Triangular decomposition of semi-algebraic systems

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joint work with
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IPM Workshop on differential algebra and related topics
24 June 2014

Plan

Triangular set

Definition

$T \subset \mathbf{k}[x_n > \cdots > x_1]$ is a *triangular set* if $T \cap \mathbf{k} = \emptyset$ and $\mathrm{mvar}(p) \neq \mathrm{mvar}(q)$ for all $p, q \in T$ with $p \neq q$.

Theorem (J.F. Ritt, 1932)

Let $V \subset \mathbf{K}^n$ be an irreducible variety and $F \subset \mathbf{k}[x_1, \cdots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

$$(\forall\ g \in \langle \mathbf{F} \rangle)\ \ \mathrm{prem}(g, T) = 0.$$

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Regular chain

Definition

Let $T \subset \mathbf{k}[x_n > \cdots > x_1]$ be a triangular set. For all $t \in T$ write $\mathrm{init}(t) := \mathrm{lc}(t, \mathrm{mvar}(t))$ and $h_T := \prod_{t \in T} \mathrm{init}(t)$. The *quasi-component* and *saturated ideal* of T are:

$$W(T) := V(T) \setminus V(h_T) \text{ and } \mathrm{sat}(T) = \langle T \rangle : h_T^{\infty}$$

Theorem (F. Boulier, F. Lemaire and M^3 2006)

We have: $\overline{W(T)} = V(\mathrm{sat}(T))$. *Moreover, if* $\mathrm{sat}(T) \neq \langle 1 \rangle$ *then* $\mathrm{sat}(T)$ *is strongly equi-dimensional.*

Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

T is a *regular chain* if $T = \emptyset$ or $T := T' \cup \{t\}$ with $\mathrm{mvar}(t)$ maximum s.t.

- T' is a regular chain,
- $\mathrm{init}(t)$ is regular modulo $\mathrm{sat}(T')$

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Regular chain: alternative definition

Regular chain: algorithmic properties

Definition

Let $T \subset \mathbf{k}[x_n > \cdots > x_1]$ be a triangular set and $p \in \mathbf{k}[x_n > \cdots > x_1]$. If T is empty then, the *iterated resultant* of p w.r.t. T is $\mathrm{res}(T, p) = p$. Otherwise, writing $T = T_{<w} \cup T_w$

$$\mathrm{res}(T, p) \quad = \quad \begin{cases} \mathrm{res}(T_{<w}, p) & \text{if } \deg(p, w) = 0 \\ \mathrm{res}(T_{<w}, \mathrm{res}(T_w, p, w)) & \text{otherwise} \end{cases}$$

Theorem (P. Aubry, D. Lazard, M^3)

T is a regular chain iff

$$\{p \mid \mathrm{prem}(p, T) = 0\} \quad = \quad \mathrm{sat}(T)$$

Theorem (L. Yang, J. Zhang 1991)

p is regular modulo $\mathrm{sat}(T)$ iff

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Triangular decomposition of an algebraic variety

Kalkbrener triangular decomposition

Let $F \subset \mathbf{k}[x_1, \ldots, x_n]$. A family of regular chains $T_1, \ldots, T_e$ of $\mathbf{k}[x_1, \ldots, x_n]$ is called a Kalkbrener triangular decomposition of $V(F)$ if

$$V(F) = \cup_{i=1}^{e} V(\mathrm{sat}(T_i)).$$

Wu-Lazard triangular decomposition

Let $F \subset \mathbf{k}[x_1, \ldots, x_n]$. A family of regular chains $T_1, \ldots, T_e$ of $\mathbf{k}[x_1, \ldots, x_n]$ is called a Wu-Lazard triangular decomposition of $V(F)$ if

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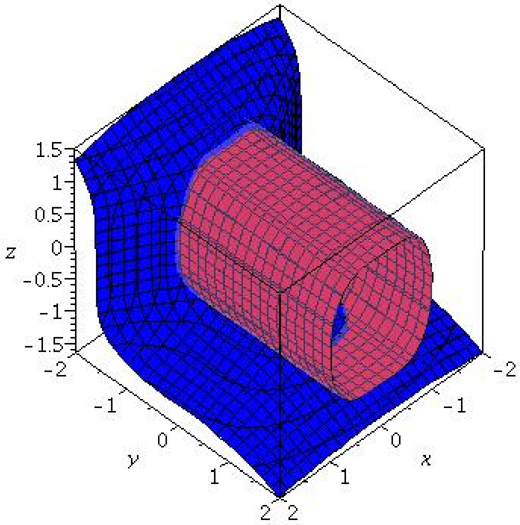
Triangularize applied to *sofa* and *cylinder* (1/2)

$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$



Triangularize applied to *sofa* and *cylinder* (2/2)

Applications Places System — Sun Feb 13, 10:36 PM changbo

/home/changbo/src/maple/triade/mapleP4/lib/cylinder.mw - [Server 1] - Maple 14

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```
> R := PolynomialRing([z, y, x]): F := [x^2+y^3+z^5, x^4+z^2-1]: dec :=
  Triangularize(F, R): map(Display, dec, R);
```

$$\left[\left\{\begin{array}{c}\left(-2x^4+x^8+1\right)z+x^2+y^3=0\\ y^6+2x^2y^3+10x^{12}-10x^8+x^{20}-5x^{16}+6x^4-1=0\\ -2x^4+x^8+1\neq 0\end{array}\right.\right]$$

```
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Ready — Memory: 0.74M — Time:

Plan

Regular chain: specialization properties

Notation

Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ *be a regular chain with* $\mathbf{y} := \{\mathrm{mvar}(t) \mid t \in T\}$ *and* $\mathbf{u} := x_1, \ldots, x_n \setminus \mathbf{y} = u_1, \ldots, u_d$. *Hence* $\mathrm{sat}(T)$ *has dimension* d.

- *Recall that* h_T *is the product of the* $\mathrm{init}(t)$, *for* $t \in T$.
- *Denote by* s_T *the product of the* $\mathrm{discrim}(\mathrm{t}, \mathrm{mvar}(t))$.

Definition

We say that T *specializes well* at a point $u \in \mathbb{R}^d$ if $h_T(u) \neq 0$ and the triangular set $T(u)$ is a regular chain generating a radical ideal.

Theorem (X. Hou, B. Xia, L. Yang, 2001)

Define $BP_T := \mathrm{res}(T, h_T)\,\mathrm{res}(T, s_T)$, *the* border polynomial *of* T. *Then*

- T *specializes well at* $u \in \mathbb{R}^d$ *if and only if* $BP_T(u) \neq 0$.
- *For each connected component* C *of* $BP_T(u) \neq 0$, *the number of real solutions of* $T(u)$ *is constant for* $u \in C$.

