r} y^2 + (-1 + 2z^2)y - 2z^2 + z + z^4 = 0 \\ y^2 + z - y - z^2 = 0 \end{array} \right.$

• Let α_1 and α_2 be the roots of $z^2 + 2z - 1 = 0$. After dropping multiplicities, we obtain $(z, y) \in \{(0, 0), (0, 1), (\alpha_1, \alpha_1), (\alpha_2, \alpha_2), (1, 0)\}$.



How to pass from one triangular decomposition to another?

$$\left\{\begin{array}{r} z=0 \\ y=1 \\ x=0 \end{array}\right. \bigcup \left\{\begin{array}{r} z=0 \\ y=0 \\ x=1 \end{array}\right. \bigcup \left\{\begin{array}{r} z=1 \\ y=0 \\ x=0 \end{array}\right. \bigcup \left\{\begin{array}{r} z^2+2z-1=0 \\ y=z \\ x=z \end{array}\right.$$

$$\downarrow \ \mathrm{CRT} \ \downarrow$$

$$\left\{\begin{array}{r} z=0 \\ y^2-y=0 \\ x+y=1 \end{array}\right. \bigcup \left\{\begin{array}{r} z=1 \\ y=0 \\ x=0 \end{array}\right. \bigcup \left\{\begin{array}{r} z^2+2z-1=0 \\ y=z \\ x=z \end{array}\right.$$

$$\downarrow \ \mathrm{CRT} \ \downarrow$$

$$\left\{\begin{array}{r} z=0 \\ y^2-y=0 \\ x+y=1 \end{array}\right. \bigcup \left\{\begin{array}{r} z^3+z^2-3z=-1 \\ 2y+z^2=1 \\ 2x+z^2=1 \end{array}\right.$$

From a lexicographical Gröbner basis to a triangular decomposition (I)

• Let us consider again (last time) the polynomials

$$\begin{cases} f_1 = y^2 + (2z^2 - 1)y - 2z^2 + z + z^4 \\ f_2 = y^2 + z - y - z^2 \end{cases}$$

• It is natural to ask how we could obtain a triangular decomposition from the reduced lexicographical Gröbner basis of $\{f_1, f_2\}$ for $y > z$. This basis is:

$$\begin{cases} g_1 = z^6 - 4z^4 + 4z^3 - z^2 \\ g_2 = 2z^2 y + z^4 - z^2 \\ g_3 = y^2 - y - z^2 + z \end{cases}$$

• We initialize $T := \{g_1\}$. We would **add** g_2 into T provided that $\mathrm{lc}(g_2, y)$ is a **unit**.

From a lexicographical Gröbner basis to a triangular decomposition (II)

- So, we compute $\gcd(2z^2, g_1, \mathbb{Q}[z]) = z^2$. This shows $g_1 = z^2(z^4 - 4z^2 + 4z - 1)$ and splits the computations into two cases.

- $\boxed{\text{Case } z^2 = 0.}$ In this case g_2 **vanishes** and $g_3 = y^2 - y + z$, leading to $T^1 := \{z^2, y^2 - y + z\}$

- $\boxed{\text{Case } z^4 - 4z^2 + 4z - 1.}$ In this case $\mathrm{lc}(g_2, y)$ has $2z^3 + (1/2)z^2 - 8z + 6$ for **inverse**. Multiplying g_2 by this inverse leads to $\tilde{g}_2 = y + (1/2)z^2 - (1/2)$. Then, we observe that $\begin{array}{c|c} g_3 & \tilde{g}_2 \\ \hline 0 & y - (1/2)z^2 - (1/2) \end{array}$ leading to a second component $T^2 := \{z^4 - 4z^2 + 4z - 1, 2y + 1z^2 - 1\}$.

- For more details: **(Gianni, 1987), (Kalkbrener, 1987), (Lazard, 1992)**.

Some notations before we start the theory (I)

NOTATION. Throughout the talk, we consider a field $\mathbb{K}$ and an ordered set $X = x_1 < \cdots < x_n$ of n variables. Typically $\mathbb{K}$ will be

- a **finite field**, such as $\mathbb{Z}/p\mathbb{Z}$ for a prime p, or
- the field $\mathbb{Q}$ of **rational numbers**, or
- a field of **rational functions** over $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$.

We will denote by $\overline{\mathbb{K}}$ the **algebraic closure** of $\mathbb{K}$.

NOTATION. We denote by $\mathbb{K}[x_1, \ldots, x_n]$ the ring of the polynomials with coefficients in $\mathbb{K}$ and variables in X. For $F \subset \mathbb{K}[x_1, \ldots, x_n]$, we write $\langle F \rangle$ and $\sqrt{\langle F \rangle}$ for the ideal generated by F in $\mathbb{K}[x_1, \ldots, x_n]$ and its radical, respectively.

NOTATION. For $F \subset \mathbb{K}[x_1, \ldots, x_n]$, we are interested in

$$V(F) = \{\zeta \in \overline{\mathbb{K}}^n \mid (\forall f \in F)\ f(\zeta) = 0\},$$

the **zero-set** of F or **algebraic variety** of F in $\overline{\mathbb{K}}^n$.

REMARK. In some circumstances $\overline{\mathbb{K}}^n$ will be denoted $A^n(\overline{\mathbb{K}})$, especially when we consider several n at the same time.

Some notations before we start the theory (II)

NOTATION. Let i and j be integers such that $1 \leq i \leq j \leq n$ and let $V \subseteq A^n(\overline{\mathbb{K}})$ be a variety over $\mathbb{K}$. We denote by π_i^j the natural projection map from $A^j(\overline{\mathbb{K}})$ to $A^i(\overline{\mathbb{K}})$, which sends $(x_1, \ldots, x_j)$ to $(x_1, \ldots, x_i)$. Moreover, we define $V_i = \pi_i^n(V)$. Often, we will restrict π_i^j from V_i to V_j.

NOTATION. The algebraic varieties in $\overline{\mathbb{K}}^n$ defined by polynomial sets of $\mathbb{K}[x_1, \ldots, x_n]$ form the **closed sets** of a topology, called **Zariski Topology**. For a subset $W \subset \overline{\mathbb{K}}^n$, we denote by $\overline{W}$ the closure of W for this topology, that is, the intersection of the $V(F)$ containing W, for all $F \subset \mathbb{K}[x_1, \ldots, x_n]$.

NOTATION. For $W \subset \overline{\mathbb{K}}^n$, we denote by $I(W)$ the ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials vanishing at every point of W.

REMARK. When $\mathbb{K} = \overline{\mathbb{K}}$ and $W = V(F)$, for some $F \subset \mathbb{K}[x_1, \ldots, x_n]$, recall the Hilbert Theorem of Zeros:

$$\sqrt{\langle F \rangle} = I(V(F)).$$

Lazard triangular sets

DEFINITION. **(Lazard, 1992)** A subset

$$T = \{T_1, \ldots T_n\} \subset \mathbb{K}[x_1 < \cdots < x_n]$$

is a Lazard triangular set if for $i = 1 \cdots n$

$$T_i = 1\,\mathbf{x_i^{d_i}} + a_{d_i-1}\,\mathbf{x_i^{d_i-1}} + \cdots + a_1\,\mathbf{x_i} + a_0$$

with

$$a_{d_i-1}, \ldots, a_1, a_0 \in \mathbf{k}[x_1, \ldots, x_{i-1}].$$

reduced w.r.t $\langle T_1, \ldots, T_{i-1}\rangle$ in the sense of Gröbner bases.

THEOREM. A family T of n polynomials in $\mathbb{K}[x_1 < \cdots < x_n]$ is a **Lazard triangular set** if and only it is the **reduced lexicographical Gröbner basis** of a **zero-dimensional** ideal.

How those triangular sets look like? (II)

NOTATION. Let $T = \{T_1, \ldots T_n\} \subset \mathbb{K}[x_1, \ldots, x_n]$ be a Lazard triangular set. Let V be its variety in $A^n(\overline{\mathbb{K}})$. Let $d_1 = \deg(T_1, x_1), \ldots, d_n = \deg(T_n, x_n)$.

NOTATION. For $1 \leq i < j \leq n$, recall that

$$\pi_i^j : \begin{array}{ccc} V_j & \longmapsto & V_i \\ (x_1, \ldots, x_j) & \to & (x_1, \ldots, x_i) \end{array}$$

where $V_i = \pi_i^n(V)$ and $V_j = \pi_j^n(V)$.

PROPOSITION. For a point $M \in V_i$ the *fiber* (i.e. the pre-image) $(\pi_i^j)^{-1}(M)$ has cardinality $d_{i+1} \cdots d_j$, that is

$$|(\pi_i^j)^{-1}(M)| = d_{i+1} \cdots d_j.$$

Equiprojectable varieties

DEFINITION. Let i and j be integers such that $1 \leq i < j \leq n$ and let $V \subseteq A^j(\overline{\mathbb{K}})$ be a variety over $\mathbb{K}$. The set V is said

(1) **equiprojectable on** V_i, its projection on $A^i(\overline{\mathbb{K}})$, if there exists an integer c such that for every $M \in V_i$ the cardinality of $(\pi_i^j)^{-1}(V_i)$ is c.

(2) **equiprojectable** if V is equiprojectable on $V_1, \ldots, V_{j-1}$.

THEOREM. **(Aubry & Valibouze, 2000)** Assume $\mathbb{K}$ is **perfect** and let $V \subset A^n(\overline{\mathbb{K}})$ be finite. Assume that there exists $F \subset \mathbb{K}[x_1, \ldots, x_n]$ such that $V = V(F)$. Then, the following conditions are equivalent:

(1) V is equiprojectable,

(2) There exists a Lazard Triangular set $T \subset \mathbb{K}[x_1, \ldots, x_n\}$ whose zero-set in $A^n(\overline{\mathbb{K}})$ is exactly V.

PROOF $\triangleright$ For proving $(1) \Rightarrow (2)$ one can use the **interpolation formulas** of **(Dahan & Schost, 2004)** to construct a Lazard triangular set in $\overline{\mathbb{K}}[x_1, \ldots, x_n]$. To conclude, one uses the hypothesis $\mathbb{K}$ perfect, $V = V(F)$ together with the Hilbert Theorem of Zeros. $\triangleleft$

The interpolation formulas: sketch (I)

- Let $V \subset A^n(\overline{\mathbb{K}})$ be (finite and) equiprojectable. Let $\mathbf{K}$ be a field, with $\mathbb{K} \subseteq \mathbf{K} \subseteq \overline{\mathbb{K}}$ such that every point of V has its coordinates in $\mathbf{K}$.

- We have $T_1 = \prod_{\alpha \in V_1}(x_1 - \alpha)$. Let $1 \leq \ell < n$. We give interpolation formulas for $T_{\ell+1}$ from the coordinates (in $\mathbf{K}$) of the points of $V_{\ell+1}$, for $1 \leq \ell < n$.

- Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in V_\ell$. We define the varieties

$$\begin{array}{lcccc}
V_\alpha^1 & = \{ & \beta = (\beta_1, \ldots, \beta_\ell, \beta_{\ell+1}) \in V_{\ell+1} & \mid & \beta_1 \neq \alpha_1 \} \\
V_\alpha^2 & = \{ & \beta = (\alpha_1, \beta_2, \ldots, \beta_\ell, \beta_{\ell+1}) \in V_{\ell+1} & \mid & \beta_2 \neq \alpha_2 \} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
V_\alpha^\ell & = \{ & \beta = (\alpha_1, \ldots, \alpha_{\ell-1}, \beta_\ell, \beta_{\ell+1}) \in V_{\ell+1} & \mid & \beta_\ell \neq \alpha_\ell \} \\
V_\alpha^{\ell+1} & = \{ & \beta = (\alpha_1, \ldots, \alpha_\ell, \beta_{\ell+1}) \in V_{\ell+1} & & \}
\end{array}$$

The sets $V_\alpha^1, V_\alpha^2, V_\alpha^3, \ldots, V_\alpha^\ell, V_\alpha^{\ell+1}$ form a partition of $V_{\ell+1}$.

