On the Extended Hensel Construction and its Application to the Computation of Limit Points

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Plan

1. Introduction
2. The Extended Hensel Construction
3. Yun-Moses Polynomials
4. Lifting the factors
5. Experimentation

Plan

Problem

Goal

Factorize $F(X, Y) \in \mathbb{C}[X, Y]$ into linear factors in X over $\mathbb{C}(\langle Y^* \rangle)$:

$$F(X, Y) = (X - \chi_1(Y))(X - \chi_2(Y)) \cdots (X - \chi_d(Y))$$

where each $\chi_i(Y)$ is a <u>Puiseux series</u>.

Puiseux Series

Series of the form

$$\chi_i(Y) = \sum_{k=a_i}^{\infty} c_k Y^{k/d_i}$$

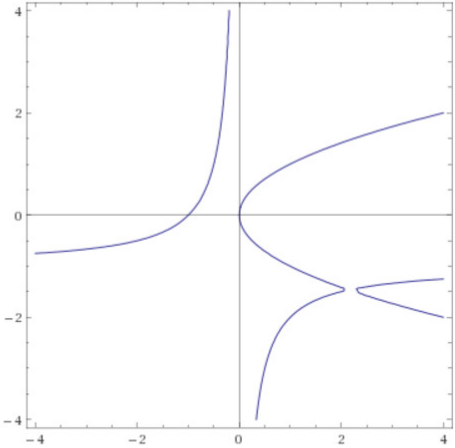
where

- $c_k \in \mathbb{C}$
- $a_i \in \mathbb{Z}$
- $d_i \in \mathbb{Z}_{>0}$

An example

$$F(X,Y) = Y^2X + Y^2 - YX^3 - YX^2 + Y - X^2.$$

- $\chi_1(Y) = \frac{-Y-1}{Y}$
- $\chi_2(Y) = Y$
- $\chi_3(Y) = -Y$



Another example

```
> P := PowerSeries([y, z]) :
  U := UnivariatePolynomialOverPowerSeries([y, z], x) :
  poly := y·x^3 + (−2 · y + z + 1)·x + y :
  U:−ExtendedHenselConstruction(poly, [0, 0], 3);
```

$$\left[\left[x = \frac{-RootOf(\_Z^2+y) + RootOf(\_Z^2+y)\, y - \frac{1}{2} RootOf(\_Z^2+y)\, z + \frac{1}{2} y^2}{y}\right],\right.$$

$$\left[x = \frac{RootOf(\_Z^2+y) - RootOf(\_Z^2+y)\, y + \frac{1}{2} RootOf(\_Z^2+y)\, z + \frac{1}{2} y^2}{y}\right],$$

$$\left.[x = -y]\right]$$

Previous works (1/2)

1. Extended Hensel Construction (EHC):
   - Introduction: F. Kako and T. Sasaki, 1999
   - Extensions:
     - M. Iwami, 2003,
     - D. Inaba, 2005,
     - D. Inaba and T. Sasaki 2007,
     - D. Inaba and T. Sasaki 2016.
2. Newton-Puiseux:
   - H. T. Kung and J. F. Traub, 1978,
   - D. V. Chudnovsky and G. V. Chudnovsky, 1986
   - A. Poteaux and M. Rybowicz, 2015.

Previous works (2/2)

- The Extended Hensel Construction (EHC) compute all branches concurrently
- while approaches based on Newton-Puiseux computes one branch after another.

For $F(X, Y) := -X^3 + Y\,X + Y$:

1. the EHC produces
   1. $\chi_1(Y) := Y^{\frac{1}{3}} + \frac{1}{3}\,Y^{\frac{2}{3}} + O(Y)$,
   2. $\chi_2(Y) := \frac{-1+\sqrt{-3}}{2}\,Y^{\frac{1}{3}} + \frac{1}{3}\,(\frac{-1-\sqrt{-3}}{2})\,Y^{\frac{2}{3}} + O(Y)$,
   3. $\chi_3(Y) := (\frac{-1-\sqrt{-3}}{2})\,Y^{\frac{1}{3}} + \frac{1}{3}\,(\frac{-1+\sqrt{-3}}{2})\,Y^{\frac{2}{3}} + O(Y)$.