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Border polynomial and specialization

Example (bad specialization of a regular chain)

$$T := \begin{cases} x_4 x_5{}^2 + 2x_5 + 1 \\ (x_1 + x_2) x_3{}^2 + x_3 + 1 \\ x_1{}^2 - 1. \end{cases} \qquad T_{x_2, x_4 = -1, 1} := \begin{cases} x_5{}^2 + 2x_5 + 1 \\ (x_1 - 1) x_3{}^2 + x_3 + 1 \\ x_1{}^2 - 1. \end{cases}$$

Example (border polynomial)

$$\mathrm{res}(\mathrm{dis}(t_2), t_1)\, \mathrm{res}(\mathrm{res}(\mathrm{dis}(t_3), t_2), t_1) .\, \mathrm{res}(\mathrm{init}(t_2), t_1)\, \mathrm{res}(\mathrm{res}(\mathrm{init}(t_3), t_2), t_1).$$

For the above regular chain, it is

$$(4x_2 + 3)(4x_2 - 5)(x_2^2 - 1)(x_4 - 1)x_4$$

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Regular semi-algebraic system

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- *Let* P *be a finite set of polynomials, s.t. every* $f \in P$ *is regular modulo* $\mathrm{sat}(T)$.
- *Let* $\mathcal{Q}$ *be a quantifier-free formula of* $\mathbb{Q}[\mathbf{u}]$.

Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(*i*) $\mathcal{Q}$ defines a non-empty open semi-algebraic set S in $\mathbb{R}^d$,

(*ii*) the regular system $[T, P]$ specializes well at every point u of S

(*iii*) at each point u of S, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

Define

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

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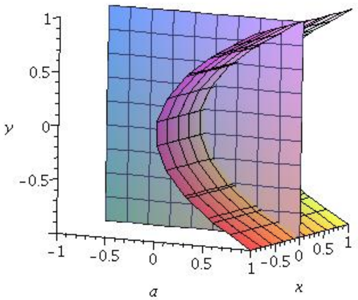
Example

The system $[\mathcal{Q}, T, P_>]$, where

$$\mathcal{Q} := a > 0, \;\; T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases}, \;\; P_> := \{y > 0\}$$

is a regular semi-algebraic system.



Triangular decompositions of semi-algebraic systems (1/2)

Proposition

Let $R := [\mathcal{Q}, T, P_>]$ be a regular semi-algebraic system of $\mathbb{Q}[u_1, \ldots, u_d, \mathbf{y}]$. Then the zero set of R is a nonempty semi-algebraic set of dimension d.

Theorem

Every semi-algebraic system $\mathcal{S}$ can be decomposed as a finite union of regular semi-algebraic systems such that the union of their zero sets is the zero set of $\mathcal{S}$. We call it a (full) triangular decomposition of $\mathcal{S}$.

Triangular decompositions of semi-algebraic systems (2/2)

Notation

Let $\mathcal{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$ be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Let c be the dimension of the constructible set of $\mathbb{C}^n$ corresponding to $\mathcal{S}$.

Definition

A finite set of regular semi-algebraic systems R_i is called a lazy triangular decomposition of $\mathcal{S}$ if

- for each i, $Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(\mathcal{S})$ holds, and
- there exists $G \subset \mathbb{Q}[\mathbf{x}]$ such that

$$Z_{\mathbb{R}}(\mathcal{S}) \setminus \left(\cup_{i=1}^{t} Z_{\mathbb{R}}(R_i)\right) \subseteq Z_{\mathbb{R}}(G),$$

where the complex zero set $V(G)$ has dimension less than c.

A detailed example

Original problem

Consider the following question (Brown, McCallum, ISSAC'05): when does $p(z) = z^3 + az + b$ have a non-real root $x + iy$ satisfying $xy < 1$.

The equivalent quantifier elimination problem

$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[f = g = 0 \wedge y \neq 0 \wedge xy - 1 < 0]$, where

- $f = \mathrm{Re}(p(x + iy)) = x^3 - 3xy^2 + ax + b$
- $g = \mathrm{Im}(p(x + i))/y = 3x^2 - y^2 + a$

The semi-algebraic system to solve

$$\mathcal{S} := \begin{cases} f = 0, \\ g = 0, \\ y \neq 0, \\ xy - 1 < 0 \end{cases}$$

A lazy triangular decomposition

The command `LazyRealTriangularize`$([f, g, y \neq 0, xy-1 < 0], [y, x, b, a])$ returns the following:

$$
\begin{cases}
[\{t_1 = 0, t_2 = 0, 1 - xy > 0\}] & h_1 > 0, h_2 \neq 0 \\
\texttt{\%LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, \\ h_1 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & h_1 = 0 \\
\texttt{\%LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, \\ h_2 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & h_2 = 0 \\
[\ ] & \text{otherwise}
\end{cases}
$$

where

$$
\begin{aligned}
t_1 &= 8x^3 + 2ax - b, t_2 = 3x^2 - y^2 + a, \\
h_1 &= 4a^3 + 27b^2, \\
h_2 &= -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096.
\end{aligned}
$$

A full triangular decomposition

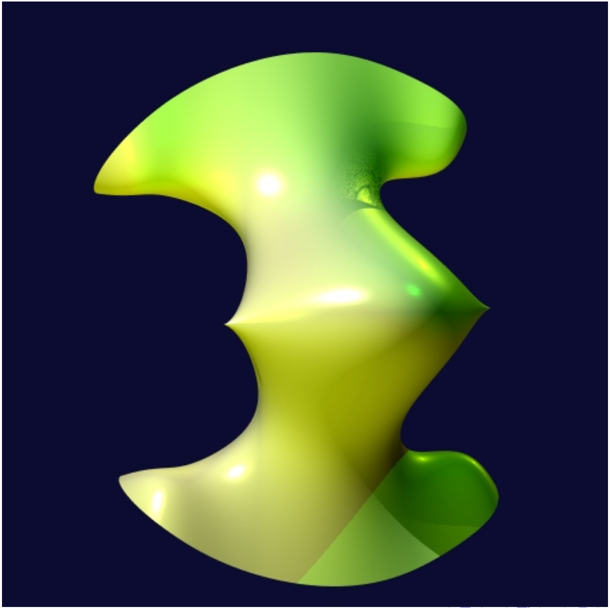
Evaluate the output with the `value` command, which yields

$$\begin{cases} [\{t_1 = 0, t_2 = 0, 1 - xy > 0\}] & h_1 > 0, h_2 \neq 0 \\ [\,] & h_1 = 0 \\ [\{t_3 = 0, t_4 = 0, h_2 = 0\}] & h_2 = 0 \\ [\,] & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} t_3 &= (2a^3 + 32a + 18b^2)x - a^2b - 48b \\ t_4 &= xy + 1 \\ h_1 &= 4a^3 + 27b^2, \\ h_2 &= -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096 \end{aligned}$$

Computing th real points of an algebraic variety (1/2)



Computing th real points of an algebraic variety (2/2)