- The intermediate goal is to build $T_{\alpha,\ell+1} = T_i(\alpha_1, \ldots, \alpha_\ell, x_{\ell+1}) \in \mathbf{K}[x_{\ell+1}]$.

The interpolation formulas: sketch (II)

- We consider also the projections

$$
\begin{array}{ccccccc}
v_\alpha^1 & = & \pi_1^{\ell+1}(V_\alpha^1) & = & \{(\beta_1) \in V_1 & \mid & \beta_1 \neq \alpha_1\} \\
v_\alpha^2 & = & \pi_2^{\ell+1}(V_\alpha^2) & = & \{(\alpha_1, \beta_2) \in V_2 & \mid & \beta_2 \neq \alpha_2\} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
v_\alpha^\ell & = & \pi_\ell^{\ell+1}(V_\alpha^\ell) & = & \{(\alpha_1, \ldots, \alpha_{\ell-1}, \beta_\ell) \in V_\ell & \mid & \beta_\ell \neq \alpha_\ell\}
\end{array}
$$

- For $1 \leq i \leq \ell$, define $e_{\alpha,i} := \prod_{\beta \in v_\alpha^i} (x_i - \beta_i) \ \in \ \mathbf{K}[x_i]$ and

$\boxed{E_\alpha := \prod_{1 \leq i \leq \ell} e_{\alpha,i} \ \in \ \mathbf{K}[x_1, \ldots, x_\ell].}$

- Then, we have:

$$
\begin{array}{rcl}
T_{\alpha,\ell+1} & = & \prod_{\beta \in V_\alpha^{\ell+1}} (x_{\ell+1} - \beta_{\ell+1}) \\
T_{\ell+1} & = & \Sigma_{\alpha \in V_\ell} \frac{E_\alpha T_{\alpha,\ell+1}}{E_\alpha(\alpha)}
\end{array}
$$

- Related work: **(Abbot, Bigatti, Kreuzer & Robbiano, 1999)**, …

Direct product of fields, the D5 Principle (I)

PROPOSITION. Let $f \in \mathbb{K}[x]$ be a non-constant and **square-free** univariate polynomial. Then $\mathbb{L} = \mathbb{K}[x]/\langle f \rangle$ is a direct product of fields (DPF).

PROOF $\triangleright$ The factors of f are **pairwise coprime**. Then, apply the **Chinese Remaindering Theorem**. (If $f = f_1 f_2$ then $\mathbb{L} \simeq \mathbb{K}[x]/\langle f_1 \rangle \times \mathbb{K}[x]/\langle f_2 \rangle$. $\triangleleft$

PRINCIPLE. **(Della Dora, Dicrescenzo & Duval, 1985)** If $\mathbb{L}$ is a DPF, then one can compute with $\mathbb{L}$ as **if it were a field**: it suffices to **split** the computations into cases whenever a **zero-divisor** is met.

PROPOSITION. Let $\mathbb{L}$ be a DPF and $f \in \mathbb{L}[x]$ be a non-constant monic polynomial such that f and its derivative generate $\mathbb{L}[x]$, that is, $\langle f, f' \rangle = \mathbb{L}[x]$. Then $\mathbb{L}[x]/\langle f \rangle$ is another DPF.

PROOF $\triangleright$ It is convenient to establish the following more general theorem: *A Noetherian ring is isomorphic with a direct product of fields if and only if every non-zero element is either a unit or a non-nilpotent zero-divisor.* $\triangleleft$

Direct product of fields, the D5 Principle (II)

PROPOSITION. Let $T \subset \mathbb{K}[x_1, \ldots, x_n]$ be a Lazard triangular set such that $\langle T \rangle$ is **radical**. Then, we have

- $\mathbb{K}[x_1, \ldots, x_n]/\langle T \rangle$ is a DPF,
- if $\mathbb{K}$ is **perfect** then $\overline{\mathbb{K}}[x_1, \ldots, x_n]/\langle T \rangle$ is a DPF.

REMARK. **Recall the trap!** Consider $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}(t)$, for a prime p. Consider the polynomial $f = x^p - t \in \mathbb{F}[x]$ and $\overline{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$.

Since f is not constant, it has a root $\alpha \in \overline{\mathbb{F}}$ and we have

$$f = x^p - t = x^p - \alpha^p = (x - \alpha)^p \qquad (1)$$

in $\overline{\mathbb{F}}[x]$, which is clearly not square-free. However f is irreducible, and thus squarefree, in $\mathbb{F}[x]$.

Polynomial GCDs over DPF, quasi-inverses (I)

DEFINITION. **(M^3 & Rioboo, 1995)** Let $\mathbb{L}$ be a DPF. The polynomial $h \in \mathbb{L}[y]$ is a **GCD** of the polynomials $f, g \in \mathbb{L}[y]$ if the ideals $\langle f, g \rangle$ and $\langle h \rangle$ are equal.

REMARK. **Another trap!** Even if f, g are both **monic**, there **may not exist a monic** polynomial h in $\mathbb{L}[y]$ such that $\langle f, g \rangle = \langle h \rangle$ holds. Consider for instance $f = y + \frac{a+1}{2}$ (assuming that 2 is invertible in $\mathbb{L}$) and $g = y + 1$ where $a \in \mathbb{L}$ satisfies $a^2 = a$, $a \neq 0$ and $a \neq 1$.

REMARK. In practice, polynomial GCDs over DPF are computed via the D5 Principle. Moreover, only monic GCDs are useful. So, we generalize:

DEFINITION. Let $\mathbb{L}$ be a DPF and $f, g \in \mathbb{L}[y]$. A **GCD** of f, g in $\mathbb{L}[y]$ is a sequence of pairs $((h_i, \mathbb{L}_i), 1 \leq i \leq s)$ such that

- $\mathbb{L}_i$ is a DPF, for all $1 \leq i \leq s$ and the direct product of $\mathbb{L}_1, \ldots, \mathbb{L}_s$ is isomorphic to $\mathbb{L}$,
- h_i is a null or monic polynomial in $\mathbb{L}_i[y]$, for all $1 \leq i \leq s$,
- h_i is a GCD (in the above sense) of the projections of f, g to $\mathbb{L}_i[y]$, for all $1 \leq i \leq s$.

Polynomial GCDs over DPF, quasi-inverses (II)

DEFINITION. Let $\mathbb{L}$ be a DPF and let $f \in \mathbb{L}$. A **quasi-inverse** of f is a sequence of pairs $((g_i, \mathbb{L}_i), 1 \leq i \leq s)$ such that

- $\mathbb{L}_i$ is a DPF, for all $1 \leq i \leq s$ and the direct product of $\mathbb{L}_1, \ldots, \mathbb{L}_s$ is isomorphic to $\mathbb{L}$
- $g_i \in \mathbb{L}_i$, for all $1 \leq i \leq s$,
- let f_i be the projection of f to $\mathbb{L}_i$; either $f_i = g_i = 0$ or $f_i g_i = 1$ hold, for all $1 \leq i \leq s$.

PROPOSITION. Let $T \subset \mathbb{K}[x_1, \ldots, x_n]$ be a Lazard triangular set such that $\langle T \rangle$ is **radical**. We define $\mathbb{L} = \mathbb{K}[x_1, \ldots, x_n]/\langle T \rangle$.

(1) For all $f \in \mathbb{K}[x_1, \ldots, x_n]$ (reduced w.r.t. T) one can compute a **quasi-inverse** in $\mathbb{L}$ of f (regarded as an element of $\mathbb{L}$).

(1) For all $f, g \in \mathbb{L}[y]$ one can compute a **GCD** of f and g in $\mathbb{L}[y]$.

Equiprojectable decomposition

REMARK. Not every variety is equiprojectable, for instance $V = \{(0,1),(0,0),(1,0)\}$.

DEFINITION. Let $V \subset A^n(\overline{\mathbb{K}})$ be finite. Consider the projection $\pi : V \longmapsto \overline{\mathbb{K}}^{n-1}$ which forgets x_n. To every $x \in V$ we associate

$$N(x) = \#\pi^{-1}(\pi(x)).$$

We write $V = C_1 \cup \cdots \cup C_d$ where $C_i = \{x \in V \mid N(x) = i\}$. This splitting process is applied recursively to all varieties $C_1, \ldots, C_d$.

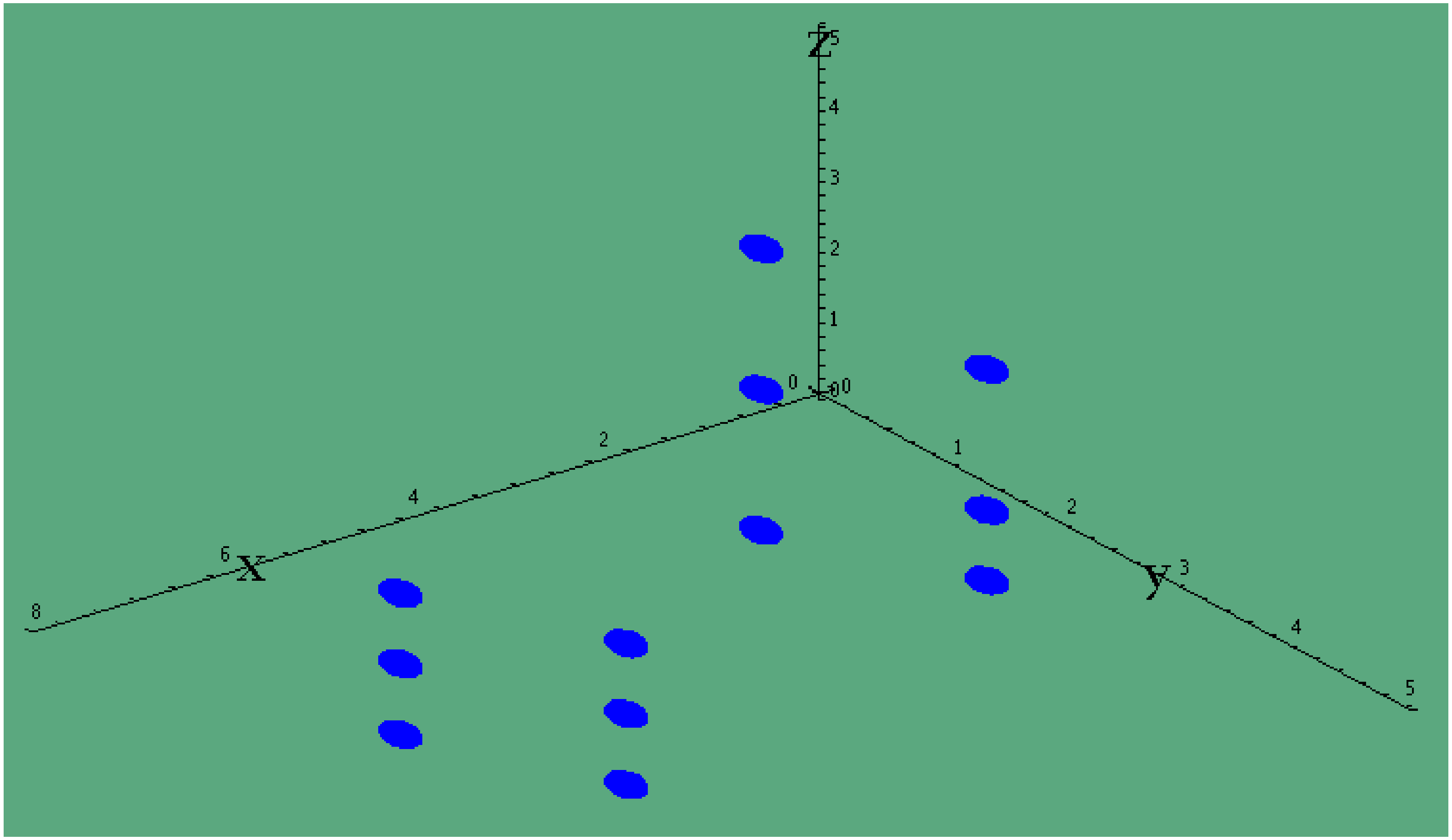
In the end, we obtain a family of pairwise disjoint, equiprojectable varieties, whose reunion equals V. This is the **equiprojectable decomposition** of V.

