Computation of Canonical Forms for Ternary Cubics

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ABSTRACT

In this paper we conduct a careful study of the equivalence classes of ternary cubics under general complex linear changes of variables. Our new results are based on the method of moving frames and involve triangular decompositions of algebraic varieties. We provide a computationally efficient algorithm that matches an arbitrary ternary cubic with its canonical form and explicitly computes a corresponding linear change of coordinates. We also describe a classification of the symmetry groups of ternary cubics.

Keywords. Method of moving frames, classical invariant theory, ternary cubics, elliptic curves, systems of algebraic equations, triangular decompositions.

1. INTRODUCTION

One of the central problems of classical invariant theory is the equivalence and symmetry of multivariable polynomials under linear changes of variables. The standard action of the general linear group on the m dimensional complex linear space $\mathbb{C}^m$ induces an action $F \to \bar{F}$ on the linear space of polynomials $\mathbb{C}[\mathbf{x}]$ defined by

$$\bar{F}(g \cdot \mathbf{x}) = F(\mathbf{x}), \tag{1}$$

where $g \in GL(m, \mathbb{C})$, $\mathbf{x} \in \mathbb{C}^m$ and $F, \bar{F} \in \mathbb{C}[\mathbf{x}]$ are polynomials in m variables.

The set of homogeneous polynomials of degree n is mapped to itself under the $GL(m, \mathbb{C})$ action defined by (1). Two polynomials are said to be equivalent if there is a linear change of variables that transforms one into the other. The group of symmetries of a polynomial consists of all linear changes of variables that leave the polynomial unchanged. It is desirable to describe the classes of equivalent polynomials, or, in other words, classify the orbits of the action of a general linear group on the linear space of polynomials.

In this paper we consider *ternary cubics*, that is, homogeneous polynomials in three variables of degree three. Their classification is well known, see for instance [22], [16]. In each orbit we identify a "simple" canonical form and provide an algorithm, that matches an arbitrary cubic with its canonical form. A corresponding linear change of variables is computed explicitly. Note that the most naive approach to the latter problem, solving (1) for the group parameters, requires very hard computations. Our algorithm is based on the novel differential geometry approach first introduced by Olver [28]. The solution of the equivalence problem over real linear transformations can be approached by similar techniques but is more subtle. We will indicate how to adapt our new results to the real case in a future paper.

Polynomials of degree n in m variables form a linear space W of dimension $\binom{m+n-1}{n}$ isomorphic to the n-th symmetric tensor product of $\mathbb{C}^m$. The coefficients of polynomials can serve as coordinates on this space and formula (1) induces an irreducible representation of $GL(m, \mathbb{C})$ on W. One can try to classify polynomials by computing continuous *invariants*, that is, functions $H(\dots, a_{i_1,\dots,i_m}, \dots)$ in the coefficients of a polynomial, invariant under the action (1). However, the action (1) is not regular: the dimensions of the orbits vary and all orbits, except the orbit of the zero polynomial, are not closed. Hence the orbits can not be distinguished by continuous invariants. More information can be obtained from *covariants*, $H(\mathbf{x}, \dots, a_{i_1,\dots,i_m}, \dots)$, continuous functions that are invariant under the simultaneous action of the general linear group on the variables and the coefficients of polynomials. The simplest example of a covariant is the polynomial $F(\mathbf{x})$ itself (see formula (1)). Computation of *rational* invariants and covariants, as well as polynomial relations, or *sysygies*, among them is the main subject of the nineteenth century classical theory. The overview of the classical methods for constructing rational covariants as well their application to the classification of polynomials can be found in [12], [16], [28]. A set of rational covariants is called *fundamental* if any other rational covariant can be expressed as a rational function of the fundamental ones. The existence of a finite fundamental set of covariants follows from the finite basis theorem for the actions of linear reductive groups proved by Hilbert in 1890 [1]. The number of fundamental covariants, however, grows dramatically, with the increase of the degree of polynomials even for a fixed number of variables. The Hilbert's proof of the finite basis theorem became a turning point from classical computational approach in the invariant theory to its modern formulation in terms of algebraic geometry [25], [22], [31] and representation theory [17].

Despite the enormous amount of results obtained by classical and modern methods, the general classification of homogeneous polynomials and their symmetry groups remains unknown even for the case of polynomials in two or three variables (over $\mathbb{C}$ or $\mathbb{R}$) except for polynomials of low degree.