Applications Places System moreno

Untitled (1)* - [Server 1] - Maple 14

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$R := PolynomialRing([x, y, z]);\ F := [5*x^2 + 2*x*z^2 + 5*y^6 + 15*y^4 + 5*z^2 - 15*y^5 - 5*y^{\wedge}$

$$polynomial\_ring$$

$$[5x^2 + 2xz^2 + 5y^6 + 15y^4 + 5z^2 - 15y^5 - 5y^3]$$

$RealTriangularize(F, R, output = record);$

$$\left\{\begin{array}{l} 5x^2 + 2z^2x + 5y^6 + 15y^4 - 5y^3 - 15y^5 + 5z^2 = 0 \\ 25y^6 - 75y^5 + 75y^4 - z^4 - 25y^3 + 25z^2 < 0 \end{array}\right.,$$

$$\left\{\begin{array}{c} 5x + z^2 = 0 \\ 25y^6 - 75y^5 + 75y^4 - 25y^3 - z^4 + 25z^2 = 0 \\ 64z^4 - 1600z^2 + 25 > 0 \\ z \neq 0 \\ z - 5 \neq 0 \\ z + 5 \neq 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x + 5 = 0 \\ y - 1 = 0 \\ z - 5 = 0 \end{array}\right.,$$

$$\left\{\begin{array}{c} x + 5 = 0 \\ y = 0 \\ z - 5 = 0 \end{array}\right., \left\{\begin{array}{c} x + 5 = 0 \\ y - 1 = 0 \\ z + 5 = 0 \end{array}\right., \left\{\begin{array}{c} x + 5 = 0 \\ y = 0 \\ z + 5 = 0 \end{array}\right., \left\{\begin{array}{c} 5x + z^2 = 0 \\ 2y - 1 = 0 \\ 64z^4 - 1600z^2 + 25 = 0 \end{array}\right.$$

Plan

Outline of the algorithm

Definition

Let $[T, P]$ be as before and $B \subset \mathbb{Q}[\mathbf{u}]$. We say that $[B_{\neq}, T, P_{>}]$ is a pre-regular semi-algebraic system (PRSAS) of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$ if $[T, P]$ specializes well at every point of $B(\mathbf{u}) \neq 0$.

Computation in complex space

$$Z_{\mathbb{R}}(F, N_{\geq}, P_{>}, H_{\neq})$$
$$\downarrow$$
$$\bigcup Z_{\mathbb{R}}(B_{\neq}, T, P_{>})$$

Computation in real space

$$[B_{\neq}, T, P_{>}]$$
$$\downarrow$$
$$\mathcal{Q} := \exists \mathbf{y}\, (B(\mathbf{u}) \neq 0, T(\mathbf{u}, \mathbf{y}) = 0, P(\mathbf{u}, \mathbf{y}) > 0)$$
$$\downarrow$$
$$\text{output } [\mathcal{Q}, T, P_{>}], \text{ where } \mathcal{Q} \neq \text{false}$$

Fingerprint polynomial set

Definition

Let $R := [B_{\neq}, T, P_{>}]$. Let $D \subset \mathbb{Q}[\mathbf{u}]$. Let dp and b be the product of D and B. We call D a *fingerprint polynomial set* (FPS) of R if:

(*i*) for all $\alpha \in \mathbb{R}^d$, $b \in B$: $dp(\alpha) \neq 0 \quad \Rightarrow \quad b(\alpha) \neq 0$,

(*ii*) for all $\alpha, \beta \in \mathbb{R}^d$ with $\alpha \neq \beta$, $dp(\alpha) \neq 0$, $dp(\beta) \neq 0$: if for all $p \in D$, $\mathrm{sign}(p(\alpha)) = \mathrm{sign}(p(\beta))$, then $Z_{\mathbb{R}}(R(\alpha)) \neq \emptyset \iff Z_{\mathbb{R}}(R(\beta)) \neq \emptyset$.

Open projection operator (Brown-McCalumn operator)

Let A be a squarefree basis in $\mathbb{Q}[u_1 < \cdots < u_d]$. Define

$$\mathrm{oproj}(A, u_d) := \bigcup_{f \in A} \mathrm{lc}(f, u_d) \cup \bigcup_{f \in A} \mathrm{discrim}(f, u_d) \cup \bigcup_{f,g \in A} \mathrm{res}(f, g, u_d).$$

Theorem

For $A \subset \mathbb{Q}[u_1, \ldots, u_d]$*, let* $\mathrm{oaf}(A) = \mathrm{der}(A, u_d) \cup \mathrm{oaf}(\mathrm{oproj}(\mathrm{der}(A, u_d), u_{d-1}))$*. If* $R := [B_{\neq}, T, P_{>}]$ *is a PRSAS, then,* $\mathrm{oaf}(B)$ *is an FPS of* R*.*

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A detailed example (1/3)

Applications Places System moreno

/home/moreno/src/Talks/ApplMath-UWO-2011/bp-fps.mw* - [Server 1] - Maple 14

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RealRootIsolate, RefineBox, RefineListBox, RemoveRedundantComponents, RepresentingBox, RepresentingChain, RepresentingQuantifierFreeFormula, RepresentingRootIndex, VariableOrdering]

```
> F := [x^2+y+a, y^2+b*x+a];
```

$$F := [y + x^2 + a, y^2 + b\,x + a]$$

```
> R := PolynomialRing([y, x, b, a]);
```

$$R := polynomial\_ring$$

```
> dec := Triangularize(F, R);
  rc := dec[1];
```

$$dec := [regular\_chain]$$

$$rc := regular\_chain$$

```
> Display(rc, R);
```

$$\begin{cases} y + x^2 + a = 0 \\ x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0 \end{cases}$$

```
> rceqs := Info(rc, R);
```

$$rceqs := [y + x^2 + a, x^4 + 2\,a\,x^2 + b\,x + a^2 + a]$$

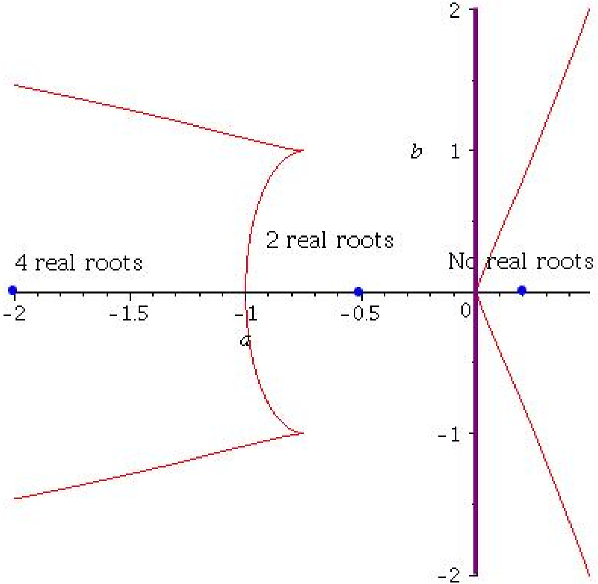
```
> bp := discrim(rceqs[2],x);
```

$$bp := 256\,b^2\,a^3 - 27\,b^4 + 288\,b^2\,a^2 + 256\,a^4 + 256\,a^3$$

```
> with(plots);
```