PROPOSITION. Let $V(F) \subset A^n(\overline{\mathbb{K}})$ be finite with $F \subset \mathbb{K}[x_1, \ldots, x_n]$. There exist Lazard triangular sets $T^1, \ldots, T^s \subset \mathbb{K}[x_1, \ldots, x_n]$ such that

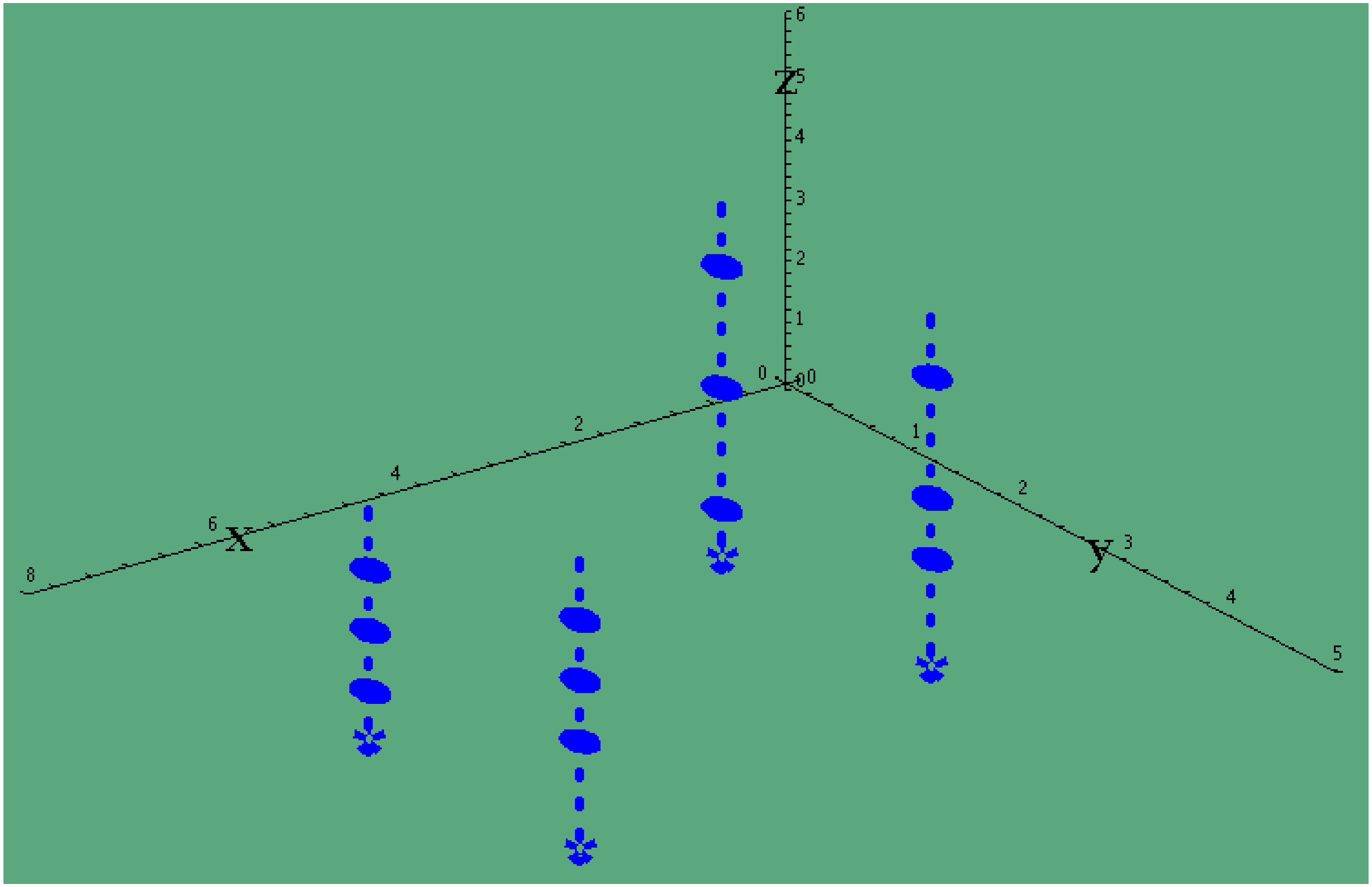
$$V(F) = V(T^1) \cup \cdots \cup V(T^s) \text{ and } i \neq j \Rightarrow V(T^i) \cap V(T^j) = \varnothing.$$

They form a **triangular decomposition** of $V(F)$.

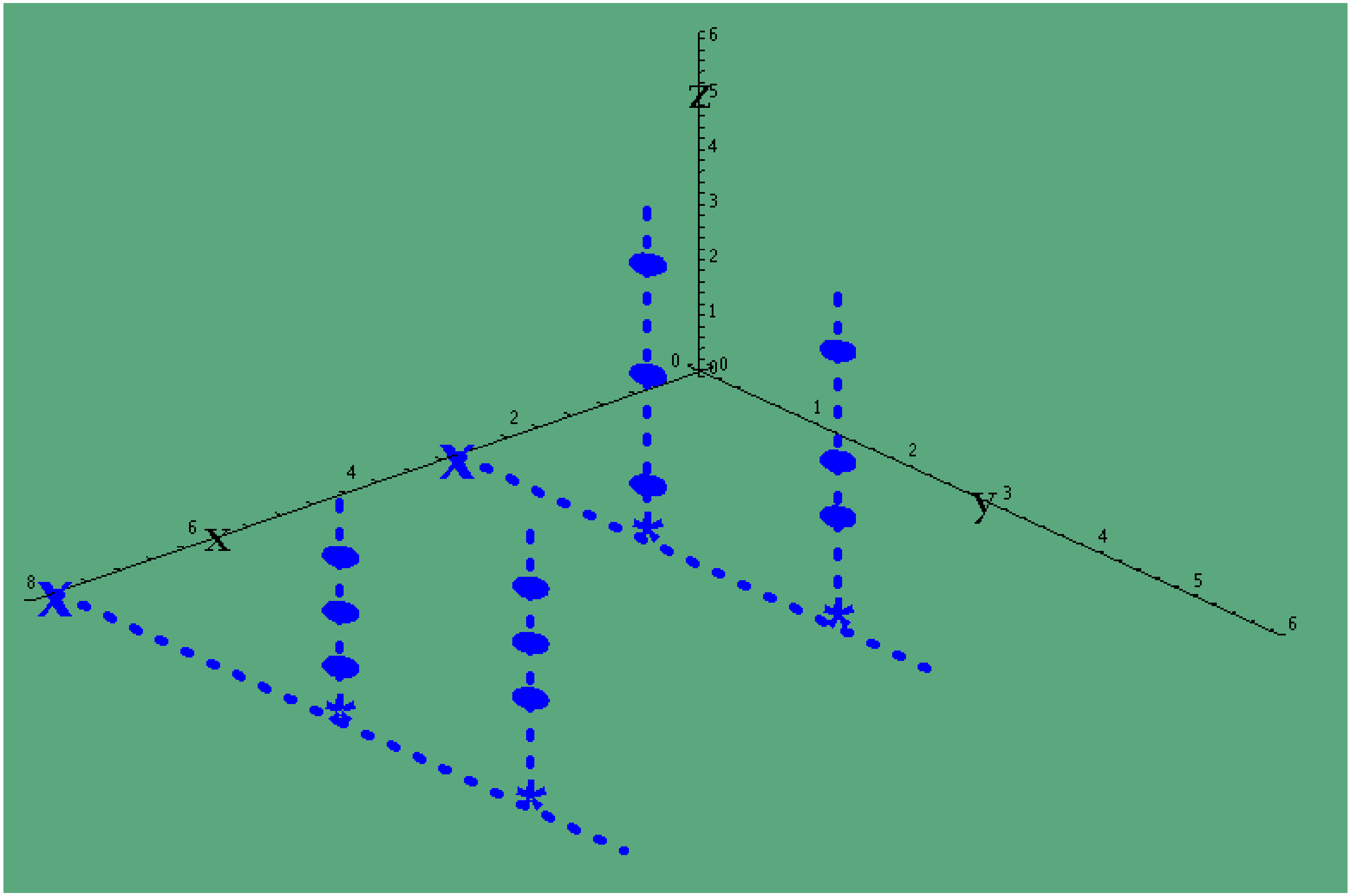
Equiprojectable variety definition (1/3)



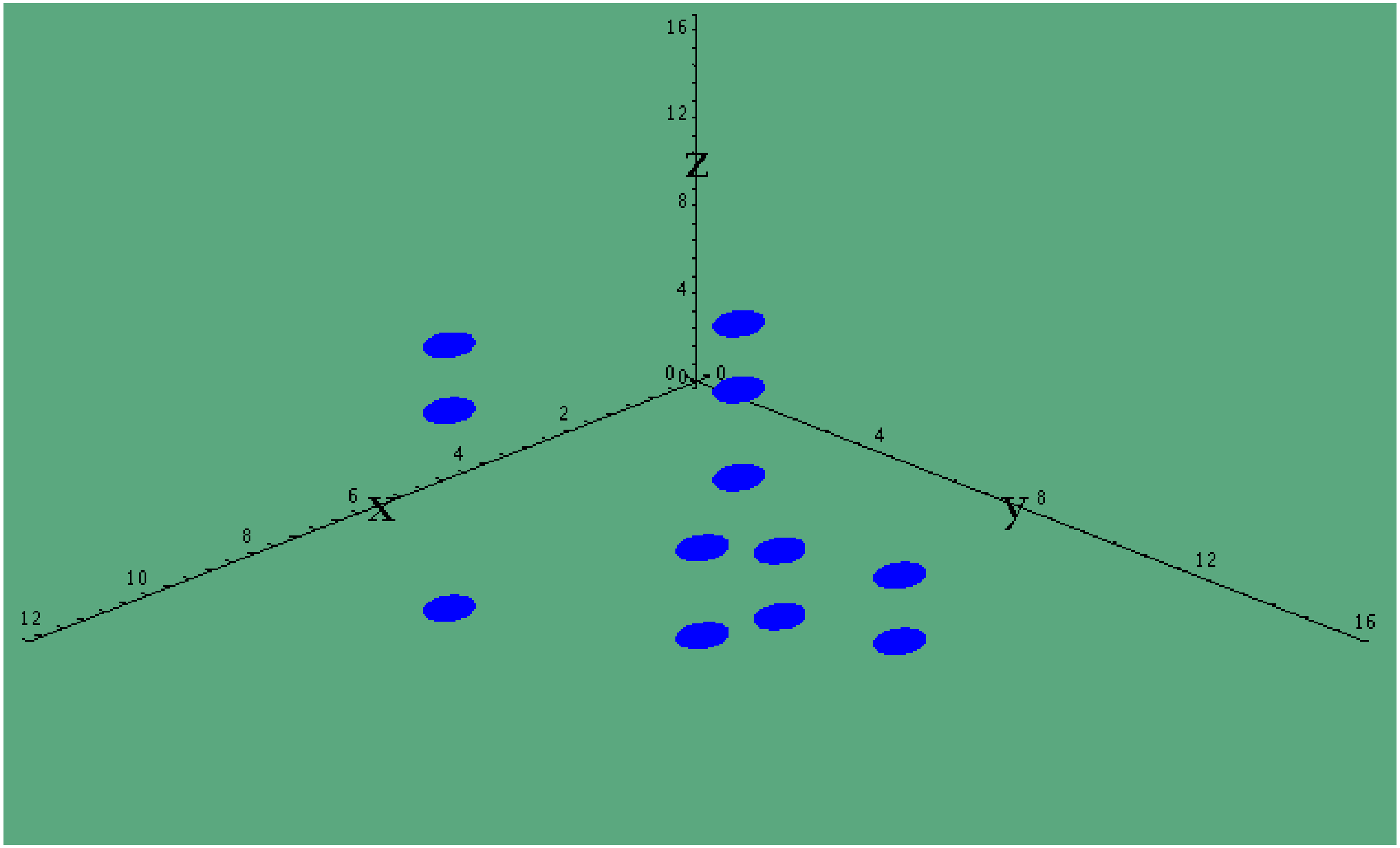
Equiprojectable variety definition (2/3)