The main idea of moving frame approach, appeared in [28], is to consider the graph of a polynomial $F(\mathbf{x})$ in m variables as a submanifold in $(m+1)$-dimensional complex (or real) space therefore reducing the question to equivalence problems for submanifolds. The latter problem can be resolved by Cartan's method of moving frames, which involves computation of a fundamental set of differential invariants. We note that, in contrast with the classical approach, a set of differential invariants is called *fundamental* if any other differential invariant can be at least locally expressed as a smooth function of the fundamental ones. Fundamental differential invariants parameterize the *signature manifold*, or the *classifying manifold*, of F [27]. Two equivalent polynomials have the same signature manifold. Moreover the signatures can be used to determine the symmetry groups (see Section 5) and to describe the geometry of the orbits for the $GL(3,\mathbb{C})$-action on the linear space of polynomials (see Section 4).

In the joint paper by Peter Olver and the first author [3] the moving frame method has been applied to polynomials in two variables (binary forms). A MAPLE package which determines the dimension of the symmetry group of a given polynomial and, in the case when the symmetry group is finite, computes it explicitly has been provided. Computing differential invariants becomes more challenging for the homogeneous polynomials in more than two variables, even in the next case of polynomials in three variables. This work has been completed by the first author in her thesis [20], using the inductive moving frame construction presented in [21]. These invariants were used to classify the symmetry groups for ternary cubics and to solve some equivalence problems. For instance, a necessary and sufficient condition for a homogeneous polynomial in three variables to be equivalent to $x^n + y^n + z^n$ was computed in [20] and [30]. The general problem of expressing a binary or higher order form as a sum of powers of linear forms has applications in signal processing and is discussed in [8], [29].

To the advantage of the moving frames approach, it is applicable to polynomials of any degree in any number of variables, as well as one can consider various group actions. Moreover, one can use the same invariants for polynomials of any degree, as long as the number of variables and the group action remain the same. In practice, however, we are confronted with challenging elimination problems for multivariable polynomials, not solvable by the standard algorithm based on Gröbner bases computations. The results for the case of ternary cubics, presented here, were obtained by using triangular decompositions. More precisely, the computations of the signature manifolds of the canonical ternary forms were obtained by means of the PARDI algorithm by Boulier, Lemaire and the second author [4]. The justification of these computations follows from the theory of regular chains introduced by Kalkbrener [18] and its recent developments ([2], [5]). The linear transformation from an arbitrary ternary cubic into its canonical form involves the triangular decomposition algorithm by the second author [24] and its implementation in ALDOR [6].

The paper is organized as follows.

Section 2 contains a formulation of the equivalence and symmetry problems for multivariable polynomials under linear changes of variables together with a description of the equivalence classes of ternary cubics under this action.

Section 3 contains a derivation of a moving frame and the corresponding differential invariants for the action considered in the previous section. These invariants are not rational. However we use them to build a new set of fundamental rational invariants which leads to exceptionally simple signature manifolds.

Section 4 describes the signature manifolds for each of the equivalence classes. We also make some observations about the geometry of the orbits.

Section 5 contains an algorithm for computing the symmetry groups listed in the previous section.

Section 6 provides an algorithm that matches an arbitrary ternary cubic with its canonical form and computes a corresponding linear change of coordinates.

2. EQUIVALENCE AND SYMMETRY

We consider the action of the general linear group on the space of polynomials over $\mathbb{C}$ in m variables:

$$\bar{F}(g \cdot \mathbf{x}) = F(\mathbf{x}), \text{ equivalently } \bar{F}(\mathbf{x}) = F(g^{-1}\mathbf{x}) \qquad (2)$$

where $g \in GL(m,\mathbb{C}),\ \mathbf{x} \in \mathbb{C}^m$ and $F, \bar{F} \in \mathbb{C}[\mathbf{x}],$.

DEFINITION 1. *A polynomial F is said to be equivalent to a polynomial $\bar{F}$ if there exists $g \in GL(m,\mathbb{C})$ such that $\bar{F}(g \cdot \mathbf{x}) = F(\mathbf{x})$. In this case we will also write $\bar{F} = g \cdot F$*

EXAMPLE 1. *The polynomial $x^3+y^3+z^3$ is equivalent to $x^3+6\,x\,y^2-6\,x\,y\,z-3\,y^2\,z+3\,y\,z^2$ under the transformation*

$$x \mapsto x-y; \quad y \mapsto x+y-z; \quad z \mapsto z-x.$$

DEFINITION 2. *The symmetry group of F is the subgroup $G_F \subset GL(m,\mathbb{C})$ consisting of all linear transformations that map F to itself.*

EXAMPLE 2. *The symmetry group of the cubic $z\,(x^2+y\,z)$ is a two-dimensional noncommutative (affine) group, generated by two linear transformations:*

$$\begin{pmatrix} 1 & 0 & \alpha \\ -2\alpha & 1 & -\alpha^2 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \beta & & \\ & \beta^4 & \\ & & \beta^{-2} \end{pmatrix}.$$

We note that the symmetry groups of equivalent polynomials are related by matrix conjugation:

$$\bar{F}(\mathbf{x}) = F(g \cdot \mathbf{x}) \Longrightarrow G_{\bar{F}} = g G_F g^{-1}. \qquad (3)$$

Thus the problem of the classification of polynomials is closely related to the problem of the classification of their symmetry groups up to matrix conjugation. In Section 5 we describe an algorithm (first appeared in [28] and used in [20]) to determine the dimension of the symmetry group of a given polynomials and, in the case when the cardinality of the symmetry group is finite, to compute its order.