[*animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal,*

A detailed example (2/3)



A detailed example (3/3)

$$[x^2 + y + a,\ y^2 + b\,x + a]$$

$dec := Triangularize(F, R) : rc := dec[1] : Display(rc, R);$

$$\begin{cases} y + x^2 + a = 0 \\ x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0 \end{cases}$$

$LazyRealTriangularize(F, R, output = record);$

$$\left[\begin{cases} y + x^2 + a = 0 \\ x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0 \\ 256\,b^2\,a^3 - 27\,b^4 + 288\,b^2\,a^2 + 256\,a^4 + 256\,a^3 < 0 \\ a \neq 0 \end{cases},\right.$$

$$\begin{cases} y + x^2 + a = 0 \\ x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0 \\ 256\,b^2\,a^3 - 27\,b^4 + 288\,b^2\,a^2 + 256\,a^4 + 256\,a^3 > 0 \\ a < 0 \end{cases}, \%LazyRealTriangularize([a = 0, y + x^2 + a$$

$= 0, y^2 + b\,x + a = 0, x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0], polynomial\_ring, output = record),$

$\%LazyRealTriangularize([y + x^2 + a = 0, y^2 + b\,x + a = 0, 256\,b^2\,a^3 - 27\,b^4 + 288\,b^2\,a^2 + 256\,a^4 + 256\,a^3$

$= 0, x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0], polynomial\_ring, output = record)$

Plan

LazyRealTriangularize for a system of equations

Algorithm 1: LazyRealTriangularize($\mathcal{S}$)

Input: a semi-algebraic system $\mathcal{S} = [F, \emptyset, \emptyset, \emptyset]$
Output: a lazy triangular decomposition of $\mathcal{S}$
$\mathcal{T} :=$ Triangularize(F)
for $T_i \in \mathcal{T}$ **do**
  $Bp_i :=$ BorderPolynomial($T_i, \emptyset$)
  solve $\exists \mathbf{y}(Bp_i(\mathbf{u}) \neq 0, T_i(\mathbf{u}, \mathbf{y}) = 0)$,
  and let $\mathcal{Q}_i$ be the resulting quantifier-free formula
  if $\mathcal{Q}_i \neq$ *false* **then** output $[\mathcal{Q}_i, T_i, \emptyset]$

Complexity results (1/2)

Assumptions

($\mathbf{H_0}$) $V(F)$ is equidimensional of dimension d,

($\mathbf{H_1}$) $x_1, \ldots, x_d$ are algebraically independent modulo each associated prime ideal of the ideal generated by F in $\mathbb{Q}[\mathbf{x}]$,

($\mathbf{H_2}$) F consists of $m := n - d$ polynomials, $f_1, \ldots, f_m$.

Geometrical formulation

Hypotheses ($\mathbf{H_0}$) and ($\mathbf{H_1}$) are equivalent to the existence of regular chains $T_1, \ldots, T_e$ of $\mathbb{Q}[x_1, \ldots, x_n]$ such that

- $x_1, \ldots, x_d$ are free w.r.t. each T_i
- $V(F) = V(\mathrm{sat}(T_1)) \cup \ldots \cup V(\mathrm{sat}(T_e))$.

Complexity results (2/2)

Notation

Let n, m, δ, $\hbar$ be respectively the number of variables, number of polynomials, maximum total degree and height of polynomials in F.

Proposition

Within $m^{O(1)}(\delta^{O(n^2)})^{d+1} + \delta^{O(m^4)O(n)}$ operations in $\mathbb{Q}$, one can compute a Kalkbrener triangular decomposition $E_1, \ldots, E_e$ of $V(F)$, where each polynomial of each E_i

- has total degree upper bounded by $O(\delta^{2m})$,
- has height upper bounded by $O(\delta^{2m}(m\hbar + d m \log(\delta) + n \log(n)))$.

From which, a lazy triangular decomposition of F can be computed in $\left(\delta^{n^2} n 4^n\right)^{O(n^2)} \hbar^{O(1)}$ bit operations.

Plan

Notations

Table 1 Notions for Tables 2 and 3

symbol	meaning
#e	number of equations in the system
#v	number of variables in the equations
d	max total degree of the equations
G	Groebner:-Basis (with plex order) in MAPLE 13
T	Triangularize in REGULARCHAINS library of MAPLE
LR	lazy RealTriangularize implemented in MAPLE
R	complete RealTriangularize implemented in MAPLE
Q	QEPCAD B
$> 1h$	the examples cannot be solved in 1 hour
FAIL	QEPCAD B failed due to prime list exhausted

Timings for algebraic varieties

Table 2 Timings for algebraic varieties

system	#v/#e/d	G	T	LR
Hairer-2-BGK	13/ 11/ 4	25	1.924	2.396
Collins-jsc02	5/ 4/ 3	876	0.296	0.820
Leykin-1	8/ 6/ 4	103	3.684	3.924
8-3-config-Li	12/ 7/ 2	109	5.440	6.360
Lichtblau	3/ 2/ 11	126	1.548	11
Cinquin-3-3	4/ 3/ 4	64	0.744	2.016
Cinquin-3-4	4/ 3/ 5	$> 1h$	10	22
DonatiTraverso-rev	4/ 3/ 8	154	7.100	7.548
Cheaters-homotopy-1	7/ 3/ 7	3527	174	$> 1h$
hereman-8.8	8/ 6/ 6	$> 1h$	33	62
L	12/ 4/ 3	$> 1h$	0.468	0.676
dgp6	17/19/ 2	27	60	63
dgp29	5/ 4/ 15	84	0.008	0.016

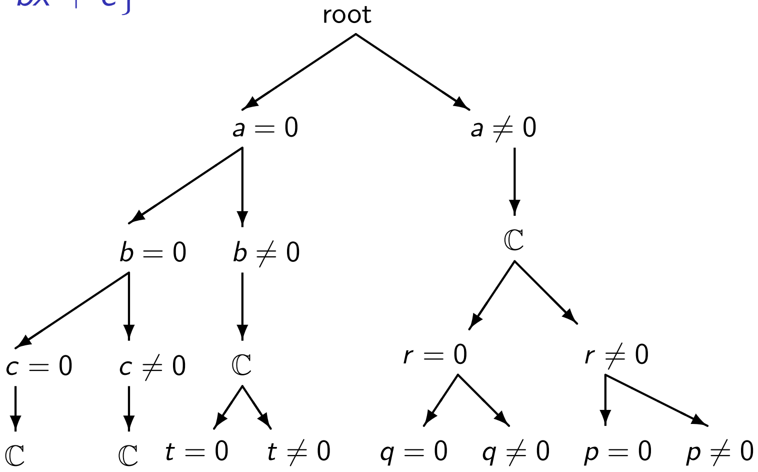
Timings for semi-algebraic systems

Table 3 Timings for semi-algebraic systems

system	#v/#e/d	T	LR	R	Q
BM05-1	4/ 2/ 3	0.008	0.208	0.568	86
BM05-2	4/ 2/ 4	0.040	2.284	$> 1h$	FAIL
Solotareff-4b	5/ 4/ 3	0.640	2.248	924	$> 1h$
Solotareff-4a	5/ 4/ 3	0.424	1.228	8.216	FAIL
putnam	6/ 4/ 2	0.044	0.108	0.948	$> 1h$
MPV89	6/ 3/ 4	0.016	0.496	2.544	$> 1h$
IBVP	8/ 5/ 2	0.272	0.560	12	$> 1h$
Lafferriere37	3/ 3/ 4	0.056	0.184	0.180	10
Xia	6/ 3/ 4	0.164	2.192	230.198	$> 1h$
SEIT	11/ 4/3	0.400	33.914	$> 1h$	$> 1h$
p3p-isosceles	7/ 3/ 3	1.348	$> 1h$	$> 1h$	$> 1h$
p3p	8/ 3/ 3	210	$> 1h$	$> 1h$	FAIL
Ellipse	6/ 1/ 3	0.012	0.904	$> 1h$	$> 1h$