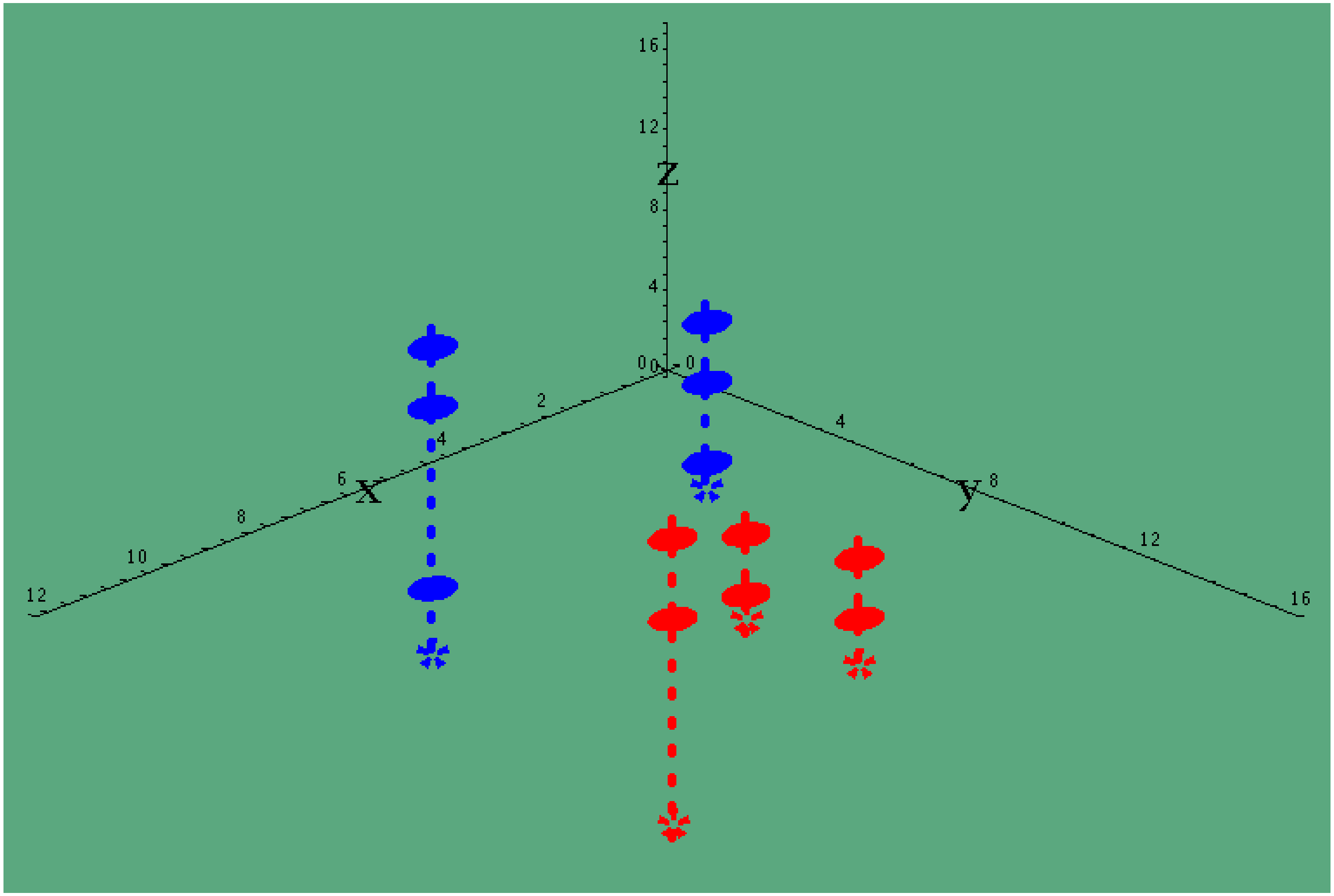
Equiprojectable variety definition (3/3)



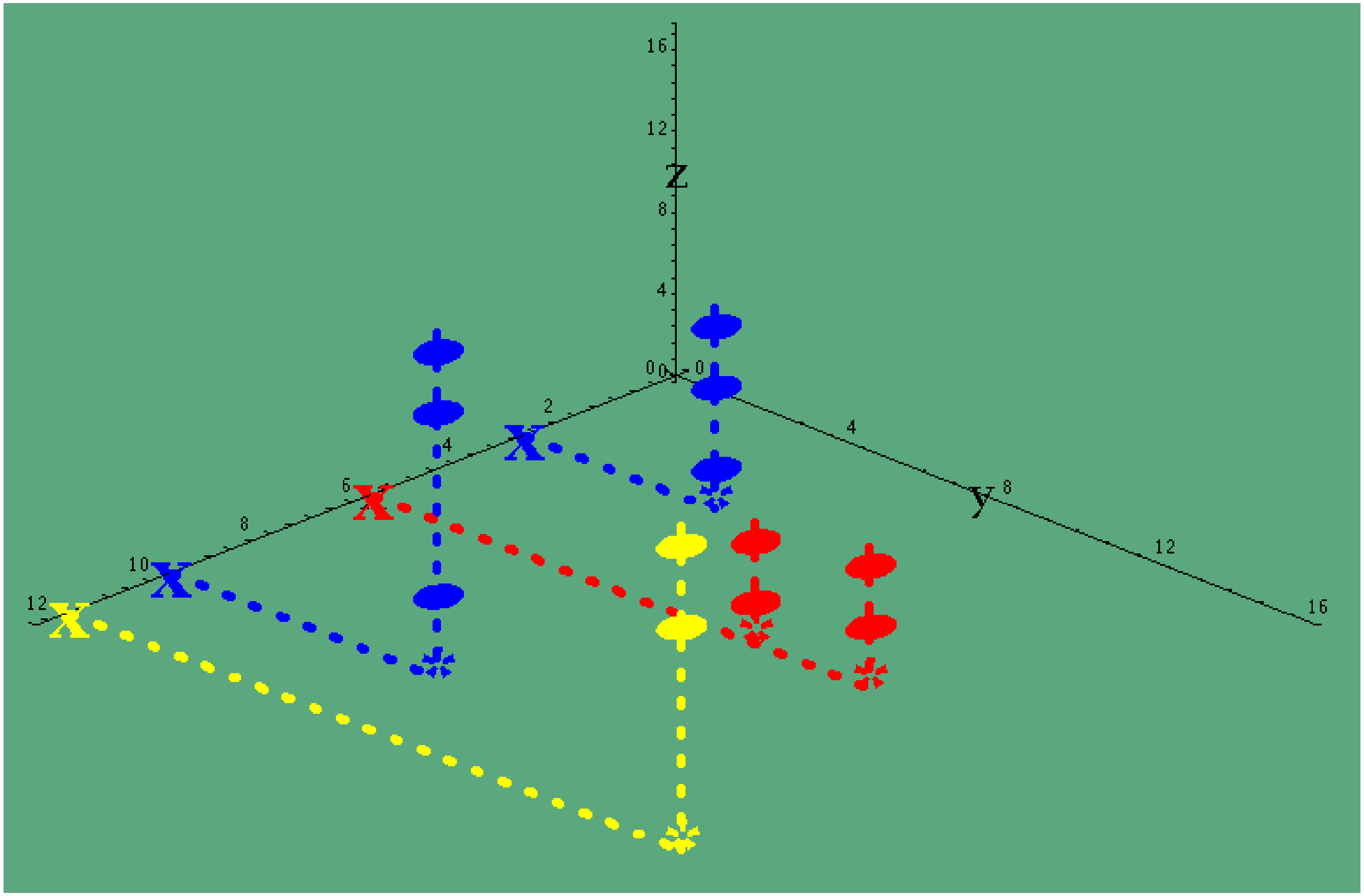
Equiprojectable decomposition definition (1/3)



Equiprojectable decomposition definition (2/3)



Equiprojectable decomposition definition (3/3)



From triangular to equiprojectable decomposition

NOTATION. Let $V(F) \subset A^n(\overline{\mathbb{K}})$ be finite with $F \subset \mathbb{K}[x_1, \ldots, x_n]$. Let Δ be a triangular decomposition of $V(F)$.

PROPOSITION. We compute from Δ another triangular decomposition $\{T^1, \ldots, T^d\}$ of V such that $V(T^1), \ldots, V(T^d)$ is the **equiprojectable decomposition** of V.

PROOF $\triangleright$ We proceed into two steps:

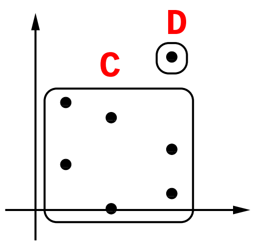
- split: reducing what we call **critical pairs** by means of **GCD** computations modulo Lazard triangular sets,
- merge: reducing what we call **solvable pairs** by means of **CRT** computations modulo Lazard triangular sets.

$\triangleleft$

REMARK. Among all possible triangular decompositions of $V(F)$, the equiprojectable decomposition is a **canonical choice**: it depends only on the variable order and $V(F)$.

Example: *split* + *merge* modulo 7

$$C \left| \begin{array}{l} C_2 = y^2 + 6yx^2 + 2y + x \\ C_1 = x^3 + 6x^2 + 5x + 2 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

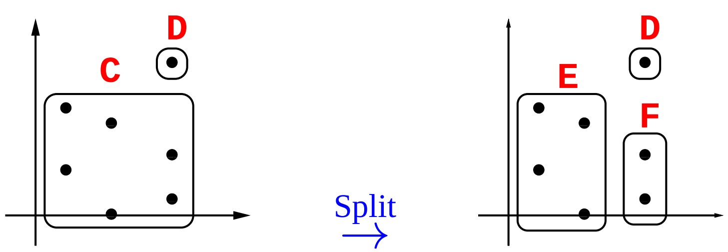


Example: *split+merge* modulo 7

$$C \left| \begin{array}{l} C_2 = y^2 + 6yx^2 + 2y + x \\ C_1 = x^3 + 6x^2 + 5x + 2 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$$\downarrow \ \text{Split C : GCD} \ \downarrow$$

$$E \left| \begin{array}{l} {C_2}' = y^2 + x \\ {C_1}' = x^2 + 5 \end{array} \right. , \quad F \left| \begin{array}{l} C_2'' = y^2 + y + 1 \\ C_1'' = x + 6 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$