We focus on homogeneous polynomials of degree three with three variables, also called ternary cubics. We reproduce below the known classification of ternary cubics up to a linear transformation ([16], [22]). We start with irreducible cubics. A non-singular projective variety V defined by an irreducible ternary cubic $F(x,y,z)$ over $\mathbb{C}$ is an *elliptic curve*. Elliptic curves are well studied and of great importance in number theory and the theory of modular forms ([19], [23]). The canonical form, also called Weierstrass normal form, can be obtained by transforming one of the inflection points of V to the infinite point $(0,1,0)$, and the tangent line at this point to the line $(k,1,0)$ at infinity [19].

THEOREM 1. *An irreducible cubic $F(x,y,z)$ can be transformed under a linear change of variables to one of the following forms:*

1 *If $F(x,y,z)$ defines a nonsingular projective variety then it either equivalent to:*

a) *a cubic in one-parametric family:*

$$x^3 + a\,x\,z^2 + z^3 - y^2\,z,$$

where $a \neq 0$ (otherwise it is an irreducible cubic of type c) below) and $a^3 \neq -27/4$ (otherwise it is equivalent to the reducible cubic of type b) below).
or either

$$\textbf{b)}\quad x^3 + x\,z^2 - y^2\,z \quad \textit{or} \quad \textbf{c)}\quad x^3 + z^3 - y^2\,z.$$

2. *If $F(x,y,z)$ defines a singular projective variety then it is equivalent to either*

$$\textbf{a)}\quad x^3 - y^2\,z \quad \textit{or} \quad \textbf{b)}\quad x^2(x+z) - y^2\,z.$$

THEOREM 2. *A reducible cubic $F(x,y,z)$ is equivalent under a linear change of variables to one of the following forms:*

1. *If it is a product of quadratic and linear factors then it is equivalent to either*

$$\textbf{a)}\quad z\,(x^2 + yz) \qquad \textit{or} \qquad \textbf{b)}\quad z\,(x^2 + y^2 + z^2).$$

2. *If it is a product of three linear factors and*

a) *three factors are linearly independent factors, then the cubic is equivalent to xyz.*

b) *three factors are linearly dependent, but any pair of them is linearly independent, then the cubic is equivalent to $xy(x+y)$;*

c) *two factors are the same, then the cubic is equivalent to x^2y;*

d) *all three factors are the same, then it is equivalent to x^3;*

The classification of reducible cubics can be obtained by very elementary methods [20].

In the next two sections, we address the problem of matching a cubic F with one of the canonical forms listed above. Once the canonical form $\bar{F}$ is identified, we would like to determine a linear transformation such that $\bar{F} = F(g \cdot \mathbf{x})$. The most straightforward approach, solving the equation

$$\bar{F}(x,y,z) = F\left(\alpha x + \beta y + \lambda z\,, \gamma x + \delta y + \mu z\,, ax + by + \eta z\right).$$

for the group parameters, turns out to be computationally impractical. We propose a practical solution in Section 6.

3. DIFFERENTIAL INVARIANTS AND SIGNATURE MANIFOLDS

The classification algorithms presented in the next sections are based on the moving frame method first developed by E. Cartan [7] for the solution of equivalence and symmetry problems in differential geometry. We refer the reader to the above reference as well as to [15] for the classical formulation of the method, and to [11], [14], [13] for its modern algorithmic formulation. In this section we implement the inductive variation [21] of the moving frame algorithm, presented in [11], to compute a fundamental set of differential invariants for ternary cubics.

Following the ideas of [28], we consider the graph of a homogeneous cubic polynomial $u = F(x,y,z)$ as a smooth three-dimensional submanifold in $\mathbb{C}^4$. Coordinate functions in $\mathbb{C}^4$ are chosen to be (x,y,z,u). A group element

$$g = \begin{pmatrix} \alpha & \beta & \lambda \\ \gamma & \delta & \mu \\ a & b & \eta \end{pmatrix} \in GL(3,\mathbb{C}) \tag{4}$$

acts on $\mathbb{C}^4$ by leaving coordinate u unchanged and transforming linearly x,y,z:

$$u \to u,$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} \alpha & \beta & \lambda \\ \gamma & \delta & \mu \\ a & b & \eta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5}$$