Plan

Recall: cylindrical algebraic decomposition of $\{ax^2 + bx + c\}$



The cylindrical algebraic decomposition of $\{ax^2 + bx + c\}$ is given by the tree above, where $t = bx + c$, $q = 2ax + b$, and $r = 4ac - b^2$. This is the best possible output for that method.

Cylindrical algebraic decomposition of $\mathbb{R}^n$ (1/2)

Definition

A CAD of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$, where

- all the cells are cylindrically arranged, that is, for all $1 \leq j < n$ the projections on the first j coordinates $(x_1, \ldots, x_j)$ of any two cells are either identical or disjoint.
- each cell is a connected semi-algebraic subset, called a region

Complexity of CAD

Unfortunately the number of cells can be **doubly exponential** in n.

Case of $n = 1$

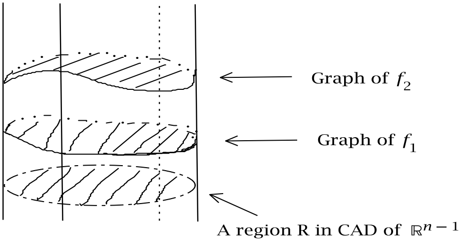
This is a finite partition of the real line into points and open intervals.

Cylindrical algebraic decomposition of $\mathbb{R}^n$ (2/2)

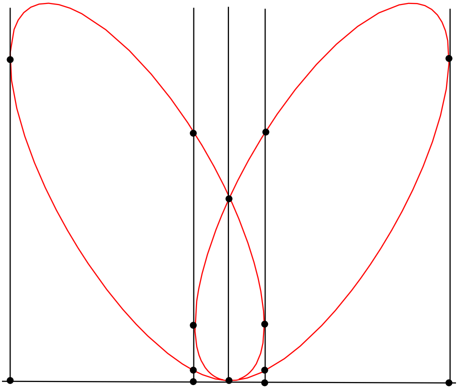
Case of $n > 1$

From a CAD D' of $\mathbb{R}^{n-1}$, one builds a CAD D of $\mathbb{R}^n$. Above each $R \in D'$:

- consider finitely many disjoint graphs (called *sections*) of continuous real-valued algebraic functions,
- decomposing the cylinder $R \times \mathbb{R}^1$, into sections and *sectors* (located between two consecutive sections), which form a stack over R,
- then all the sections and sectors are the elements of D.



A Cylindrical Algebraic Decomposition of $\mathbb{R}^2$ Induced by the Tacnode Curve



Tacnode curve: $y^4 - 2y^3 + y^2 - 3x^2y + 2x^4 = 0.$

RealTriangularize applied to the Tacnode Curve

```
> R := PolynomialRing([x,y]);
> F := [y^4-2*y^3+y^2-3*x^2*y+2*x^4];
> RealTriangularize(F, R, output=record);
{    4          2    4      3    2
{ 2 x  - 3 y x  + y  - 2 y  + y  = 0
{
{              0 < y                             { x = 0
{                                              , {              ,
{            y - 1 <> 0                          { y = 0
{
{            2
{         8 y  - 16 y < 1

    {   x = 0                {    2
    {                      , { 2 x  - 3 = 0          ,
    { y - 1 = 0              {
                             {   y - 1 = 0

    {        2
    { 32 y x  - 48 y - 3 = 0
    {
    {      2
    {  8 y  - 16 y - 1 = 0
```

`RealTriangulaeize`: summary and notes

- We have proposed adaptations of the notions of regular chains and triangular decompositions in order to solve semi-algebraic systems symbolically.
- We have shown that any such system can be decomposed into finitely many *regular semi-algebraic systems*.
- We propose two specifications of such a decomposition and present corresponding algorithms:
- Under some assumptions, one type of decomposition (`LazyRealTriangularize`) can be computed in singly exponential time w.r.t. the number of variables.
- We have implemented both types of decompositions and reported on comparative benchmarks.
- Our experimental results suggest that these approaches are promising.

Recent work

- We have obtained geometrical invariants for the notion of border polynomial.
- We have improved the performances of our algorithms by avoiding unnecessary recursive calls
- We have developed an incremental algorithms for decomposing semi-algebraic systems
- We have procedures for performing set theoretical operations on semi-algebraic sets.
- As a consequence we can produce decomposition free of redundant components.

Plan

Laurent's model for the mad cow disease (1/4)

The dynamical system ruling the transformation

The normal form PrP^C is harmless, while the infectious form PrP^{S_c} catalyzes a transformation from the normal form to the infectious one.

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} & = & k_1 - k_2 x - ax\frac{(1+by^n)}{1+cy^n} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & ax\frac{(1+by^n)}{1+cy^n} - k_4 y \end{cases}$$

where $x = \left[PrP^C\right]$, $y = \left[PrP^{S_c}\right]$ and where b, c, n, a, k_4, k_1 are biological constants which can be set as follows:

$$b = 2, \quad c = 1/20, \quad n = 4, \quad a = 1/10, \quad k_4 = 50 \ \text{ and } \ k_1 = 800.$$

The dynamical system to study

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} & = & \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{2(x+2xy^4-500y-25y^5)}{20+y^4} \end{cases}$$

Laurent's model for the mad cow disease (2/4)

The semi-algebraic system to be solved

$$\mathcal{S} := \begin{cases} 16000 + 800y^4 - 20k_2x - k_2xy^4 - 2x - 4xy^4 &= 0 \\ 2(x + 2xy^4 - 500y - 25y^5) &= 0 \\ k_2 &> 0 \end{cases}$$

Computations (1/5)

LazyRealTriangularize to this system, yields the following regular semi-algebraic system (and unevaluated recursive calls)

$$\begin{cases} (2y^4+1)x - 500y - 25y^5 = 0 \\ (k_2+4)y^5 - 64y^4 + (20k_2+2)y - 32 = 0 \\ (k_2 > 0) \ \wedge \ (R_1 \neq 0) \end{cases}$$

where

$$R_1 = 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 - 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056.$$

Laurent's model for the mad cow disease (3/4)

Computations (2/5)

Through the computation of sample points, we easily obtain the following observation. Whenever $R_1 > 0$ holds, the system has 1 equilibrium, while $R_1 < 0$ implies that the system has 3 equilibria.