Example: *split+merge* modulo 7

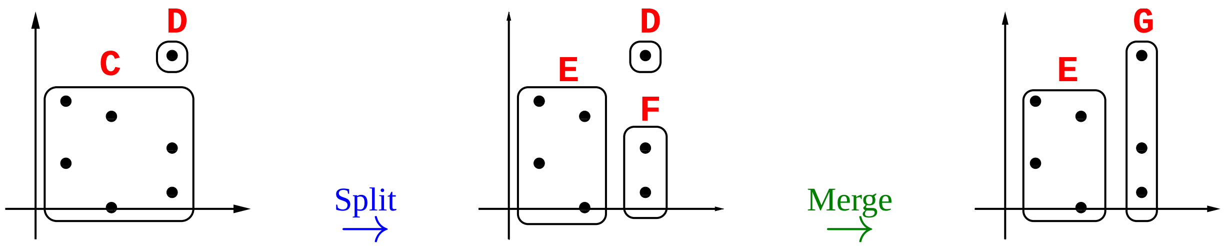
$$C \left| \begin{array}{l} C_2 = y^2 + 6yx^2 + 2y + x \\ C_1 = x^3 + 6x^2 + 5x + 2 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$$\downarrow \ \text{Split C : GCD} \ \downarrow$$

$$E \left| \begin{array}{l} {C_2}' = y^2 + x \\ {C_1}' = x^2 + 5 \end{array} \right. , \quad F \left| \begin{array}{l} C_2'' = y^2 + y + 1 \\ C_1'' = x + 6 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$$\downarrow \ \text{Merge F and D : CRT} \ \downarrow$$

$$E \left| \begin{array}{l} C_2' = y^2 + x \\ C_1' = x^2 + 5 \end{array} \right. , \quad G \left| \begin{array}{l} G_2 = y^3 + 6 \\ G_1 = x + 6 \end{array} \right.$$

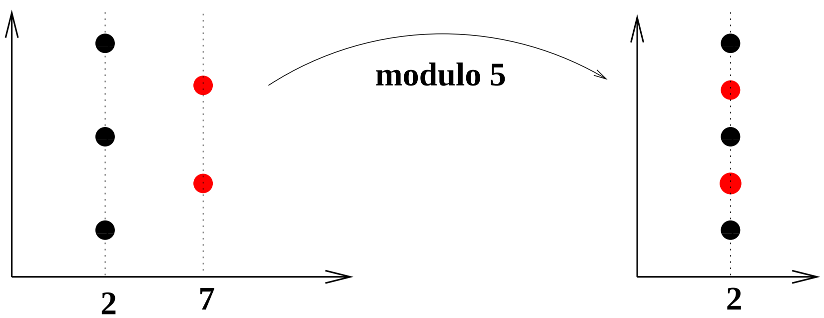


Specialization properties: sketch

Oversimplified case: Assume all points $V(F)$ are in $\mathbb{Q}^n$. Let $p \in \mathbb{Z}$ prime. if

1. p divides no denominator of the coordinates; $\boxed{(V \mod p \text{ is well defined})}$
2. the cardinality of none of the projections of V decreases mod p;

then the equiprojectable decomposition specializes mod p. Below, is a bad case.



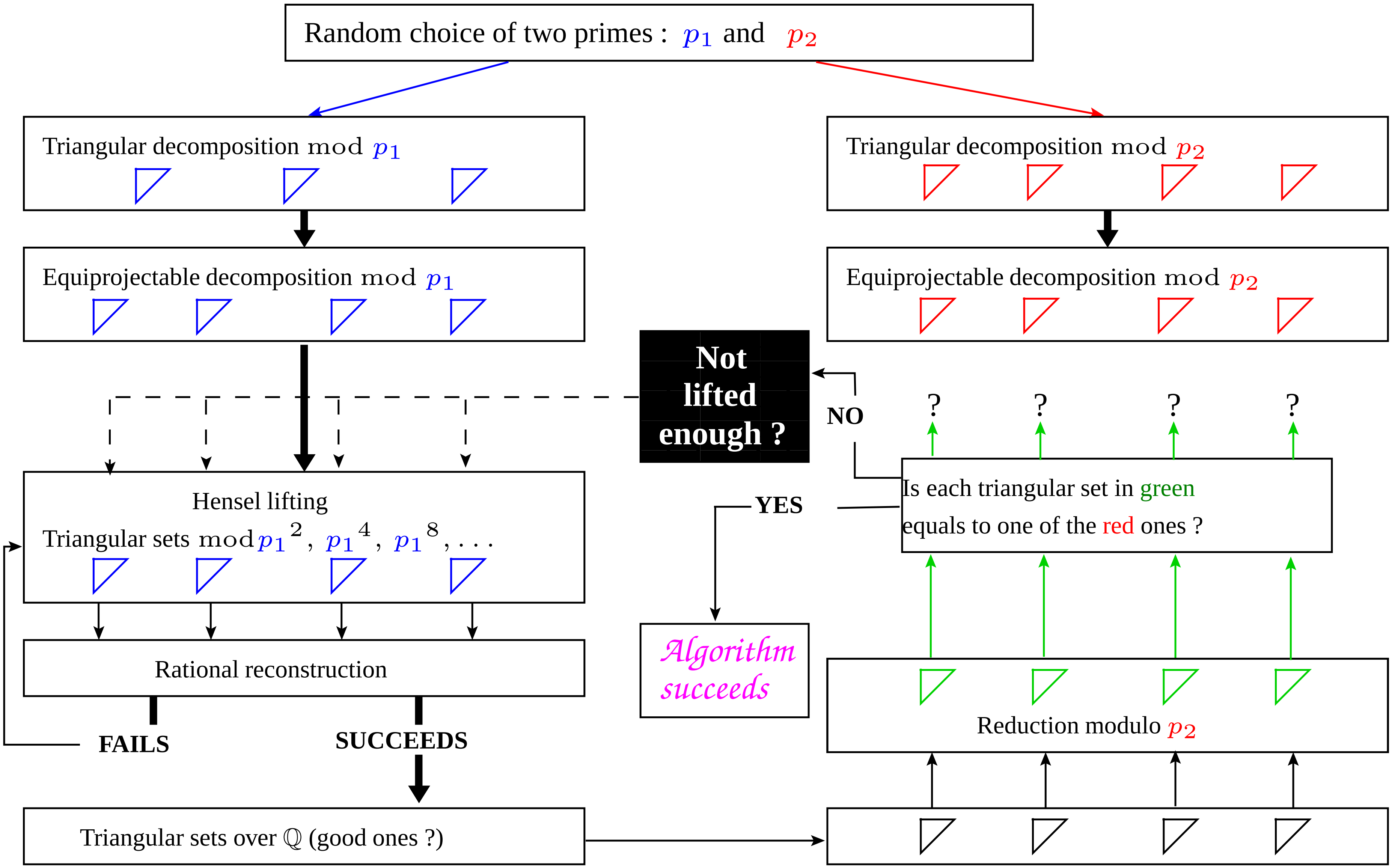
General case: Under *similar* assumptions, every coordinate of every point of V lies in a direct sum $\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$ where $\mathbb{Z}_p$ is the ring of p-adic integers.

THEOREM. **(Dahan, M[3] , Schost, Wu & Xie, 2005)** Let h the maximum length of a coefficient in F, and d the maximum degree in F. There exists $A \in \mathbb{N}$ s. t.:

(1) $h(A) \leq 2n^2 d^{2n+1}(3h + 7\log(n+1) + 5n\log d + 10)$.

(1) If $p \not| A$, then the equiprojectable decomposition specializes well mod p.

A probabilistic algorithm



Generalizing Lazard triangular sets

REMARK. Let $T = \{T_1, \ldots, T_n\} \subset \mathbb{K}[x_1, \ldots, x_n]$ be a Lazard triangular set. Let $\mathcal{I} := \langle T \rangle$. We have shown that given $p \in \mathbb{K}[x_1, \ldots, x_n]$,

- one can decide whether $p \in \mathcal{I}$. Indeed T is a Gröbner basis of $\mathcal{I}$.
- assuming $\mathcal{I}$ radical, one can decide whether $p^{-1} \mod \mathcal{I}$ exists. Indeed $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$ is a DPF.

We aim at:

- first, relaxing the hypothesis $\mathrm{lc}(T_i, x_i) = 1$, for all $1 \leq i \leq n$,
- second, relaxing the **as many polynomials as variables** constraint.

while preserving a **triangular shape** together with the above **algorithmic properties**.

Zero-dimensional regular chains

DEFINITION. A subset $C = \{C_1, \ldots, C_n\} \subset \mathbb{K}[x_1 < \cdots < x_n]$ is a **zero-dimensional regular chain** if for all $i = 1 \cdots n$ we have

(1) $C_i \in \mathbb{K}[x_1, \ldots, x_i]$,

(2) $\deg(C_i, x_i) > 0$,

(3) $h_i := \mathrm{lc}(C_i, x_i)$ is **invertible** modulo the ideal $\langle C_1, \ldots, C_{i-1} \rangle$.

PROPOSITION. Let $C \subset \mathbb{K}[x_1, \ldots, x_i]$ be a **zero-dimensional regular chain**. There exists a Lazard triangular set $T \subset \mathbb{K}[x_1, \ldots, x_i]$ such that $\langle C \rangle = \langle T \rangle$.

PROOF $\triangleright$ By induction on n.

- For $n = 1$ we have $T_1 = \mathrm{lc}(C_1)^{-1} C_1$ and the claim follows clearly.
- For $n > 1$ we compute $\tilde{h}_n$ the inverse of h_n modulo $\langle T_1, \ldots, T_{n-1} \rangle$ and observe

$$\langle T_1, \ldots, T_{n-1}, \tilde{h}_n C_n \rangle = \langle T_1, \ldots, T_{n-1}, C_n \rangle.$$

$\triangleleft$

The Dahan-Schost Transform (I)

PROPOSITION. Consider $T = \{T_1, \ldots, T_n\}$ a Lazard triangular set. Assume T generates a radical ideal. Let $D_1 = 1$ and $N_1 = T_1$. For $2 \leq \ell \leq n$, define

$$\begin{array}{rcl} D_\ell & = & \prod_{1 \leq i \leq \ell - 1} \frac{\partial T_i}{\partial x_i} \mod \langle T_1, \ldots, T_{\ell-1} \rangle \\ N_\ell & = & D_\ell T_\ell \mod \langle T_1, \ldots, T_{\ell-1} \rangle \end{array}$$

Then $N = \{N_1, \ldots, N_n\}$ is a zero-dimensional regular chain with $\langle T \rangle = \langle N \rangle$.

REMARK. The results of **(Dahan & Schost, 2004)** "essentially" show that the height (or "size") of each coefficient in N is upper bounded by

- the height of $\mathbf{V}(T)$ if $\mathbb{K} = \mathbb{Q}$, that is the minimum size of a data set encoding $\mathbf{V}(T)$,
- the degree of $\mathbf{V}(T^\downarrow)$ if $\mathbb{K}$ is a field $k(t_1, \ldots, t_m)$ of rational functions and $T^\downarrow$ is T regarded in $k[t_1, \ldots, t_m, x_1, \ldots, x_n]$.

See the authors' article for precise statements.

The Dahan-Schost Transform (II)

- Consider the system F (Barry Trager).

$$-x^5 + y^5 - 3y - 1 = 5y^4 - 3 = -20x + y - z = 0$$

We solve it for $z < y < x$.