The graph Γ_F of $u = F(x,y,z)$ is transformed under this action to the graph $g \cdot \Gamma_F$ of

$$u = F\left(\tilde{\alpha}\,x + \tilde{\beta}\,y + \tilde{\lambda}\,z\,, \tilde{\gamma}\,x + \tilde{\delta}\,y + \tilde{\mu}\,z\,, \tilde{a}\,x + \tilde{b}\,y + \tilde{\eta}\,z\right), \tag{6}$$

where $\tilde{\alpha}, \ldots, \tilde{\eta}$ denote the entries of the matrix g^{-1}. In order to reduce the dimension of the jet spaces which will be introduced below, we prefer to work with inhomogeneous (projective) version of F, the unique polynomial in two variables p,q defined by:

$$f(p,q) = F(p,q,1). \tag{7}$$

For example, the polynomial $f(p,q) = pq$ is the inhomogeneous version of the cubic $F(x,y,z) = xyz$. We note that the degree of f is less than or equal to three, but we keep referring to f as a cubic, since the knowledge of the initial degree is necessary to restore F from f:

$$F(x,y,z) = z^3\,f\left(\frac{x}{z}, \frac{y}{z}\right).$$

We consider the graph Γ_f of the polynomial $u = f(p,q)$ as a surface in the three dimensional complex space. From (6) and (7) we deduce that Γ_f is transformed to the graph of:

$$u = \left(\tilde{a}\,p + \tilde{b}\,q + \tilde{\eta}\right)^3 f\left(\frac{\tilde{\alpha}\,p + \tilde{\beta}\,q + \tilde{\lambda}}{\tilde{a}\,p + \tilde{b}\,q + \tilde{\eta}}, \frac{\tilde{\gamma}\,p + \tilde{\delta}\,q + \tilde{\mu}}{\tilde{a}\,p + \tilde{b}\,q + \tilde{\eta}}\right). \tag{8}$$

This corresponds to the following local action of $GL(3,\mathbb{C})$ on $\mathbb{C}^3$, defined at each point for the group elements that do not allow the denominator vanish.

$$\begin{aligned} u &\rightarrow \bar{u} = (a\,p + b\,q + \eta)^{-3}\,u, \\ p &\rightarrow \bar{p} = \frac{\alpha\,p + \beta\,q + \lambda}{a\,p + b\,q + \eta}, \\ q &\rightarrow \bar{q} = \frac{\gamma\,p + \delta\,q + \mu}{a\,p + b\,q + \eta}. \end{aligned} \tag{9}$$

The application of the moving frame method is justified by the following simple proposition:

PROPOSITION 1. *Two homogeneous cubics $F(x,y,z)$ and $\bar{F}(x,y,z)$ are equivalent under a linear change of variables (see Definition 1) if and only if the graph $u = F(x,y,z)$ in $\mathbb{C}^4$ can be mapped to the graph $u = \bar{F}(x,y,z)$ under the transformation (5) or, equivalently, the graph $u = f(p,q)$ in $\mathbb{C}^3$ of the inhomogeneous version of F can be mapped to $u = \bar{f}(p,q)$ by (9).*

The first step is to prolong the graph $u = f(p,q)$ and the group action to the jet space of smooth functions in two variables. In our case the n-th jet space J^n is a (n^2+3n+6)-dimensional complex space parameterized by coordinates $(p, q, u, u_{1,0}, u_{0,1}, \ldots, u_{i,j}, \ldots u_{0,n})$, where $i + j \leq n$ and $u_{i,j}$ are formal coordinates that correspond to all possible partial derivatives of u with respect to p and q. Given a particular polynomial $f(p,q)$, one can actually compute these derivatives in order to obtain a two-dimensional surface in J^n:

$$u = f(p,q), \quad u_{i,j} = \frac{\partial^{i+j} u}{\partial^i p \partial^j q}, \tag{10}$$

which is called n-th prolongation of the graph $u = f(p,q)$ and is denoted by $\Gamma_f^{(3)}$. We note that since f is a polynomial of degree 3, all the derivatives of order higher than three are zero, and thus one can solve the equivalence problem on the third order jet space J^3 of dimension 12.