Computations (3/5)

Now we study the stability of those equilibria. To this end, we consider the two Hurwitz determinants.
Adding to $\mathcal{S}$ the constraints $\{\Delta_1 > 0, a_2 > 0\}$

$$\Delta_1 = 54y^8 + 40k_2y^4 + 2082y^4 - 312xy^3 + 20040 + k_2y^8 + 400k_2,$$
$$a_2 = 20000k_2 + 2000 + 50k_2y^8 + 200y^8 + 2000k_2y^4 - 312k_2xy^3 + 4100y^4.$$

we obtain a new semi-algebraic system $\mathcal{S}'$.

Laurent's model for the mad cow disease (4/4)

Computations (4/5)

Applying LazyRealTriangularize to $\mathcal{S}'$ in conjunction with sample point computations brings the following conclusion. If $R_1 > 0$, then the system has 1 asymptotically stable hyperbolic equilibria.

Computations (5/5)

If $R_1 < 0$ and $R_2 \neq 0$, then System has 2 asymptotically equilibria, where R_2 is given by:

$$
\begin{aligned}
R_2 = {} & 10004737927168k_2^9 + 624166300700672k_2^8 + 7000539052537600k_2^7 \\
& + 45135589467012800k_2^6 - 84035141185645375 0k_2^5 - 5009800435224844687 5k_2^4 \\
& - 2738816898945500000 0k_2^3 - 8675209266696000000k_2^2 \\
& + 10296091735680000000k_2 + 5932546064102400000 0.
\end{aligned}
$$

To further investigate the number of asymptotically stable hyperbolic equilibria on the hypersurface $R_2 = 0$ and the equilibria when $R_1 = 0$, one can apply SamplePoints on $\mathcal{S}'$, which produces 14 points.

Program verification: an example from Lafferriere (1/4)

Reachability computation

This problem reduces to determine the set

$$\{(y_1, y_2) \in \mathbb{R}^2 \mid (\exists a \in \mathbb{R})(\exists z \in \mathbb{R})\,(0 \leq a) \wedge (z \geq 1) \wedge (h_1 = 0) \wedge (h_2 = 0)\}$$

where

$$h_1 = 3\,y_1 - 2\,a(-z^4 + z) \;\; \text{and} \;\; h_2 = 2\,y_2 z^2 - a(z^4 - 1).$$

The semi-algebraic system to be solved

One wishes to compute the projection of the semi-algebraic set defined by

$$(0 \leq a) \wedge (z \geq 1) \wedge (h_1 = 0) \wedge (h_2 = 0)$$

onto the (y_1, y_2)-plane.
For the variable ordering $a > z > y_1 > y_2$. we obtain the five following regular semi-algebraic systems R_1 to R_5

Program verification: an example from Lafferriere (2/4)

The triangular decomposition (1/3)

$$R_2^T = \begin{cases} a \\ y_1 \\ y_2 \end{cases} \qquad R_3^T = \begin{cases} z-1 \\ y_1 \\ y_2 \end{cases} \qquad R_4^T = \begin{cases} a \\ z-1 \\ y_1 \\ y_2 \end{cases}$$

$$R_2^P = \{\ z > 1 \qquad R_3^P = \{\ 0 < a$$

The projection on the (y_1, y_2)-plane of $Z_{\mathbb{R}}(R_2) \cup Z_{\mathbb{R}}(R_3) \cup Z_{\mathbb{R}}(R_4)$ is clearly equal to the $(y_1, y_2) = (0, 0)$ point.

Program verification: an example from Lafferriere (3/4)

The triangular decomposition (2/3)

$$R_1^T = \begin{cases} (z^4 - 1)\, a - 2\, z^2 y_2 \\ 4\, y_2\, z^5 + 4\, y_2\, z^4 + (3\, y_1 + 4\, y_2)\, z^3 + 3\, y_1\, z^2 + 3\, y_1\, z + 3\, y_1 \end{cases}$$

$$R_1^Q = \begin{cases} (y_1 + y_2 < 0) \wedge (y_1 < 0) \wedge (0 < y_2) \\ 3y_1^5 - 6y_2y_1^4 - 63y_2^2y_1^3 + 192y_2^3y_1^2 + 112y_2^4y_1 + 16y_2^5 \neq 0 \end{cases}$$

$$R_1^P = \{\ z > 1$$

The projection on the (y_1, y_2)-plane of $Z_{\mathbb{R}}(R_1)$ is given by $Z_{\mathbb{R}}(R_1^Q)$.

Program verification: an example from Lafferriere (4/4)

The triangular decomposition (3/3)

$$R_5^T = \begin{cases} \left(z^4 - 1\right) a - 2\, z^2 y_2 \\ t_z \\ 3\, {y_1}^5 - 6\, y_2\, {y_1}^4 - 63\, {y_2}^2 {y_1}^3 + 192\, {y_2}^3 {y_1}^2 + 112\, {y_2}^4 y_1 + 16\, {y_2}^5 \end{cases}$$

$$R_5^Q = \{\ 0 < y_2 \qquad R_5^P = \{\ z > 1$$

where t_z is a large polynomial of degree 4 in z.
The polynomial with main variable y_1, say t_{y_1} is delineable above $0 < y_2$.
Using a sample point we check that t_{y_1} admits a single real root.

Conclusion

It follows that the projection on the (y_1, y_2)-plane of $Z_{\mathbb{R}}(R_5)$ is given by:

$$(0 < y_2) \wedge (3\, {y_1}^5 - 6\, y_2\, {y_1}^4 - 63\, {y_2}^2 {y_1}^3 + 192\, {y_2}^3 {y_1}^2 + 112\, {y_2}^4 y_1 + 16\, {y_2}^5).$$

Plan

The predator-prey biological model (1/4)

- Two species interact, one is a predator and one is its prey, according to the pair of differential equations:

$$\begin{cases} \frac{dx}{dt} & = & x(a - by) \\ \frac{dy}{dt} & = & -y(c - dx). \end{cases}$$

- Say x and y are numbers of **carnivores** and **herbivores**, while a, b, c, d are parameters.

- Population equilibria at:

$$\begin{cases} x(a - by) = 0 \\ y(c - dx) = 0. \end{cases}$$

- This gives two solutions:

$$(x, y) = (0, 0) \ \text{ and } \ (x, y) = (\frac{c}{d}, \frac{b}{a}).$$

The predator-prey biological model (2/4)

- Stability analysis of the hyperbolic equilibria via linearization (Hartman and Grobman Theorem). The Jacobian matrix of the system:

$$J(x, y) = \begin{bmatrix} a - by & -bx \\ dy & dx - c \end{bmatrix}.$$

- Its characteristic polynomial is:

$$p = \lambda^2 + (c - xd - a + yb)\lambda + xad - ac + ybc.$$

- At $(x, y) = (0, 0)$, we have a saddle point, thus instable, since:

$$p = -(\lambda + c)(-\lambda + a)$$

- At $(x, y) = (\frac{c}{d}, \frac{b}{a})$, the characteristic polynomial p has roots with zero real part, as we shall see.