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   3. $\chi_3(Y) := (\frac{-1-\sqrt{-3}}{2})\,Y^{\frac{1}{3}} + \frac{1}{3}\,(\frac{-1+\sqrt{-3}}{2})\,Y^{\frac{2}{3}} + O(Y)$.
2. Whereas Kung-Traub's method (based on Newton-Puiseux) computes
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   3. $\chi_3(Y) := (\frac{-1-\sqrt{-3}}{2})\,Y^{\frac{1}{3}} + \frac{1}{3}\,(\frac{-1+\sqrt{-3}}{2})\,Y^{\frac{2}{3}} + O(Y)$.
2. Whereas Kung-Traub's method (based on Newton-Puiseux) computes
   1. $\chi_1(Y) := Y^{\frac{1}{3}} + \frac{1}{3}\,Y^{\frac{2}{3}} + O(Y)$,
   2. $\chi_2(Y) := \theta\,Y^{\frac{1}{3}} + \frac{\theta^2}{3}\,Y^{\frac{2}{3}} + O(Y)$,
   3. $\chi_3(Y) := \theta^2\,Y^{\frac{1}{3}} + \frac{\theta}{3}\,Y^{\frac{2}{3}} + O(Y)$,

   for $\theta \in \mathbb{C}$ such that $\theta^3 = 1, \theta^2 \neq 1, \theta \neq 1$, since $F(X, Y)$ is a Weierstrass polynomial.

Our contributions

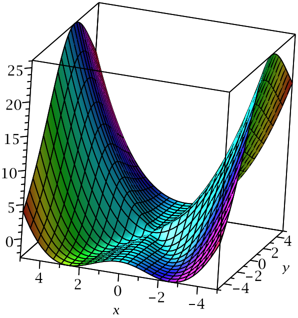
1. We show that the EHC requires only linear algebra and univariate polynomial arithmetic
2. We derive complexity estimates for the EHC
3. We obtain a competitive implementation against the original EHC and Kung-Traub's method
4. We apply the EHC to the problem of computing real limit points of quasi-components of regular chains.

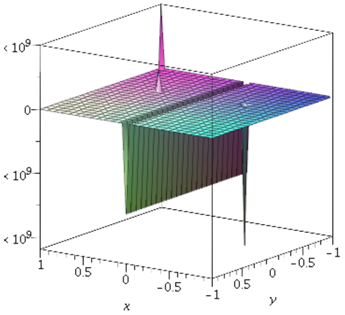
Application (1/2)

One immediate application of Extended Hensel Construction is finding the limit of a multivariate rational function i.e. for a rational function $q = \frac{P_1}{P_2}$ where P_1 and $P_2 \in \mathbb{F}[X_1, \ldots X_n]$, we can ask whether

$$\lim_{(X_1,\ldots,X_n)\to(x_1,\ldots,x_n)} q(X_1, \ldots, X_n)$$

exists and what it converges to.





Application (2/2)

```
> R := PolynomialRing([x, y, z]) :
  rc := Chain([y^(3)-2*y^(3) + y^(2) + z^(5), z^(4)*x + y^(3)-y^(2)], Empty(R), R) :
> LimitPoints(rc, R, coefficient = complex); Display(%, R);
```

[*regular_chain*, *regular_chain*]

$$\left[\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases} \right]$$

```
> LimitPoints(rc, R, coefficient = real); Display(%, R);
```

[*regular_semi_algebraic_system*]

$$\left[\begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases} \right]$$

```
> RegularChainBranches(rc, R, [z]);
```

$$\left[\left[z = T^2, y = \frac{1}{2} T^5 \left(-T^5 + 2\, RootOf\left(\_Z^2 + 1\right)\right), x = -\frac{1}{8} T^2 \left(-T^{20} + 6\, T^{15}\, RootOf\left(\_Z^2 + 1\right) + 10\, T^{10} + 8\right)\right], \left[z = T^2, y = -\frac{1}{2}\right.\right.$$

$$\left. + 2\, RootOf\left(\_Z^2 + 1\right)\right), x = \frac{1}{8} T^2 \left(T^{20} + 6\, T^{15}\, RootOf\left(\_Z^2 + 1\right) - 10\, T^{10} - 8\right)\Big], \left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right.\right.$$

```
> RegularChainBranches(rc, R, [z], coefficient = real);
```