REMARK 1. *Not every two-dimensional surface in J^3 corresponds to a prolongation of a cubic polynomial f. In fact there is one and only one such graph passing through every point of J^3. Remarkably, when polynomials of an arbitrary degree n are considered, one does not need to prolong the graph up to the order n to solve the equivalence and symmetry problems, but at most up to the order 6. Although there are several prolonged graphs of polynomials of degree > 6 passing through a point in J^6, there is at most one polynomial from each equivalence class passing through each point. That is, if $\Gamma_f^{(6)}$ and $\Gamma_{g\cdot f}^{(6)}$ are passing through the same point in J^6, then $f = g \cdot f$ (see rigidity theorems [11] Sect. 14).*

The next step is to prolong the action (9) to the jet space J^3, that is, to define a transformation for the coordinates $u_{i,j}$, in a way that it commutes with the process of the prolongation of the graph (10). According to [11], [26] Chapter 2 this can be done in the unique way:

$$u_{i,j} \rightarrow D_p^i\, D_q^j(\bar{u}),$$

where D_p and D_q are differential operators defined by:

$$D_p = C\left((p\,A + \delta\,\eta - b\,\mu)\,\frac{d}{dp} + (q\,A + a\,\mu - \gamma\eta)\,\frac{d}{dq}\right),$$

$$D_q = C\left((p\,B + \beta\,\eta - b\,\lambda)\,\frac{d}{dp} + (q\,B + a\,\lambda - \alpha\eta)\,\frac{d}{dq}\right),$$

where $C = \frac{a\,p + b\,q + \eta}{\det(g)}$, $A = \delta\,a - \gamma\,b$ and $B = \beta\,a - \alpha\,b$.

For example, $u_{1,0}$ is transformed to

$$D_p(\bar{u}) = \frac{-3A\,u + (p\,A + \delta\,\eta - b\,\mu)\,u_{1,0} + (q\,A + a\,\mu - \gamma\eta)\,u_{0,1}}{(a\,p + b\,q + \eta)^2 \det(g)}.$$

To avoid long and, in fact, unnecessary formulae we will denote the rest of the transformations symbolically:

$$u_{i,j} \rightarrow \bar{u}_{i,j} = \Phi_{i,j}(\mathbf{u}^{(3)}, g), \tag{11}$$

where $0 \leq i + j \leq 3$ and $\Phi_{i,j}$ are smooth functions of the groups parameters (4) and coordinates of J^3.

An important observation is that the action of the group $G = GL(3,\mathbb{C})$ is *locally free* on J^3, which means that the stabilizers of each points are discrete. In fact, the isotropy group of a generic point consists of three matrices in the complex case and of identity matrix only in the real case. It follows that each orbit is locally diffeomorphic to G and thus has dimension equal to $\dim G = 9$. One can find a *cross-section* $\mathcal{K}$ to the orbits, that is, a submanifold in J^3 of dimension $3 = \dim J^3 - \dim G$, that is, transversal to the orbits and intersects each orbit in an open subset of J^3 once. It turns out that the submanifold

$$\begin{aligned} \mathcal{K} = \{\mathbf{u}^{(3)} \in J^3 | p = 0, q = 0, u = 1, \\ u_{1,0} = 0, u_{0,1} = 0, u_{2,0} = 0, u_{0,2} = 0, u_{3,0} = 1, u_{0,3} = 1\} \end{aligned} \tag{12}$$

can serve this purpose on a dense subset of J^3. Computationally it is reflected in the existence of a finite set of the solutions of the following system (see (9,11)):

$$\begin{aligned} \bar{p} = 0,\ \bar{q} = 0,\ \bar{u} = 1,\ \Phi_{1,0} = 0,\ \Phi_{0,1} = 0, \\ \Phi_{2,0} = 0,\ \Phi_{0,2} = 0,\ \Phi_{3,0} = 1,\ \Phi_{0,3} = 1. \end{aligned} \tag{13}$$

for the group parameters at almost every point of J^3. A solution for this system yields a matrix $g(\mathbf{u}^{(3)})$ which maps a point $\mathbf{u}^{(3)} \in J^3$ to $\mathcal{K}$ under the prolonged transformation (9, 11). The matrix-valued function $g(\mathbf{u}^{(3)})$, defined at almost each point of J^3, is called a *moving frame* [11]. By construction the map $\pi(\mathbf{u}^{(3)}) = g(\mathbf{u}^{(3)}) \cdot \mathbf{u}^{(3)}$ defines a projection π of J^3 onto $\mathcal{K}$ along the orbits. In particular, one can restrict π to the prolonged graph of $u = f(p,q)$ defined by (10), to obtain a projection $\pi_f \colon \Gamma_f \rightarrow \mathcal{K}$. As the following two theorems indicate the image of Γ_f in $\mathcal{K}$, called the *signature manifold* of f, provides the key to the solution of the symmetry and equivalence problems.

DEFINITION 3. *The signature manifold $\mathcal{C}_f$ of a cubic polynomial $f(p,q)$ is the projection along the orbits of the third prolongation (10) of its graph $\Gamma_f^{(3)}$ to the cross-section $\mathcal{K}$.*

THEOREM 3. *Two cubics $f(p,q)$ and $\bar{f}(p,q)$ are equivalent under a linear change of variables if and only if their signature manifolds coincide, $\mathcal{C}_f = \mathcal{C}_{\bar{f}}$.*

For the absolute majority of cubic polynomials the projections of their two-dimensional graphs to $\mathcal{K}$ remains two-dimensional, but some of the cubics, whose graphs are not transversal to the orbits, will have the projections (signatures) of a smaller dimension. The drop in the dimension of the signature manifold reflects the increase in the dimension of the symmetry group of f.