The predator-prey biological model (3/4)

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```
with(RegularChains) : with(LinearAlgebra) : R := PolynomialRing([x, y, a, b, c, d]) :
F := [x * (a - b * y), y * (c - d * x)] : P := [a, b, c, d] :
Jac := Matrix(2, 2, [[a - b * y,  - b * x], [d * y,  d * x - c]]) :
p := CharacteristicPolynomial(Jac, lambda);
```

$$\lambda^2 + (c - d\,x - a + b\,y)\,\lambda + d\,x\,a - c\,a + c\,b\,y \qquad (1)$$

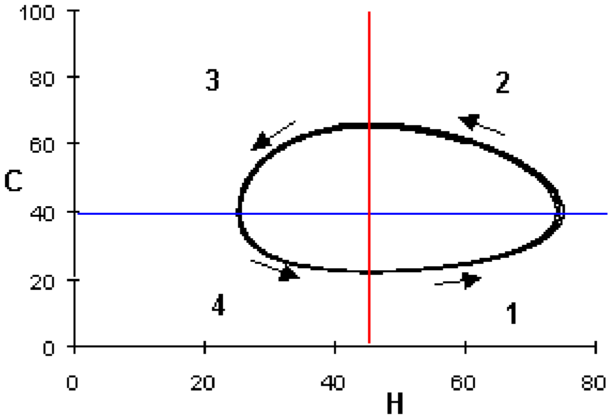
```
q := eval(p, [x = c/d, y = a/d]) : s := eval(q, [lambda = I * theta]); su := coeff(s,
    I) : sv := simplify(s - coeff(s, I) * I) : R := PolynomialRing([theta, a, b, c, d]) : F
    := [numer(su), numer(sv)] :
```

$$-\theta^2 + \mathrm{I}\left(-a + \frac{b\,a}{d}\right)\theta + \frac{c\,b\,a}{d} \qquad (2)$$

```
RealTriangularize(F, [ ], [a, b, c, d], [ ], R, output = record);
```

$$\left\{\begin{array}{l} \theta^2 - a\,c = 0 \\ a > 0 \\ b - d = 0 \\ c > 0 \\ d > 0 \end{array}\right. \qquad (3)$$

The predator-prey biological model (4/4)



Therefore, at $(x, y) = (\frac{c}{d}, \frac{b}{a})$, we have a limit cycle:

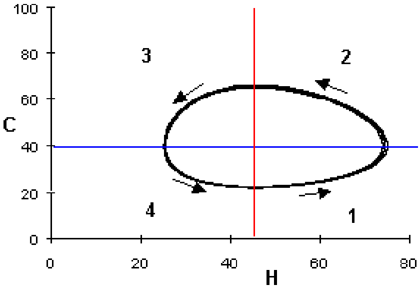
- as the number of herbivores increases, then so does that of carnivores.
- but as that of carnivores increases, that of herbivores decreases, . . .

Cycles limite dans le modèle proie-prédateur

- Two species interact, one is a predator and one is its prey, according to the pair of differential equations:

$$\begin{cases} \frac{dx}{dt} & = & x(a - by) \\ \frac{dy}{dt} & = & -y(c - dx). \end{cases}$$

- Say x and y are numbers of **carnivores** and **herbivores**, while a, b, c, d are parameters.



At $(x, y) = (\frac{c}{d}, \frac{b}{a})$, we have a limit cycle:

- as the number of herbivores increases, then so does that of carnivores.
- but as that of carnivores increases, that of herbivores decreases, ...

Hilbert 16's Problem: the statement

H 16: modern version

The (second half) of the 16th problem is one of the two remaining ones.

It asks for an upper bound of the number of limit cycles in polynomial vector fields:

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y) \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ are real polynomials of total degree n.

So far one got there:

$n = 2$ is solved and the maximum is 3.

But $n = 3$ resists, even if restricting to the nearby of isolated fixed points.

We consider the computation of small limit cycles bifurcated from a center at origin.

Problem set up

Original problem:

Consider a general normalized cubic system:

$$\dot{x} = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3$$
$$\dot{y} = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \,. \quad (2)$$

Reworked problem:

After various transformations (rescaling, etc.) aiming at reducing the number of parameters, one obtains:

$$\begin{aligned} \dot{x} =& \alpha x + y + x^2 + (b+2d)xy + cy^2 + fx^3 + gx^2y + (h-3p)xy^2 + ky^3 \\ \dot{y} =& -x + \alpha y + dx^2 + (e-2)xy - dy^2 + lx^3 + (m-h-3f)x^2y \\ & + (n-g)xy^2 + py^3. \end{aligned} \quad (3)$$

which depends on **13 variables** $\{\alpha, b, c, d, e, f, g, h, k, l, m, n, p\}$.

Using polar coordinates

Normal form

One obtains the so-called normal form:

$$\begin{aligned} \frac{dr}{dt} &= r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k}), \\ \frac{d\theta}{dt} &= 1 + \omega + t_1 r^2 + t_2 r^4 + \cdots + t_k r^{2k}, \end{aligned} \quad (4)$$

where $v_0, \ldots, v_k$ depend polynomially on $\{\alpha, b, c, d, e, f, g, h, k, l, m, n, p\}$.

Theorem (Yu Pei)

If the system

$$v_0 = v_1 = \cdots = v_{k-1} = 0, \; v_k \neq 0 \,, \quad (5)$$

is consistent, then there are at most k limit cycles. Furthermore, these are **exactly** *k limit cycles if at one* **real** *solution we have:*

$$det \left(\frac{\partial v_i}{\partial a_j} \right)_{(k-1)\times(k-1)} \neq 0 \quad (6)$$

The system to solve

13 should be the maximum!

It follows that "generically"' we need to solve

$$v_0 = v_1 = \cdots = v_{k-1} = 0, \;\; v_k \neq 0 \,, \tag{7}$$

for $k = 13$, since there are 13 variables $\{\alpha, b, c, d, e, f, g, h, k, l, m, n, p\}$.

Thus we expect to prove that 13 is an upper bound.

The system is hard to generate

So far we could only generate $v_0, v_1, \ldots, v_9$ after several days of computation with MAPLE.

However, $v_0, v_1, \ldots, v_9$ appear to be **very sparse** (linear growth w.r.t. their total degree).

A first attempt via symbolic solving

Make the origin a center!

We return to the Cartesian formulation

$$\dot{x} = ax + y + x^2 + (b + 2d)xy + cy^2 + fx^3 + gx^2y + (h - 3p)xy^2 + ky^3 ,$$
$$\dot{y} = -x + ay + dx^2 + (e - 2)xy - dy^2 + lx^3 + (m - h - 3f)x^2y + (n - g)xy^2 + py^3 \quad (8)$$

and set $\alpha = b = e = h = m = n = 0$, $p = f$ and
$n = 1/3(35c^2 + 30c - 15l - 15k - 45)$.

Experimental result

We solved the new system for $g < f < l < k < c$ modulo a 2^{58}-bit prime.
After 19 days of MAPLE, using 9506.1MB, we obtained **852 complex roots** using the `RegularChains library`.
Unfortunately, the output length is $6,355,573$ character long.
Too big for isolating the real roots on a desktop!

Hilbert 16's Problem: summary and notes

- There is hope to solve Hilbert 16's Problem, for $n = 3$, on a cluster (but not on a desktop).
- Using symbolic computation is required.
- We are currently building a software for that purpose.
- Joint work (dynamical system part) with Changbo Chen, Robert M. Corless, Pei Yu, Yiming Zhang.
- The involved `RegularChains library` algorithms are based on the following papers:
  - (Changbo Chen & M^3 , ISSAC 2011)
  - (Xavier Dahan, M^3 , Éric Schost, Yuzhen Xie, ISSAC 2005)
  - (François Boulier, Changbo Chen, François Lemaire & M^3 , ASCM 2009)
  - (Changbo Chen, James H. Davenport, M^3 , Bican Xia & Rong Xiao, ISSAC 2010 & ISSAC 2011)

Plan

Reduction to Taylor shift

The Taylor shift $\mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}+\mathbf{1})$ operation is at the core of Collins-Akritas Algorithm for real root isolation (counting).

Algorithm 2: NumberInZeroOne(p)

Input: a squarefree univariate polynomial p
Output: number of real roots of p in $(0, 1)$