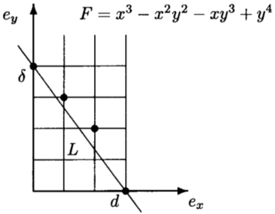
$$\left[\left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

```
>
```

Plan

Newton Polynomial

- Let $F(x,y) \in \mathbb{C}[x,y]$ be square-free, monic in x and let $d := \deg_x(F)$.
- The "south-west-most" terms $c\,x^{e_x}y^{e_y}$ of $F(x,y)$ satisfy an equation of the form $e_x/d + e_y/\delta = 1$, with $\delta \in \mathbb{Q}$ and form the Newton polynomial $F^{(0)}(x,y)$ which is homogeneous $(x, y^{\delta/d})$ of degree d.
- Let $\hat{\delta}, \hat{d} \in \mathbb{Z}^{>0}$ such that: $\hat{\delta}/\hat{d} = \delta/d, \;\; \gcd(()\hat{\delta}, \hat{d}) = 1$.
$$F(x,y) \;\; = \;\; G_1^{(0)}(x,y) \cdots G_r^{(0)}(x,y)$$
where the $G_i^{(0)}(X,Y) := (X - \zeta_i Y^{\delta/d})^{m_i}$ are the initial factors.
- We define the ideal
$$S_k = \langle X^d Y^{(k+0)/\hat{d}}, X^{d-1} Y^{(k+\hat{\delta})/\hat{d}}, \ldots, X^0 Y^{(k+d\hat{\delta})/\hat{d}} \rangle, \qquad (1)$$
for $k = 1, 2, \ldots$.



Algorithm

Algorithm 1: EHC_Lift(F, k)

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \ldots, d-1$;

for $j = 1, \ldots, k$ **do**

  Compute $\Delta F^{(j)}(X, Y) := F(X, Y) - \prod_{i=1}^{r} G_i^{(j-1)}$ mod $\bar{S}_{j+1}$;

  Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

  Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Algorithm

Algorithm 2: EHC_Lift(F, k)

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \cdots, d-1$;

for $j = 1, \ldots, k$ **do**

Compute $\Delta F^{(j)}(X, Y) := F(X, Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$;

Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Algorithm

Algorithm 3: EHC_LiftF, k

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \cdots, d-1$;

for $j = 1, \ldots, k$ **do**

Compute $\Delta F^{(j)}(X, Y) := F(X, Y) - \prod_{i=1}^{r} G_i^{(j-1)}$ mod $\bar{S}_{j+1}$;

Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Plan

Yun-Moses Polynomials (1/3)

Assume $G_1(X,Y),\dots,G_r(X,Y)$ are homogeneous polynomials. Regarding them as polynomials of $\mathbb{C}\langle Y\rangle[X]$, further assume

$$\gcd((\hat{G}_i,\hat{G}_j)) = 1 \text{ for } i \neq j,$$

Let $d := \deg(G_1(X,Y)\ \dots\ G_r(X,Y))$. Then, for each $\ell \in \{0,\dots,d-1\}$, there exists a unique set of polynomials $\{W_i^{(\ell)}(X,Y) \in \mathbb{C}\langle Y\rangle[X] \mid i = 1,\dots,r\}$ satisfying

$$W_1^{(\ell)}\left(\frac{G_1\cdots G_r}{G_1}\right) + \cdots + W_r^{(\ell)}\left(\frac{G_1\cdots G_r}{G_r}\right) = X^\ell Y^{d-\ell},$$

where $\deg_X(W_i^{(\ell)}(X,Y)) < \deg_X(G_i(X,Y)),\quad i = 1,\dots,r.$

Yun-Moses Polynomials (2/3)

Key observation

Let us fix $i := \lambda$. Writing $W_\lambda^{(\ell)} = \sum_{j=0}^{m_\lambda - 1} w_{\lambda,j}(\hat{Y}) X^j$, we have

$$\sum_{j=0}^{m_\lambda - 1} \frac{\partial^\mu}{\partial X^\mu} \left(X^j \frac{F^{(0)}}{G_\lambda^{(0)}} \right) \Big|_{X = \zeta_\lambda \hat{Y}} w_{\lambda,j}^{(\ell)} = \frac{\partial^\mu}{\partial X^\mu} (X^\ell \hat{Y}^{d-\ell}) \Big|_{X = \zeta_\lambda \hat{Y}} .$$

where ζ_λ is a root of $F^{(0)}(X, 1)$ and m_λ is its multiplicity

Consequences

- This is a system of linear equations $\mathcal{W}_\lambda \mathcal{X}_\lambda^{(\ell)} = \mathcal{B}_\lambda^{(\ell)}$.
- The matrix $\mathcal{W}_\lambda$ is a Wronskian matrix.