THEOREM 4. *The dimension of the symmetry group G_f of $f(p,q)$ equals to $\dim \Gamma_f - \dim \mathcal{C}_f = 2 - \dim \mathcal{C}_f$.*

See [11] (Theorems 14.7 and 14.9) and [27] (Theorems 8.19 and 8.22) for the proofs of Theorems 3 and 4 above. The computation of the signatures relies on the solution of the system (13) for the group parameters. However, the formulae for $\Phi_{i,j}$ in the second or third order are too long and frightening to approach. We can, however, split the problem into three feasible sub-problems, by decomposing the matrix of g into a product of three matrices $g = HRT$, where

$$H = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}, R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$

We apply the inductive construction of moving frames [21], [20] to this product and refer the reader to [20] for explicit computations leading to the following results. Formulae

$$t = -p,\ s = -q,\ c = u^{1/3},\ a = \frac{u_{1,0}}{3\,u^{\frac{2}{3}}},\ b = \frac{u_{0,1}}{3\,u^{\frac{2}{3}}} \tag{14}$$

determine the matrices R and T. Solution for the parameters of the matrix H is more complicated. We first introduce the following functions on J^3:

$$\begin{aligned}
Q_{2,0} &= u^{-\frac{4}{3}}\left[u_{2,0}\,u - \frac{2}{3}u_{1,0}^2\right],\\
Q_{1,1} &= u^{-\frac{4}{3}}\left[u_{1,1}\,u - \frac{2}{3}u_{1,0}u_{0,1}\right],\\
Q_{0,2} &= u^{-\frac{4}{3}}\left[u_{0,2}\,u - \frac{2}{3}u_{0,1}^2\right],\\
Q_{3,0} &= \frac{1}{u^2}\left[u_{3,0}\,u^2 - u_{2,0}u_{1,0}\,u + \frac{4}{9}u_{1,0}^3\right],\\
Q_{2,1} &= \frac{1}{u^2}\left[u_{2,1}\,u^2 + \frac{4}{9}u_{1,0}^2u_{0,1}\right.\\
&\qquad \left. - \frac{1}{3}(u_{2,0}u_{0,1} + 2\,u_{1,1}u_{1,0})\,u\right],\\
Q_{1,2} &= \frac{1}{u^2}\left[u_{1,2}\,u^2 + \frac{4}{9}u_{0,1}^2u_{1,0}\right.\\
&\qquad \left. - \frac{1}{3}(u_{0,2}u_{1,0} + 2\,u_{1,1}u_{0,1})u\right],\\
Q_{0,3} &= \frac{1}{u^2}\left[u_{0,3}\,u^2 - u_{0,2}u_{0,1}\,u + \frac{4}{9}u_{0,1}^3\right],
\end{aligned}$$

which are used to build the following expressions:

$$\begin{aligned}
\bar{Q}_{2,0} &= \frac{1}{A^2}\left(\delta^2 Q_{2,0} - 2\gamma\delta Q_{1,1} + \gamma^2 Q_{0,2}\right),\\
\bar{Q}_{1,1} &= \frac{1}{A^2}\left(-\delta\beta Q_{2,0} + (\gamma\beta + \alpha\delta)Q_{1,1} - \alpha\gamma Q_{0,2}\right),\\
\bar{Q}_{0,2} &= \frac{1}{A^2}\left(\beta^2 Q_{2,0} - 2\alpha\beta Q_{1,1} + \alpha^2 Q_{0,2}\right),\\
\bar{Q}_{3,0} &= \frac{1}{A^3}\left(\delta^3 Q_{3,0} - 3\gamma\delta^2 Q_{2,1} + 3\gamma^2\delta Q_{1,2} - \gamma^3 Q_{0,3}\right),
\end{aligned}$$

$$\begin{aligned}
\bar{Q}_{2,1} &= \frac{1}{A^3}\left(-\delta^2\beta Q_{3,0} + \delta(2\gamma\beta + \alpha\delta)Q_{2,1}\right.\\
&\qquad \left. - \gamma(\gamma\beta + 2\alpha\delta)Q_{1,2} + \alpha\gamma^2 Q_{0,3}\right)\\
\bar{Q}_{1,2} &= \frac{1}{A^3}\left(\delta\beta^2 Q_{3,0} - \beta(\gamma\beta + 2\alpha\delta)Q_{2,1}\right.\\
&\qquad \left. + \alpha(2\gamma\beta + \alpha\delta)Q_{1,2} - \alpha^2\gamma Q_{0,3}\right),\\
\bar{Q}_{0,3} &= \frac{1}{A^3}\left(-\beta^3 Q_{3,0} + 3\alpha\beta^2 Q_{2,1}\right.\\
&\qquad \left. - 3\alpha^2\beta Q_{1,2} + \alpha^3 Q_{0,3}\right),
\end{aligned}$$