```
begin
    p_1 := x^n p(1/x); p_2 := p_1(x + 1)
    let d be the number of sign variations of the coefficients of p_2
    if d ≤ 1 then return d
    p_1 := 2^n p(x/2); p_2 := p_1(x + 1)
    if x | p_2 then m := 1 else m := 0
    m' := NumberInZeroOne(p_1)
    m = m + NumberInZeroOne(p_2)
    return m + m'
end
```

Reformulate the problem: Pascal's triangle

Example

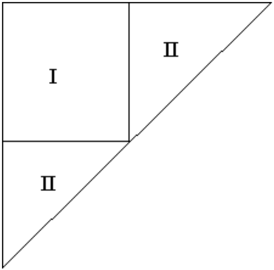
For $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, we have

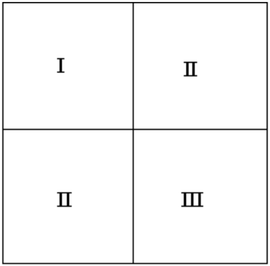
$$f(x+1) = a_3x^3 + (a_2 + 3a_3)x^2 + (a_1 + 2a_2 + 3a_3)x + (a_0 + a_1 + a_2 + a_3)$$

That is:

$$\begin{array}{ccccc}
 & 0 & 0 & 0 & 0 \\
a_3 & + & + & + & + \to c_3 \\
a_2 & + & + & + \to c_2 & \\
a_1 & + & + \to c_1 & \searrow & \\
a_0 & + \to c_0 & & &
\end{array}$$

Work, span and parallelism





For Tableau, we have

- **work:** $U_1(n) = 4U_1(n/2) + 1$, so $U_1(n) = \Theta(n^2)$.
- **span:** $U_\infty(n) = 3U_\infty(n/2) + 1$, so $U_\infty(n) = \Theta(n^{log_2 3})$.

For Pascal's triangle, we have

- **work:** $T_1(n) = 2T_1(n/2) + U_1(n/2)$, so $T_1(n) = \Theta(n^2)$.
- **span:** $T_\infty(n) = T_\infty(n/2) + U_\infty(n/2)$, so $T_\infty(n) = \Theta(n^{log_2 3})$.

The parallelism for both is $\Theta(n^{0.45})$.

Space and cache complexity

Space complexity

Since only the coefficients of $f(x+1)$ matter, computations can be done in place, so $\Theta(n)$.

Cache complexity

For two-way Tableau, we have

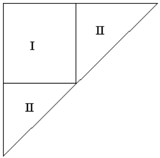
$$Q(n) = \begin{cases} 2n/L + 2 & n \leq \alpha Z \\ 4Q(n/2) + 1 & \text{otherwise} \end{cases} \quad \text{thus} \quad Q(n) = \Theta(n^2/ZL)$$

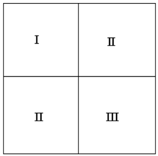
Then for the Pascal's triangle:

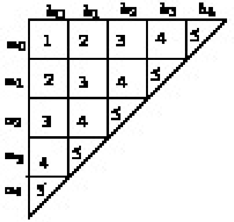
$$Q(n) = \begin{cases} 2n/L + 2 & n \leq \alpha Z \\ 2Q(n/2) + \Theta(n^2/ZL) & \text{otherwise} \end{cases} \quad \text{thus} \quad Q(n) = \Theta(n^2/ZL)$$

Using the Hong-Kung lower bound one can prove that this is optimal.

Increasing the parallelism







Using a k-way divide and conquer

Yes, but the cache complexity then depends linearly on k^2.

Using a blocking strategy

One can partition the entire Pascal Triangle into $B \times B$ blocks. Of course B should be tuned in order for a block to fit in cache.

Span and parallelism are now $\Theta(Bn)$ and $\Theta(n/B)$ respectively.

In addition, if B is well chosen, cache complexity remains optimal..

Experimental results

Table 1. Taylor shift (timings in seconds).

n	k	B	method	Bnd			Cnd			Random		
$\times 10^3$	$\times 10^3$			8p	1p	Sp	8p	1p	Sp	8p	1p	Sp
5	5	50	block	1.3	6.5	4.9	0.92	2.3	2.5	1.3	6.5	4.9
5	5	8	d-n-c	1.5	6.6	4.6	0.94	2.3	2.5	1.5	6.63	4.6
10	10	50	block	7.7	50.8	6.6	4.4	17.5	4.0	7.8	50.78	6.5
10	10	8	d-n-c	8.5	51.7	6.0	4.2	17.6	4.2	8.5	51.65	6.1
25	25	50	block	104	779	7.5	43	261	6.1	104	778.7	7.5
25	25	8	d-n-c	110	790	7.2	42	262	6.3	110	789.7	7.2

This machine has 8 GB memory and 6144 KB of L2 cache.
Each processor is Intel Xeon X5460 @3.16 GHz.
In the table, n and k denote the degree and coefficient size (number of bits) of the input polynomials.

Summary and notes

- The real roots of our polynomial (from the Hilbert 16 problem) with degree 852 and $6,355,573$ character long could be isolated on a 32-core node with 128 GB memory in 15 minutes.
- The implementation is in Cilk++.
- Work in progress includes the use of **dynamically sized blocks** to take into account the increase of work per block.
- Joint work with Changbo Chen and Yuzhen Xie.

Thank you!