- $V(F)$ is equiprojectable and its Lazard triangular set is
  - 1147412794656925600746886196713882259945463225340477687005119947622261926900489014476185343948461771260505008202862102854051702189834144507041921400912212854357946960933195335641858396501896936993494167255643877060419555161219397297718310661681373013610473433161675295215097739765468198624698033057372004369628572309403845943516901456096080945793282669881686485390936578666175235967213624577949980872265230642371971182386814553874346853792171708143077531532237850295577589142064921825588409831441292570286016853843732976447711290921201282663597873225040956392206905741146687701513841784606672511835822265889987889624672252665122778133883969304602062740935497619894651442744439433587390347755862238203761990339960554351301919398485081103440153976743524458297586182708752398894638319738859704396544591592407731579470289955844307815442694326841805687077917675761917872738339662798997128827712967353520807578712156161195412624338459316853569080754130154719452119621523713394865899777869339534459634212652323168810285894102829514014960747795605184806645733349724856391347410632777061560951110896275634940887029344611985724298328089928128704127659741470395311827709014752692114620308283759341810040325817543392095814567632394138225663551675690804005364383091912961309507299736685953680211256352496932486587513812792390171704032245316310904516304034566838688396641645490945090868618366582490420637673970853279869471018348887091817749546675847593377481568238007075259306520563109135581811542014656070637988617107330377650533573060376552912562641546080455275692923387543379737978438247137018552307587682361742927801505920906300566302345120641246953858195786422852752879754020156689945022004770650946405155986011151301751670637053436652396615265985718824532042488802422296773818429373789169917697659429318767468848486488142387103357675735987149201249564746107188031507033768129784171791787755761173195000000778571292329588891041932397871086492879872864247556074824548646907868278411846969762861333860575738177220989978593224804

- 5737063973284628003734430983569411299727316126702388435025599738111309634502445072380926719742333
  1771260505008202862102854051702189834144507041921400912212854357946960933195335641858396501896933
  6993494167255643877060419555161219397297718310661681373013610473433161675295215097739765468198622
  4698033057372004369628572309403845943516901456096080945793282669881686485390936578666175235967211
  3624577949980872265230642371971182386814553874346853792171708143077531532237850295577589142064922
  1825588409831441292570286016853843732976447711290921201282663597873225040956392206905741146687700
  1513841784606672511835822265889987889624672252665122778133883969304602062740935497619894651442744
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  2397871086492879872864247556074824548646907868278411846969762861333860575738177220989978593224800
- $3125z^{20} - 9375z^{16} - 40000000000z^{15} - 2015999988750z^{12} - 1560000000000z^{11} + 192000000000000000z^{10} - 1216512535680000675 0z^{8} - 1474560223200000000z^{7} - 652800000000000000z^{6} - 40960000000000000000000z^{5} - 16986908639233347839997975z^{4} - 14155767152640302400000000z^{3} - 5898238732800000000000000z^{2} - 1228800000000000000000000z - 6195303619231982878732441600243$

- Applying the transformation of Dahan and Schost leads to 1787 characters.
  - $(20z^{19} + (-48z^{15}) + (-192000000z^{14}) + (-(38707199784/5)z^{11}) + (-5491200000z^{10}) + 614400000000000z^{9} + (-(778568022835200432/25)z^{7}) + (-33030148999680000z^{6}) + (-1253376000000000z^{5}) + (-655360000000000000000z^{4}) + (-(27179053822773356543996 76/125)z^{3}) + (-1358953646653469030400 0z^{2}) + (-37748727889920000000000z) - 393216000000000000000)x +$

$3200000z^{15} + 161280000z^{12} + 124800000z^{11} + (-30720000000000z^{10}) + 1946419628544000z^{8} + 2359296178560000z^{7} + 10444800000000000z^{6} + 98304000000000000000z^{5} + 4076859878277227827200z^{4} + 3397384824422424192000z^{3} + 14155773972480000000z^{2} + 29491200000000000000z + 1982496995079656780596195328$

- $(20z^{19} + (-48z^{15}) + (-192000000z^{14}) + (-(38707199784/5)z^{11}) + (-5491200000z^{10}) + 614400000000000z^{9} + (-(778568022835200432/25)z^{7}) + (-33030148999680000z^{6}) + (-125337600000000000z^{5}) + (-655360000000000000000z^{4}) + (-(2717905382277335654399676/125)z^{3}) + (-13589536466534690304000z^{2}) + (-3774872788992000000000z) - 393216000000000000000)y + (-12z^{1}6) + (-(9676799856/5)z^{12}) + (-1996800000z^{11}) + (-(194642219980800648/25)z^{8}) + (-14155781713920000z^{7}) + (-83558400000000000z^{6}) + (-(679471833416273049598704/125)z^{4}) + (-9059676821914761216000z^{3}) + (-566230715596800000000z^{2}) + (-157286400000000000000z) + (-2038432221757477324800972/625)$

- $z^{2}0 + (-3z^{1}6) + (-12800000z^{15}) + (-(3225599982/5)z^{12}) + (-499200000z^{11}) + 61440000000000z^{10} + (-(97321002854400054/25)z^{8}) + (-4718592714240000z^{7}) + (-20889600000000000z^{6}) + (-131072000000000000000z^{5}) + (-(679476345569333913599919/125)z^{4}) + (-4529845488844896768000z^{3}) + (-188743639449600000000z^{2}) + (-393216000000000000000z) + (-6195303619231982878732441600243/3125)$

- One can do better! Here's the regular chain produced by the `Triangularize` algorithm of the `RegularChains` library, counting 963 characters.

  - $20x - 1y + z$

  - $\Big((4375z^{12} + 52800011625z^{8} + 32000000000z^{7} + 11059192080002925z^{4} + 61439980800000000z^{3} + 128000000$
$1875z^{13} - 9600010125z^{9} + 2000000000z^{8} - 7372714752004545z^{5} + 30720002400000000z^{4} + 12800000000000000z^{3} - 22118403456000135z + 23592963686400144000000$

  - $3125z^{20} - 9375z^{16} - 40000000000z^{15} - 2015999988750z^{12} - 15600000000000z^{11} + 192000000000000000z^{10} - 12165125356800006750z^{8} - 14745602232000000000z^{7} - 65280000000000000000z^{6} - 409600000000000000000000z^{5} - 16986908639233347839997975z^{4} - 14155767152640302400000000z^{3} - 589823873280000000000000z^{2} - 122880000000000000000000z - 6195303619231982878732441600243$

Gröbner bases (I)

NOTATION. Fix $\leq$ a term order on $M = \{x_1^{i_1} \dots x_n^{i_n} \mid i_j \geq 0\}$, i.e., a total order on M satisfying $1 \leq u$ and $u \leq v \;\Rightarrow uw \leq vw$ for all $u, v, w \in M$.

For $f \in \mathbb{K}[x_1, \dots, x_n]$, $f \neq 0$, the **leading (= greatest) monomial** w.r.t. $\leq$ in f is denoted $\boxed{\operatorname{lm} f}$ and its coefficient in f is the **leading coefficient** of f, denoted $\operatorname{lc} f$.

For $F \subset \mathbb{K}[X] \setminus \{0\}$, we write $\boxed{\operatorname{lm} F = \{\operatorname{lm} f \mid f \in F\}.}$

DEFINITION. $f \in \mathbb{K}[X]$ is **reduced** w.r.t. $g \in \mathbb{K}[X]$, $g \neq 0$ if $\operatorname{lm} g$ does not divide any monomial in f.

NOTATION. If f is not reduced w.r.t. one of the polynomials $b_1, \dots, b_k \in \mathbb{K}[X]$, then the operation $\operatorname{Reduce}(f, \{b_1, \dots, b_k\})$

(1) computes polynomials $r, q_1, \dots, q_k \in \mathbb{K}[X]$ such that $f = q_1 b_1 + \cdots + q_k b_k + r$ holds and r is reduced w.r.t. all $b_1, \dots, b_k \in \mathbb{K}[X]$,

(2) if r is not zero, then replaces r by $r/(\operatorname{lc} f)$,

(3) and returns r.

Gröbner bases (II)

NOTATION. For A, B finite subsets of $\mathbb{K}[X] \setminus \{0\}$ the collection of the $\mathrm{Reduce}(a, B)$, for $a \in A$, is denoted by $\mathrm{Reduce}(A, B)$.

DEFINITION. A subset $B \subset \mathbb{K}[X] \setminus \{0\}$ is **auto-reduced** if for all $b \in B$ the polynomial b is reduced w.r.t. $B \setminus \{b\}$ and $\mathrm{lc} b = 1$.

PROPOSITION. (**Dickson's Lemma**) Every auto-reduced set is finite.

DEFINITION. For $A, B \subseteq F$ auto-reduced sets, we write $A \leq B$ whenever

$$[\mathrm{lm} B \subseteq \mathrm{lm} A] \;\text{ or }\; [\min(\mathrm{lm} A \setminus \mathrm{lm} B) < \min(\mathrm{lm} B \setminus \mathrm{lm} A)].$$

DEFINITION. For an ideal $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$, a minimal auto-reduced subset $B \subset I$ is called a **reduced Gröbner basis** of $\mathcal{I}$.

PROPOSITION. Every ideal $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_n]$ admits a reduced Gröbner basis; moreover an auto-reduced subset $B \subset \mathcal{I}$ is a reduced Gröbner basis of $\mathcal{I}$ iff we have for all $f \in \mathbb{K}[x_1, \ldots, x_n]$

$$f \in \mathcal{I} \quad \Longleftrightarrow \quad \mathrm{Reduce}(f, B) = 0.$$

Buchberger's Algorithm for computing Gröbner bases

Input: $F \subset \mathbb{K}[X]$ and a term order $\leq$.

Output: G a reduced Gröbner basis w.r.t. $\leq$ of the ideal $\langle F \rangle$ generated by F.

repeat

(S) $B := \mathrm{MinimalAutoreducedSubset}(F, \leq)$

(R) $A := \mathrm{S\_Polynomials}(B) \cup F$;

$R := \mathrm{Reduce}(A,\ B,\ \leq)$

(U) $R := R \setminus \{0\}$; $F := F \cup R$

until $R = \emptyset$

return B

NOTATION. For $f, g \in \mathbb{K}[X]\{0\}$, let $L = \mathrm{lcm}(\mathrm{lm} f, \mathrm{lm} g)$; then

$$S(f,g) := \frac{L}{\mathrm{lm}_{\leq}\, f} f - \frac{L}{\mathrm{lm}_{\leq}\, g} g$$

and $\mathrm{S\_Polynomials}(F)$ returns the $S(f,g)$ for all pairs $\{f,g\} \subseteq F$.

A recursive vision of polynomials

DEFINITION. Let $f, g \in \mathbb{K}[X]$ with $g \notin \mathbb{K}$.

mvar(g)**:** the greatest variable in g is the **leader** or **main variable** of g,

init(g)**:** the leading coefficient of g w.r.t. mvar(g) is the **initial** of g,

mdeg(g)**:** the degree of g w.r.t. mvar(g),

rank(g) $= v^d$ where $v = \text{mvar}(g)$ and $d = \text{mdeg}(g)$,

pdivide(f, g) $= (q, r)$ with $q, r \in \mathbb{K}[X]$, $\deg(r, v_g) < d_g$ and $h_g^e f = qg + r$ where $h_g = \text{init}(g)$, $e = \max(\deg(f, v) - d_g + 1, 0)$, $v_g = \text{mvar}(g)$ and $d_g = \text{mdeg}(g)$,

prem(f, g) $= r$ if pdivide$(f, g) = (q, r)$. $f \in \mathbb{K}[X]$ is said **(pseudo-)reduced** w.r.t. $g \in \mathbb{K}[X] \notin \mathbb{K}$ if $\deg(f, \text{mvar}(g)) < \text{mdeg}(g)$.

EXAMPLE.

Assume $n \geq 3$. If $p = x_1 x_3^2 - 2x_2x_3 + 1$, then we have mvar$(p) = x_3$, mdeg$(p) = 2$, init$(p) = x_1$ and rank$(p) = x_3^2$.

Triangular sets and auto-reduced sets

DEFINITION. A finite subset $B \subset \mathbb{K}[X] \setminus \mathbb{K}$ is

- a **triangular set** if for all $f, g \in B$ we have $f \neq g \;\Rightarrow\; \mathrm{mvar}(f) \neq \mathrm{mvar}(g)$,

- **auto-(pseudo-)reduced** if all $b \in B$ is pseudo-reduced w.r.t. $B \setminus \{b\}$.

PROPOSITION. Every auto-reduced set is finite and is a triangular set.