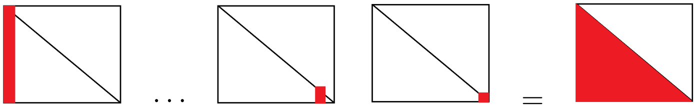
Yun-Moses Polynomials (3/3)

The inverse of $\mathcal{W}_\lambda$ is $\mathcal{W}_\lambda^{-1} = M_2 M_1$ where M_1 and M_2 are square matrices of order m_λ, defined as follows. The matrix M_1 writes $M_1 = M_{1(m_\lambda-1)} \cdots M_{11} M_{10}$ such that, for $j = 0, \cdots, m_\lambda - 1$, we have

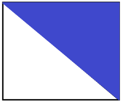
$$M_{1j} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{j!f} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \binom{j+1}{j}\frac{-f'}{f} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \binom{m_\lambda-1}{j}\frac{-f^{(m_\lambda-1-j)}}{f} & 0 & \cdots & 1 \end{bmatrix}.$$

Hence, the matrix M_{1j} differs from the identity matrix only in its $(j+1)$-th column. The matrix M_2 is an upper triangular matrix $M_2 = [\gamma_{j,k}]$ with $\gamma_{j,k} = (-1)^{j+k}\binom{k}{k-j}\zeta_\lambda^{k-j}\hat{Y}^{k-j}$ if $j \leq k$ and $\gamma_{j,k} = 0$ if $j > k$, for $j, k \in \{0, 1, \ldots, m_\lambda - 1\}$.

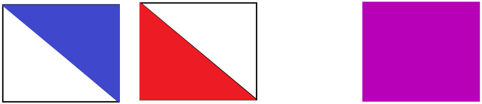
Matrix M_1



Matrix M_2



Matrix $\mathcal{W}_i^{-1} = M_2 M_1$



Complexity Result:

Theorem 1:

One can compute all the Yun-Moses polynomials $W_i^{(\ell)}$ $(0 \leq \ell \leq d-1,\ 1 \leq i \leq r)$, within

- $\mathcal{O}(d^3)$ operations in $\mathbb{C}$, or
- $\mathcal{O}(d^3\,\mathsf{M}(d))$ operations in the field of coefficients of $F(X, Y)$.

Plan

Algorithm

Algorithm 4: EHC_LiftF, k

begin

Compute the Newton polynomial $F^{(0)}$ and $\hat{\delta}, \hat{d}$;

Compute $G_i^{(0)} = (X - \zeta_i Y)^{m_i}$, with $1 \leq i \leq r$;

Compute the Yun-Moses polynomial $W_i^{(\ell)}$ for $i = 1, \cdots, r$ and $\ell = 0, \cdots, d-1$;

for $\underline{j = 1, \ldots, k}$ **do**

Compute

$\Delta F^{(j)}(X, Y) := F(X, Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$;

Compute $\Delta G_i^{(j)} = \sum_{\ell=0}^{m-1} W_i^{(\ell)} f_\ell^{(j)}$, for $i = 1, \cdots, r$;

Let $G_i^{(j)} = G_i^{(j-1)} + \Delta G_i^{(j)}$ for $i = 1, \cdots, r$;

return $G_1^{(k)}, \ldots, G_r^{(k)}$;

Computing $\Delta F^{(j)}(X, Y)$

Goal

$$\Delta F^{(j)}(X, Y) := F(X, Y) - \prod_{i=1}^{r} G_i^{(j-1)} \mod \bar{S}_{j+1}$$

Oobservation

- $G_i^{(j-2)} := G_i^{(0)} + \Delta G_i^{(1)} + \cdots + \Delta G_i^{(j-2)}$
- $G_i^{(j-1)} := G_i^{(0)} + \Delta G_i^{(1)} + \cdots + \Delta G_i^{(j-2)} + \Delta G_i^{(j-1)}$

Hence, we aim at recycling terms in the product $\prod_{i=1}^{r} G_i^{(j-1)}$ mod $\bar{S}_{j+1}$ computed from previous iterations.