where, as above, $A = \delta\alpha - \gamma\beta$. Equations

$$\Phi_{2,0} = 0,\ \Phi_{0,2} = 0,\ \Phi_{3,0} = 1,\ \Phi_{0,3} = 1.$$

are equivalent to

$$\bar{Q}_{2,0} = 0,\ \bar{Q}_{0,2} = 0,\ \bar{Q}_{3,0} = 1,\ \bar{Q}_{0,3} = 1. \tag{15}$$

From the first pair of equations (15) the ratios $\frac{\delta}{\gamma}$ and $\frac{\beta}{\alpha}$ are two roots of the same quadratic equation so we have:

$$\frac{\delta}{\gamma} = r_1 = \frac{Q_{1,1} + \sqrt{d}}{Q_{2,0}}, \qquad \frac{\beta}{\alpha} = r_2 = \frac{Q_{1,1} - \sqrt{d}}{Q_{2,0}}, \tag{16}$$

where $d = Q_{1,1}^2 - Q_{2,0}\,Q_{0,2}$.

By subtracting these expressions one obtains that

$$r_1 - r_2 = \frac{\alpha\delta - \beta\gamma}{\alpha\gamma} = \frac{2\sqrt{d}}{Q_{2,0}} \Longrightarrow A = \alpha\gamma\,\frac{2\sqrt{d}}{Q_{2,0}}.$$

From the second pair of equations (15) we obtain that:

$$\begin{aligned}
\alpha &= C_\alpha\left(r_1^3 Q_{3,0} - 3r_1^2 Q_{2,1} + 3r_1 Q_{1,2} - Q_{0,3}\right)^{1/3}, \quad (17)\\
\gamma &= C_\gamma\left(-r_2^3 Q_{3,0} + 3r_2^2 Q_{2,1} - 3r_2 Q_{1,2} + Q_{0,3}\right)^{1/3}.
\end{aligned}$$

where $C_\alpha = Q_{2,0}/d'$, and $C_\gamma = Q_{2,0}/d'$ and $d' = 2\sqrt{d}$. Since $\beta = \alpha\, r_2$, $\delta = \gamma\, r_1$ then:

$$\begin{aligned}
\beta &= C_\beta\left(r_1^3 Q_{3,0} - 3r_1^2 Q_{2,1} + 3r_1 Q_{1,2} - Q_{0,3}\right)^{1/3}, \quad (18)\\
\delta &= C_\delta\left(-r_2^3 Q_{3,0} + 3r_2^2 Q_{2,1} - 3r_2 Q_{1,2} + Q_{0,3}\right)^{1/3}.
\end{aligned}$$

where $C_\beta = (Q_{1,1} - \sqrt{d})/d'$ and $C_\delta = (Q_{1,1} + \sqrt{d})/d'$. The substitution of the group parameters (17, 18) into $\bar{Q}_{1,1}$, $\bar{Q}_{2,1}$ and $\bar{Q}_{1,2}$ produces three *fundamental differential invariants* $I_{1,1}, I_{2,1}, I_{1,2}$ of the action (9). This means that these functions, defined on J^3, are unchanged under prolonged action (9, 11) and, moreover, any other invariant function can be locally expressed in terms of $I_{1,1}, I_{2,1}, I_{1,2}$. Equivalently, $I_{1,1}, I_{2,1}, I_{1,2}$ can serve as local coordinate functions on J^3, which are constant along the orbits. Note that their restrictions to $\mathcal{K}$ are just the standard coordinate functions: $I_{1,1}|_{\mathcal{K}} = u_{1,1}$, $I_{2,1}|_{\mathcal{K}} = u_{2,1}$, $I_{1,2}|_{\mathcal{K}} = u_{1,2}$. Evaluation of functions $I_{1,1}, I_{2,1}, I_{1,2}$ at a point $\mathbf{u}^{(3)} \in J^3$ produces the image under the projection π of $\mathbf{u}^{(3)}$ to $\mathcal{K}$ along its orbit. Thus one can compute the signature manifold of a cubic $f(p,q)$ by evaluation $I_{1,1}, I_{2,1}, I_{1,2}$ at f, that is, by substitution of $u_{i,j}$ with the actual derivatives $\frac{\partial^{i+j} u}{\partial^i p \partial^j q}$. As the result we obtain three functions in p and q which define the signature manifold $\mathcal{C}_f$ parametrically. Since two different parameterizations can define the same manifold, in order to compare the signatures of two different cubics f and $\bar{f}$, we need to eliminate p and q and compare the corresponding implicit equations $\Psi_1\left(I_{1,1}|_f, I_{2,1}|_f, I_{1,2}|_f\right)=0$ and $\Psi_2\left(I_{1,1}|_{\bar{f}}, I_{2,1}|_{\bar{f}}, I_{1,2}|_{\bar{f}}\right)=0$ (see Remark 2 below). The difficulty is that the formulae for $I_{1,1}, I_{2,1}, I_{1,2}$ involve radicals and thus are unsuitable for polynomial elimination algorithms. They can be used, however, to build three independent rational invariants that can serve the same purpose. The most immediate choice consists of $I_{1,1}^3$, $I_{1,2}I_{2,1}$ and $I_{2,1}^3 + I_{1,2}^3$. Unfortunately these invariants have quite complicated formulae in terms of the coordinates on J^3. By applying some ideas of the classical invariant theory we have obtained simpler invariants