NOTATION. Let $f \in \mathbb{K}[X]$ and $B \subset \mathbb{K}[X] \setminus \mathbb{K}$ an auto-reduced set. If $B = \varnothing$ we write $\mathrm{prem}(f, B) = f$. Otherwise let $b \in B$ with largest main variable; we write $\mathrm{prem}(f, B) = \mathrm{prem}(\mathrm{prem}(f, b), B \setminus \{b\})$. For $A \subset \mathbb{K}[X]$ write $\mathrm{prem}(A, B) = \{\mathrm{prem}(a, B) \mid a \in A\}$.

EXAMPLE. For instance, with $T_4 = \{x_1(x_1 - 1), x_1 x_2 - 1\}$ and $p = x_2^2 + x_1 x_2 + x_1^2$, we have

$$\mathrm{prem}(p, T) = \mathrm{prem}(\mathrm{prem}(p,\ T_{x_2}),\ T_{x_1}) = \mathrm{prem}(x_1^4 + x_1^2 + 1,\ T_{x_1}) = 2\,x_1 + 1.$$

where $T_{x_1} = x_1(x_1 - 1)$ and $T_{x_2} = x_1 x_2 - 1$.

The saturated ideal of a triangular set (I)

DEFINITION. Let $T \subset \mathbb{K}[X]$ be a triangular set. The set

$$\mathrm{Sat}(T) = \{f \in \mathbb{K}[X] \mid (\exists e \in \mathbb{N})\ h_T^e f \in \langle T \rangle\}$$

is the **saturated ideal** of T. (**Clearly** $\mathrm{Sat}(T)$ **is an ideal.**)

PROPOSITION. Let $T \subset \mathbb{K}[X]$ be a triangular set and $f \in \mathbb{K}[X]$. We have

$$\mathrm{prem}(f, T) = 0 \ \Rightarrow\ f \in \mathrm{Sat}(T).$$

REMARK. The **converse is false.** Consider $n \geq 2$ and

$$T = \{x_1(x_1 - 1), x_1 x_2 - 1\}.$$

Consider $p = (x_1 - 1)(x_1 x_2 - 1)$ and $q = -(x_1 - 1)x_1 x_2$. We have:

$$\mathrm{prem}(p, T) = \mathrm{prem}(q, T) = 0.$$

However, we have $p + q = 1 - x_1$, $\mathrm{prem}(p + q, T) \neq 0$ but $p + q \in \mathrm{Sat}(T)$, since $\mathrm{Sat}(T)$ is an ideal. Note that $\mathrm{Sat}(T) = \langle x_1 - 1, x_2 - 1 \rangle$.

The saturated ideal of a triangular set (II)

- Consider again for $x > y > a > b > c > d > e > f > g > h > i$

$$F = \begin{cases} ax + by - c \\ dx + ey - f \\ gx + hy - i \end{cases} \quad \text{and} \quad T = \begin{cases} gx + hy - i \\ (hd - eg)\, y - id + fg \\ (ie - fh)\, a + (ch - ib)\, d + (fb - ce)\, g \end{cases}$$

- Using Gröbner basis computations, one can check the following assertions for this example:

  - $\mathrm{Sat}(T) = \langle F \rangle$.
  - $\mathrm{Sat}(T)$ is an ideal stricly larger than $\langle T \rangle$.
  - In fact $\langle T \rangle \ \subset \ \mathrm{Sat}(T) \cap \langle g, h, i \rangle$,
  - and none of $\mathrm{Sat}(T)$ or $\langle g, h, i \rangle$ contains the other.

Relations between Gröbner bases and regular chains

$$(\mathcal{P}) = \begin{cases} ax + by - c \\ dx + ey - f \\ gx + hy - i \end{cases} \quad \text{and} \quad T = \begin{cases} gx + hy - i \\ (hd - eg)\, y - id + fg \\ (ie - fh)\, a + (ch - ib)\, d + (fb - ce)\, g \end{cases}$$

$$\begin{array}{rcl} \mathbf{V}(\mathcal{P}) & = & \mathbf{W}(T) \cup \mathbf{W} \begin{cases} dx + ey - f \\ hy - i \\ (ie - fh)\, a + (-ib + ch)\, d \\ g \end{cases} \cup \mathbf{W} \begin{cases} gx + hy - i \\ (ha - bg)\, y - ia + cg \\ hd - eg \\ ie - fh \end{cases} \\ & & \cup \mathbf{W} \begin{cases} x \\ (hd - eg)\, y - id + fg \\ fb - ce \\ ie - fh \end{cases} \cup \mathbf{W} \begin{cases} ax + by - c \\ hy - i \\ d \\ g \\ ie - fh \end{cases} \cup \cdots \end{array}$$

Lex base (P):

$$\begin{cases} xa + yb - c & xd + ye - f & \boxed{xg + yh - i} \\ yae - ydb - af + dc & yah - ygb - ai + gc & \boxed{ydh - yge - di + gf} \\ \boxed{aei - ahf - dbi + dhc + gbf - gec} & & \end{cases}$$

- For more details see **(Aubry, Lazard & M^3 , 1997)**.

The quasi-component of a triangular set

DEFINITION. Let $T \subset \mathbb{K}[X]$ be a **triangular set**. Let h_T be the product of the initials of T. The set $\boxed{W(T) = V(T) \setminus V(\{h_T\})}$ is the **quasi-component** of T.

REMARK. Clearly $W(T)$ may not be variety. Consider $n = 2$ and $T = \{x_1 x_2\}$. We have $h_T = x_1$ and $W(T)$ is the line $x_2 = 0$ minus the point $(0, 0)$.

Observe that Sat$(T) = \langle x_2 \rangle$.

PROPOSITION. For any **triangular set** $T \subset \mathbb{K}[X]$ we have

$$\overline{W(T)} = V(\mathrm{Sat}(T)).$$

REMARK. Consider

$$T = \{x_2^2 - x_1, x_1 x_3^2 - 2x_2 x_3 + 1, (x_2 x_3 - 1)x_4 + x_2^2\}.$$

We have $W(T) = \varnothing = V(T)$.

Characteristic sets (I)

NOTATION. If $f, g \notin \mathbb{K}$, we write $\operatorname{rank}(f) < \operatorname{rank}(g)$ if $\operatorname{mvar}(f) < \operatorname{mvar}(g)$ or, $\operatorname{mvar}(f) = \operatorname{mvar}(g)$ and $\operatorname{mdeg}(f) < \operatorname{mdeg}(g)$. For $F \subset \mathbb{K}[X] \setminus \mathbb{K}$, we write $\boxed{\operatorname{rank}(F) = \{\operatorname{rank}(f) \mid f \in F\}.}$

DEFINITION. For A, B auto-reduced sets, we write $A \leq B$ whenever

$$[\operatorname{rank}(B) \subseteq \operatorname{rank}(A)] \text{ or } [\min(\operatorname{rank}(A) \setminus \operatorname{rank}(B)) < \min(\operatorname{rank}(B) \setminus \operatorname{rank}(A))].$$

DEFINITION. For an ideal $\mathcal{I} \subset \mathbb{K}[X]$, a minimal auto-pseudo-reduced subset $B \subset I$ is called a **Ritt (or Kolchin) characteristic set** of $\mathcal{I}$.

PROPOSITION. Every ideal $\mathcal{I} \subset \mathbb{K}[X]$ admits a **Ritt characteristic set**; an auto-reduced $B \subset \mathcal{I}$ is a Ritt characteristic set of $\mathcal{I}$ iff $\operatorname{prem}(f, B) = 0$ for all $f \in \mathcal{I}$.

Characteristic sets (II)

DEFINITION. For a set $F \subset \mathbb{K}[X]$, an auto-pseudo-reduced subset $B \subseteq F$ such that $\mathrm{prem}(F, B) \subset \mathbb{K}$ is called a **Wu characteristic set** of F.

PROPOSITION. If $B \subseteq F$ is a **Wu characteristic set** of $F \subset \mathbb{K}[X]$, then

- If $\mathrm{prem}(F, B)$ contains a non-zero constant then $V(F) = \varnothing$,
- If $\mathrm{prem}(F, B) = \{0\}$ then

$$V(F) = W(B) \cup \bigcup_{b \in B} V(F \cup \{\mathrm{init}(b)\}).$$

PROOF $\triangleright$ Indeed, $\mathrm{prem}(f, B) = 0$ implies that there exists a product h of the initials of B such that $hf \in \langle B \rangle$. Hence $W(B) \subseteq V(F)$. Thus any $\zeta \in V(F)$ either belongs to $W(B)$ or cancels one of the initials of B. $\triangleleft$

THEOREM. **(Wu, 1987)** For any $F \subset \mathbb{K}[X]$, one can compute finitely many triangular sets $T^1, \ldots, T^s$ such that

$$V(F) = W(T^1) \cup \cdots \cup W(T^s).$$

Wu's Method

Input: $F \subset \mathbb{K}[X]$ and a variable ordering $\leq$.

Output: C a Wu characteristic set of F.

repeat
(S) $B := \mathrm{MinimalAutoreducedSubset}(F, \leq)$
(R) $A := F \setminus B$;
$R :=$ prem(A, B)
(U) $R := R \setminus \{0\}$; $F := F \cup R$
until $R = \emptyset$
return B

- Repeated calls to this procedure computes a decomposition of $V(F)$.

- Cannot detect whether a quasi-component is empty.

$\Rightarrow$ This leads to the theory of **regular chains.** **(Kalkbrener, 1991)** and **(Yang & Zhang, 1991)**.

Regular chains

DEFINITION. Let $\mathcal{I}$ be a proper ideal of $\mathbb{K}[X]$. We say that a polynomial $p \in \mathbb{K}[X]$ is **regular** modulo $\mathcal{I}$ if for every prime ideal $\mathcal{P}$ associated with $\mathcal{I}$ we have $p \notin \mathcal{P}$, equivalently, this means that p is neither null modulo $\mathcal{I}$, nor a zero-divisor modulo $\mathcal{I}$.

DEFINITION. Let $T = \{T_1, \ldots, T_s\}$ be a triangular set where polynomials are **sorted by increasing main variables.**

The triangular set T is a **regular chain** if for all $i = 2 \cdots s$ the initial of T_i is **regular modulo the saturated ideal** of $T_1, \ldots T_{i-1}$.

PROPOSITION. If T is a regular chain then $\mathrm{Sat}(T)$ is a proper ideal of $\mathbb{K}[X]$ and, thus, $W(T) \neq \varnothing$.

Reduction to dimension zero (I)

THEOREM. **(Chou & Gao, 1991), (Kalkbrener, 1991), (Aubry, 1999), (Boulier, Lemaire & M^3 , 2006)** Let $T = \{T_{d+1}, \ldots, T_n\}$ be a triangular set. Assume that mvar$(T_i) = x_i$ for all $d+1 \leq i \leq n$ and assume Sat(T) is a proper ideal of $\mathbb{K}[X]$. Then, every prime ideal $\mathcal{P}$ associated with Sat(T) has dimension d and satisfies

$$\mathcal{P} \cap \mathbb{K}[x_1, \ldots, x_d] = \langle 0 \rangle.$$

COROLLARY. With T as above. Consider the localization by $\mathbb{K}[x_1, \ldots, x_d] \setminus \{0\}$; in other words, we map our polynomials from $\mathbb{K}[x_1, \ldots, x_n]$ to $\mathbb{K}(x_1, \ldots, x_d)[x_{d+1}, \ldots, x_n]$.

Let T_0 be the image of T. Let $p \in \mathbb{K}[x_1, \ldots, x_n]$ and p_0 its image in $\mathbb{K}(x_1, \ldots, x_d)[x_{d+1}, \ldots, x_n]$. Assume p non-zero modulo Sat(T). Then, the following conditions are equivalent:

(1) p is regular w.r.t. Sat(T),

(2) p_0 is invertible w.r.t. Sat(T_0).