Notations

1. $P_2^{k+1} := \prod_{i=1}^{2} G_i^{(k)} \mod \overline{S}_{k+1}$
2. $P_j^{k+1} := \prod_{i=1}^{j} G_i^{(k)} \mod \overline{S}_{k+1}$, for $j = 3, \ldots, r$.

We want

$$P_r^{k+1} = \prod_{i=1}^{r} G_i^{(k)} \mod \overline{S}_{k+2}$$

Computing $\Delta F^{(j)}(X, Y)$

Initially define: $P_j^1 \equiv G_1^{(0)} \cdots G_j^{(0)} \mod S_2$, for $j = 2, \cdots, r$. and recursively compute:

$$P_2^{k+1} = P_2^k + (\Delta_1^0 \Delta_2^k + \Delta_1^k \Delta_2^0)\tilde{Y}^k + (\Delta_1^1 \Delta_2^k + \cdots + \Delta_1^k \Delta_2^1)\tilde{Y}^{k+1} = \prod_{i=1}^{2} G_i^{(k)}$$

Now for $j = 3, \ldots, r$, define

$$P_j^k \equiv P_{j-1}^k G_j^{(k-1)} \mod S_{k+1}$$

and assume q_j^{k+1} is recursively given by

$$q_j^{k+1} = p_{j-1}^{k+1,0} \Delta_j^k + q_{j-1}^{k+1} \Delta_j^0 \quad \text{with} \quad q_2^{k+1} = \Delta_2^k \Delta_1^0 + \Delta_2^0 \Delta_1^k. \quad (2)$$

where $p_{j-1}^{k+1,0}$ is the coefficient of $\tilde{Y}^0$ in P_{j-1}^{k+1}. We can compute

$$P_j^{k+1} = P_j^k + q_j^{k+1}\tilde{Y}^k + \left(p_{j-1}^{k+1,1}\Delta_j^k + \cdots + p_{j-1}^{k+1,k+1}\Delta_j^0\right)\tilde{Y}^{k+1} = \prod_{i=1}^{j} G_i^{(k)}$$

Computing $\Delta F^{(j)}(X, Y)$

$$
\begin{tikzcd}
G_1^{(0)}\,G_2^{(0)} & P_2^1 \arrow[r] \arrow[d] & P_2^2 \arrow[r] \arrow[d] & P_2^3 \arrow[r] \arrow[d] & P_2^4 \arrow[r] \arrow[d] & P_2^5 \arrow[r] \arrow[d] & \cdots \\
& P_3^1 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & \cdots \\
& P_4^1 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & \cdots \\
& P_5^1 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & \cdots \\
& P_6^1 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & \cdots
\end{tikzcd}
$$

Computing $\Delta F^{(j)}(X, Y)$

$$
\begin{tikzcd}
G_1^{(0)} G_2^{(0)} & P_2^1 \arrow[r] \arrow[d] & P_2^2 \arrow[r] \arrow[d] & P_2^3 \arrow[r] \arrow[d] & P_2^4 \arrow[r] \arrow[d] & P_2^5 \arrow[r] \arrow[d] & \cdots \\
P_2^1 G_3^{(0)} & P_3^1 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & \cdots \\
 & P_4^1 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & \cdots \\
 & P_5^1 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & \cdots \\
 & P_6^1 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & \cdots
\end{tikzcd}
$$

Computing $\Delta F^{(j)}(X, Y)$

$$
\begin{tikzcd}
G_1^{(0)} G_2^{(0)} & P_2^1 \arrow[r] \arrow[d] & P_2^2 \arrow[r] \arrow[d] & P_2^3 \arrow[r] \arrow[d] & P_2^4 \arrow[r] \arrow[d] & P_2^5 \arrow[r] \arrow[d] & \cdots \\
P_2^1 G_3^{(0)} & P_3^1 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & \cdots \\
P_3^1 G_4^{(0)} & P_4^1 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & \cdots \\
 & P_5^1 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & \cdots \\
 & P_6^1 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & \cdots
\end{tikzcd}
$$

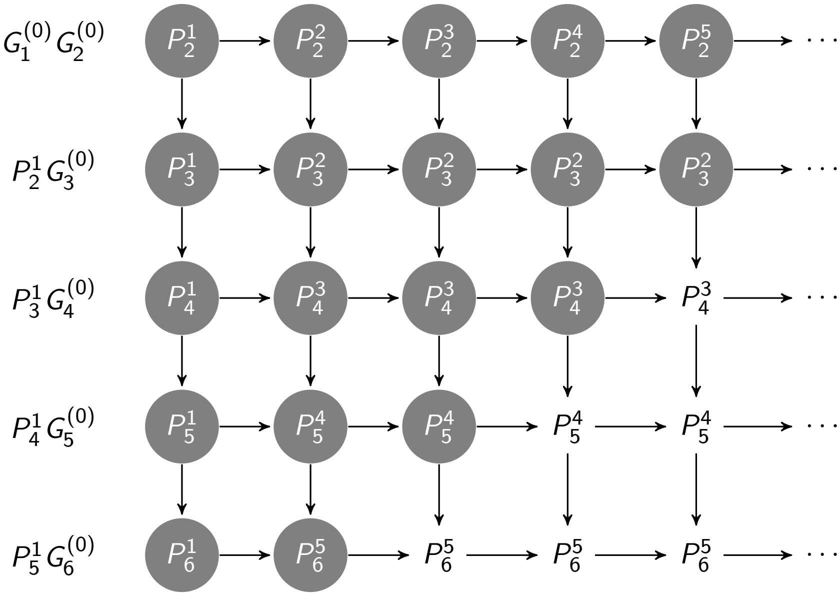
Computing $\Delta F^{(j)}(X, Y)$

$$
\begin{tikzcd}
G_1^{(0)}\, G_2^{(0)} & P_2^1 \arrow[r] \arrow[d] & P_2^2 \arrow[r] \arrow[d] & P_2^3 \arrow[r] \arrow[d] & P_2^4 \arrow[r] \arrow[d] & P_2^5 \arrow[r] \arrow[d] & \cdots \\
P_2^1\, G_3^{(0)} & P_3^1 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & \cdots \\
P_3^1\, G_4^{(0)} & P_4^1 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & \cdots \\
P_4^1\, G_5^{(0)} & P_5^1 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & \cdots \\
 & P_6^1 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & \cdots
\end{tikzcd}
$$

Computing $\Delta F^{(j)}(X, Y)$

$$\begin{tikzcd}
G_1^{(0)} G_2^{(0)} & P_2^1 \arrow[r] \arrow[d] & P_2^2 \arrow[r] \arrow[d] & P_2^3 \arrow[r] \arrow[d] & P_2^4 \arrow[r] \arrow[d] & P_2^5 \arrow[r] \arrow[d] & \cdots \\
P_2^1 G_3^{(0)} & P_3^1 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & \cdots \\
P_3^1 G_4^{(0)} & P_4^1 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & \cdots \\
P_4^1 G_5^{(0)} & P_5^1 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & \cdots \\
P_5^1 G_6^{(0)} & P_6^1 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & \cdots
\end{tikzcd}$$