In particular T is a regular chain iff T_0 is a (zero-dimensional) regular chain.

Reduction to dimension zero (II)

REMARK. Consequently, we can generalize to positive dimension our computations of **polynomial GCDs** defined previously over zero-dimensional regular chains. (Indeed, It is also possible to relax the condition $\mathrm{Sat}(T_0)$ radical.)

NOTATION. Let T is a regular chain and $F \subset \mathbb{K}[X]$ be a polynomial set. We denote by $Z(F, T)$ the intersection $V(F) \cap W(T)$, that is the set of the zeros of F that are contained in the quasi-component $W(T)$. If $F = \{p\}$, we write $Z(p, T)$ for $Z(F, T)$.

PROPOSITION. Let T be a regular chain. If p is regular modulo $\mathrm{Sat}(T)$, then $Z(p, T)$ is either empty or it is contained in a variety of dimension strictly less than the dimension of $\overline{W(T)}$.

Regular chains and characteristic sets

THEOREM. **(Aubry, Lazard & M^3, 1997)** Let $C \subset \mathbb{K}[X]$ be an auto-(pseudo-)reduced set. Then, we have

$$\mathrm{Sat}(C) = \{p \mid \mathrm{prem}(p, C) = 0\}$$

$$\Updownarrow$$

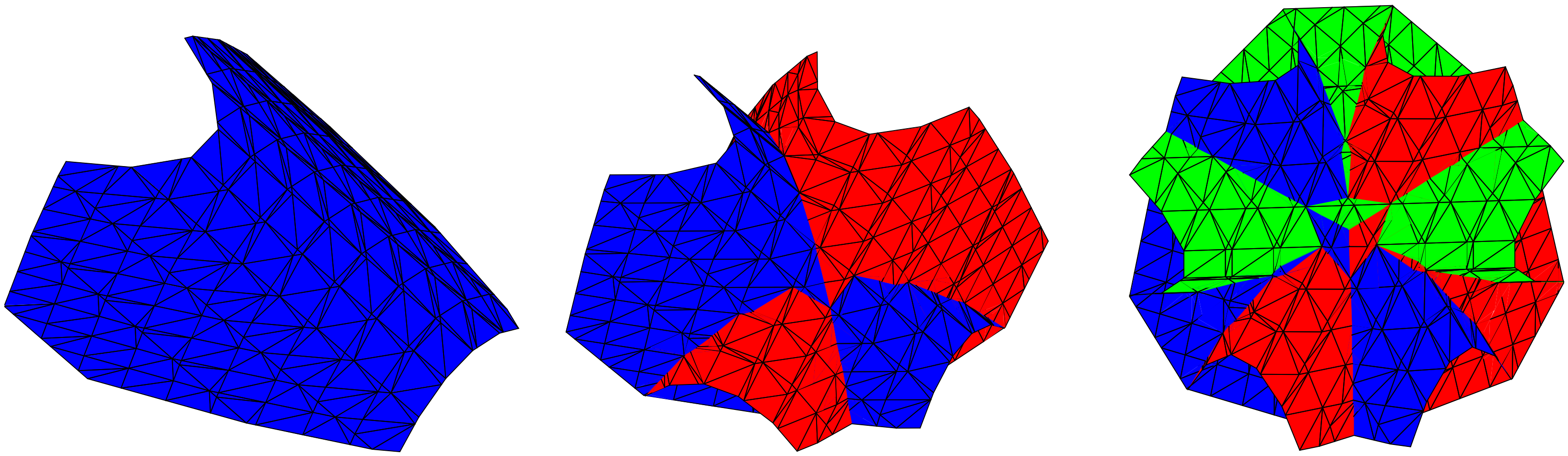
$$C \text{ regular chain}$$

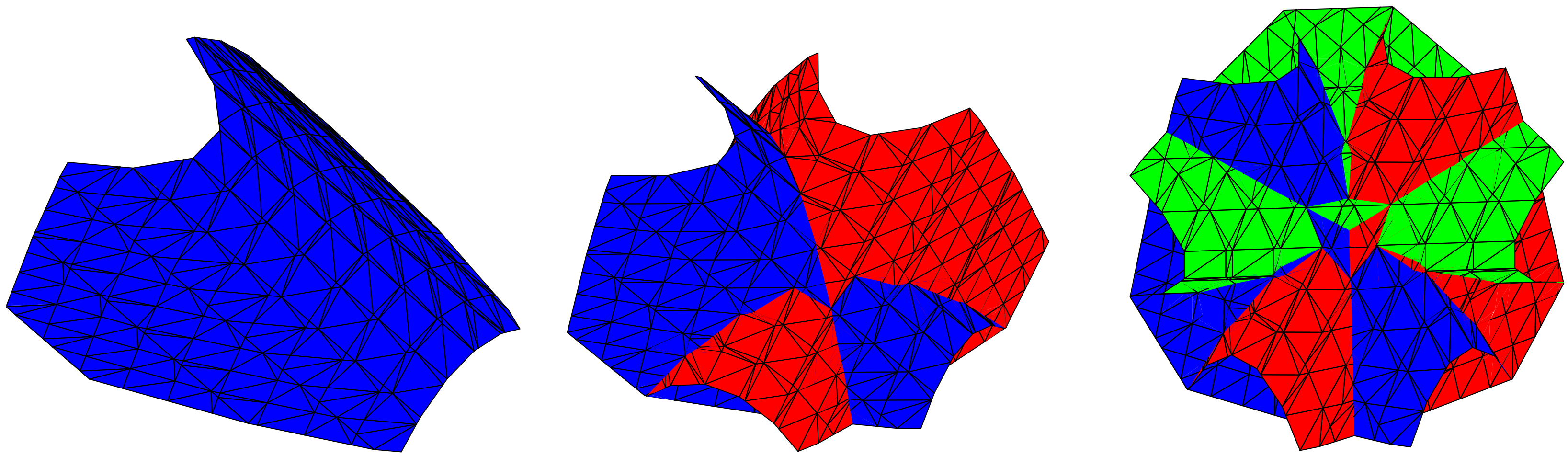
$$\Updownarrow$$

$$C \text{ characteristic set of } \mathrm{Sat}(C)$$

Incremental triangular decompositions: a geometrical approach

$$\left\{ x^2 + y + z = 1 \right. \quad \left\{ \begin{array}{l} x^2 + y + z = 1 \\ x + y^2 + z = 1 \end{array} \right. \quad \left\{ \begin{array}{l} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1 \end{array} \right.$$





$$\left\{\ x^2+y+z=1\right.$$

$$\begin{cases} x+y^2+z=1 \\ y^4+(2z-2)y^2+y-z+z^2=0 \end{cases}$$

$$\begin{cases} x+y=1 \\ y^2-y=z=0 \end{cases}$$

$$\begin{cases} 2x+z^2=2y+z^2=1 \\ z^3+z^2-3z=-1 \end{cases}$$

Incremental Solving

- Let $F \subset \mathbb{K}[x_1, \ldots, x_n]$, $f \in \mathbb{K}[x_1, \ldots, x_n]$, $T, T^m \ldots, T^e \subset \mathbb{K}[x_1, \ldots, x_n]$ reg. chains. Assume we have *solved* F as $V(F) = W(T^i) \cup \cdots \cup W(T^e)$.

- Assume that we have an operation $(f, T) \longmapsto \mathrm{Intersect}(f, T) = (C_1, \ldots, C_d)$ such that

$$V(f) \cap W(T) \ \subseteq \ \cup_i W(C_i) \ \subseteq \ V(f) \cap \overline{W(T)}.$$

Then solving $F \cup f$ reduces to $\mathrm{Intersect}(f, T^i)$ for all i.

$\Longrightarrow$ the core routine operates on well behaved objects.

$\Longrightarrow$ the decomposition can be reduced to regular GCD computation, allowing modular methods and fast arithmetic.

Incremental Solving

- Let $F \subset \mathbb{K}[x_1, \ldots, x_n]$, $f \in \mathbb{K}[x_1, \ldots, x_n]$, $T, T^m \ldots, T^e \subset \mathbb{K}[x_1, \ldots, x_n]$ reg. chains. Assume we have *solved* F as $V(F) = W(T^i) \cup \cdots \cup W(T^e)$.

- Assume that we have an operation
  $(f, T) \longmapsto \mathrm{Intersect}(f, T) = (C_1, \ldots, C_d)$ such that

$$V(f) \cap W(T) \;\subseteq\; \cup_i W(C_i) \;\subseteq\; V(f) \cap \overline{W(T)}.$$

Then solving $F \cup f$ reduces to Intersect(f, T^i) for all i.

$\Longrightarrow$ the core routine operates on well behaved objects.

$\Longrightarrow$ the decomposition can be reduced to regular GCD computation, allowing modular methods and fast arithmetic.

REMARK. (D. Lazard 91) proposes the principle. (M^3 . 00) gives a complete incremental algorithm which, in addition, generates components by decreasing order of dimension.

The notion of a Regular GCD

- Let $P, Q, G \in \mathbb{K}[x_1 < \cdots < x_n][y]$ and $T \subset \mathbb{K}[x_1 < \cdots < x_n]$ reg. chain. G is a *regular GCD* of P, Q modulo $\mathrm{sat}(T)$ if

  (i) $\mathrm{lc}(G, y)$ is a regular modulo $\mathrm{sat}(T)$,

  (ii) $G \in \langle P, Q \rangle$ modulo $\mathrm{sat}(T)$,

  (iii) $\deg_y(G) > 0 \Rightarrow \mathrm{prem}_y(P, G), \mathrm{prem}_y(Q, G) \in \mathrm{sat}(T)$.

- If both $T \cup P$ and $T \cup Q$ are regular chains and if G is a GCD of P, Q modulo $\mathrm{sat}(T)$ with $\deg_y(G) > 0$ then we have

$$
\begin{aligned}
W(T \cup P) \cap V(Q) \subseteq \quad & W(T \cup G) \cup \\
& W(T \cup P) \cap V(Q, h_G) \subseteq \overline{W(T \cup P)} \cap V(Q).
\end{aligned}
$$

One can compute $T^1, \ldots, T^e$ and $G_1, \ldots, G_e$ such that G_i is a reg. GCD of P, Q mod $\mathrm{sat}(T_i)$ and $\sqrt{\mathrm{sat}(T)} = \cap_{i=0}^{e} \sqrt{\mathrm{sat}(T^i)}$.

Regularity test

- Regularity test is a fundamental operation:

$$\mathrm{Regularize}(p, \mathcal{I}) \;\; \longmapsto \;\; (\mathcal{I}_1, \ldots, \mathcal{I}_e)$$

  such that:

$$\sqrt{\mathcal{I}} = \cap_{i=0}^{e} \sqrt{\mathcal{I}_i} \;\text{ and }\; p \in \mathcal{I}_i \text{ or } p \text{ regular modulo } \mathcal{I}_i$$

- Regularity test reduces to regular GCD computation.

Related work

- This notion of a regular GCD was proposed in (M^3 2000)
- In previous work (Kalkbrener 1993) and (Rioboo & M^3 1995), other regular GCDs modulo regular chains were introduced, but with limitations.
- In other work (Wang 2000), (Yang etc. 1995) and (Jean Della Dora, Claire Dicrescenzo, Dominique Duval 85), related techniques are used to construct triangular decompositions.
- Regular GCDs modulo regular chains generalize GCDs over towers of field extentions for which specialized algorithms are available, (van Hoeij and Monagan 2002 & 2004).
- Asymptotically fast algorithms (when $\mathrm{sat}(T)$ is zero-dimensional and radical) appear in (Xavier Dahan, M^3 , Éric Schost, Yuzhen Xie, 2006) and (Xin Li, M^3 , Wei Pan, 2009).