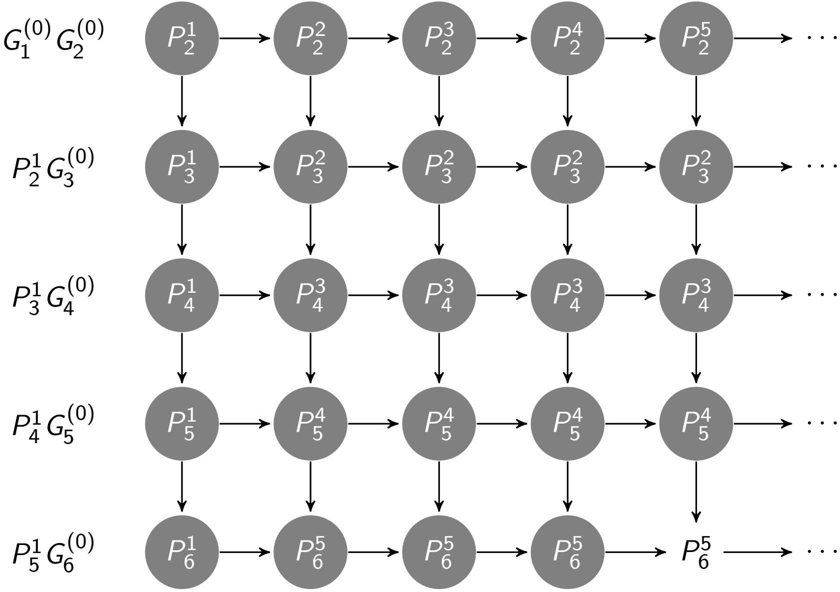
Computing $\Delta F^{(j)}(X, Y)$



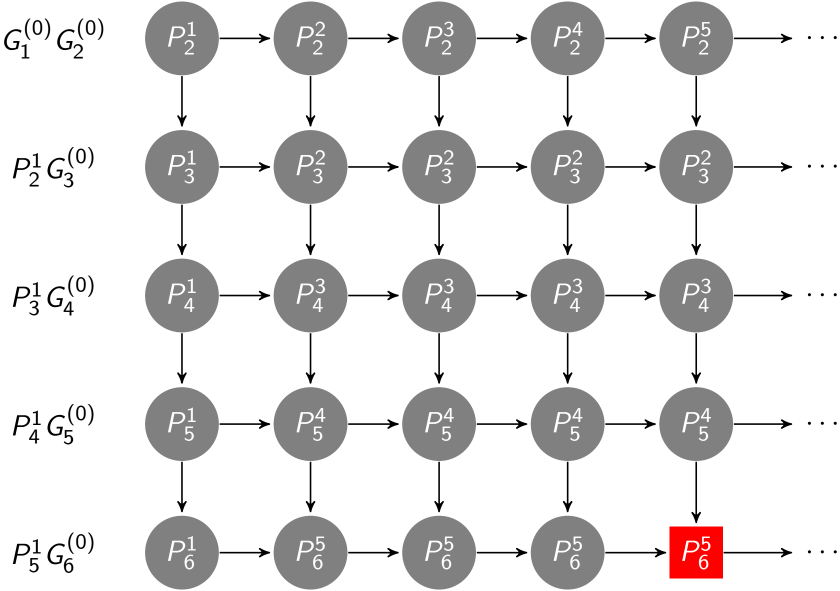
Computing $\Delta F^{(j)}(X, Y)$

$$
\begin{tikzcd}
G_1^{(0)} G_2^{(0)} & P_2^1 \arrow[r] \arrow[d] & P_2^2 \arrow[r] \arrow[d] & P_2^3 \arrow[r] \arrow[d] & P_2^4 \arrow[r] \arrow[d] & P_2^5 \arrow[r] \arrow[d] & \cdots \\
P_2^1 G_3^{(0)} & P_3^1 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & P_3^2 \arrow[r] \arrow[d] & \cdots \\
P_3^1 G_4^{(0)} & P_4^1 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & P_4^3 \arrow[r] \arrow[d] & \cdots \\
P_4^1 G_5^{(0)} & P_5^1 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & P_5^4 \arrow[r] \arrow[d] & \cdots \\
P_5^1 G_6^{(0)} & P_6^1 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & P_6^5 \arrow[r] & \cdots
\end{tikzcd}
$$

Computing $\Delta F^{(j)}(X, Y)$



Computing $\Delta F^{(j)}(X, Y)$



Complexity result:

Theorem 2:

he k-th iteration of Step 9 in the Algorithm 4 runs within

- $\mathcal{O}(k\, d\mathsf{M}(d))$ operations in $\mathbb{C}$,
- $\mathcal{O}(k\, d\mathsf{M}(d)^2)$ operations in the field of coefficients of $F(X, Y)$.

Comparative complexity results

Theorem 3:

Our enhancement of the EHC computes all the branches in $\mathcal{O}(k^2\, d\, \mathsf{M}(d))$ operations in $\mathbb{C}$, using a linear lifting scheme.

Kung-Traub, 1987

The first k iterations of Newton-Puiseux on an input bivariate polynomial of degree d computes all branches within

- $\mathcal{O}(d^2\, k\, \mathsf{M}(k))$ operations in $\mathbb{C}$ using a linear lifting scheme (Theorem 5.2 in their paper)
- $\mathcal{O}(d^2\, \mathsf{M}(k))$ operations in $\mathbb{C}$ using a quadratic lifting scheme (Corollary 5.1 in their paper)

D. V. Chudnovsky and G. V. Chudnovsky, 2015

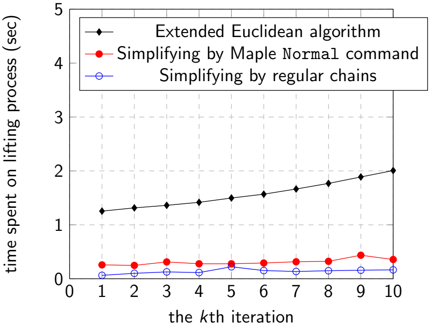
The latter estimate reported by Kung and Traub is improved to $\mathcal{O}(d^2\, k)$ operations in $\mathbb{C}$ for computing all the branches.

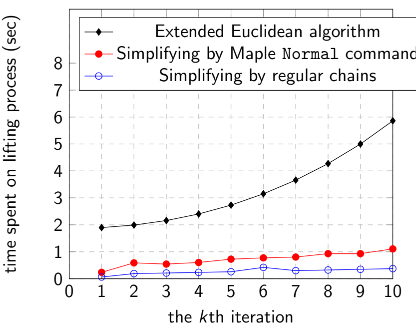
Remark

A quadratic lifting scheme for the EHC is work in progress.

Plan

Experimentation for lifting





The y axis is in square-root scale.

Experimentation for factoring (1/2)

Ex	MD	KT Lin		KT Quad		EHCWM			EHCEEA	
		KT10	KT20	KT10	KT20	EHC10	YM1	EHC20	EHC10	YM2
1	5	2.22	18.6	4.93	4.91	0.48	0.22	0.73	0.90	0.21
7	4	5.60	65.8	0.56	0.58	0.22	0.14	0.23	0.34	0.13
8	4	14.9	230	1.25	1.25	0.23	0.13	0.28	0.36	0.12
9	3	5.53	114	1.51	1.56	0.30	0.11	0.39	0.88	0.10
10	3	2.71	42.0	0.28	0.63	0.16	0.08	0.20	0.32	0.12
11	3	0.46	2.34	0.21	0.21	0.16	0.08	0.17	0.26	0.12
12	3	0.50	6.86	0.28	0.32	0.16	0.08	0.18	0.30	0.12
13	4	0.86	10.9	0.50	0.48	0.26	0.15	0.28	0.46	0.24
14	4	3.21	34.8	0.69	0.71	0.26	0.15	0.34	0.52	0.24
15	6	27.6	535	4.85	4.85	0.64	0.42	0.82	2.05	1.08
16	7	45.6	836	8.45	9.91	0.64	0.43	0.92	2.33	1.74

Table: Comparing EHC versus Kung-Traub's method

Experimentation for factoring (2/2)

Ex	MD	KT Lin		KT Quad		EHCWM			EHCEEA	
		KT10	KT20	KT10	KT20	EHC10	YM1	EHC20	EHC10	YM2
17	7	145	∞	23.4	23.2	0.78	0.43	3.37	4.12	1.77
19	4	0.14	0.16	0.16	0.14	0.39	0.26	0.45	0.51	0.15
20	4	2.79	7.98	0.77	0.82	0.26	0.15	0.29	0.50	0.24
21	4	8.58	143	1.96	1.93	0.23	0.12	0.31	0.47	0.16
24	5	2.90	24.8	1.11	1.11	0.26	0.15	0.35	0.49	0.17
25	7	1.83	9.45	0.90	1.00	0.46	0.31	0.50	0.73	0.42
26	8	2.35	12.3	3.09	3.29	0.66	0.53	0.74	2.18	1.80
27	8	60.8	2876	23.1	27.1	0.77	0.53	1.20	2.31	1.28
28	9	215	∞	73.8	123	1.88	1.03	2.11	7.03	4.92
30	17	∞	∞	∞	∞	39.8	6.70	41.3	53.8	16.5
31	32	∞	∞	∞	∞	599	24.9	∞	∞	∞
32	33	∞	∞	∞	∞	224	25.0	∞	∞	∞

Table: Comparing EHC versus Kung-Traub's method

Concluding remarks

- We have shown that the EHC requires only linear algebra and univariate polynomial arithmetic
- We have derive complexity estimates for the EHC which are comparable to those of Kung-Traub's method in linear lifting scheme
- Experimentally, this enhanced EHC is competitive and sometimes outperforms Kung-Traub's method in its linear and quadratic lifting schemes.
- A quadratic lifting scheme for the EHC is work in progress.
- Source code and experimental data are available in the `PowerSeries` library from `www.regularchains.org`.