$$\begin{aligned} I_1 &= \frac{I_{2,1}\, I_{1,2} - 1}{{I_{1,1}}^3}, \qquad I_2 = -4\,\frac{1 + 9\, I_{2,1}\, I_{1,2}}{{I_{1,1}}^3}, \\ I_3 &= \frac{1 - 3\, {I_{2,1}}^2\, {I_{1,2}}^2 - 6\, I_{2,1}\, I_{1,2} + 4\, {I_{1,2}}^3 + 4\, {I_{2,1}}^3}{{I_{1,1}}^6}. \end{aligned}$$

It follows from Remark 3 below that $I_{1,1}|_{\bar{f}}$ is zero if and only if the ternary form f can be transformed into a binary form. From now on let us assume that $I_{1,1}|_{\bar{f}} \neq 0$ holds. The restriction of I_1, I_2, I_3 to f gives a parametric description of its signature manifold $\mathcal{C}_f$. The corresponding implicit equations can be found as follows. By clearing denominators, the relations defining $I_1|_f$, $I_2|_f$, $I_3|_f$ leads to a set Σ of 3 polynomials $D_k\, I_k|_f - N_k$ (for $k = 1 \cdots 3$) in the polynomial ring $\mathbb{C}[I_1|_f, I_2|_f, I_3|_f, p, q]$ where D_k, N_k are polynomials in $\mathbb{C}[p,q]$ and such that $D_k \neq 0$ holds. Clearly Σ is a regular chain (see Section 2 in [5]) in $\mathbb{C}[I_1|_f, I_2|_f, I_3|_f, p, q]$ w.r.t. the variable ordering $I_1|_f > I_2|_f > I_3|_f > p > q$. Let $\mathcal{J}$ be its saturated ideal. Observe that the signature manifold $\mathcal{C}_f$ is the set of regular zeros of Σ over $\mathbb{C}$. Observe also that the tower of simple extensions associated with Σ is a field. (See [2] for these notions related to regular chains). It follows that $\mathcal{J}$ is prime and that Σ is a characteristic set of $\mathcal{J}$ w.r.t $I_1|_f > I_2|_f > I_3|_f > p > q$. Let T be the regular chain which is a characteristic set of $\mathcal{J}$ w.r.t $p > q > I_1|_f > I_2|_f > I_3|_f$ computed by means of the PARDI algorithm [4]. Let R be the regular chain $T \cap \mathbb{C}[I_1|_f, I_2|_f, I_3|_f]$. Relation (2) in [24] shows that the saturated ideal of R is the elimination ideal $\mathcal{J} \cap \mathbb{C}[I_1|_f, I_2|_f, I_3|_f]$. Observe that $R \neq \emptyset$ holds. Finally, Th .2 p. 130 in [9] shows that the variety associated with $\mathcal{J} \cap \mathbb{C}[I_1|_f, I_2|_f, I_3|_f]$ is the smallest variety containing $\mathcal{C}_f$.

Remark 2. *Let us call the smallest variety containing $\mathcal{C}_f$ the* signature variety *V_f of f. The signature manifold $\mathcal{C}_f$ may not fill the entire signature variety V_f neither in complex nor and real case. Over $\mathbb{C}$, however, there exist varieties $U_1, \ldots, U_\ell$ with dimension smaller than that of V_f such that $V_f = \mathcal{C}_f \cup U_1 \cup \cdots \cup U_\ell$. See for instance [2]. It follows that distinct signature varieties correspond to distinct signature manifolds. This is not the case over the real numbers (see Example 8.69 [28]). Since we address here the problem over the complex numbers, we will make no distinction between the signature manifold and the signature variety.*

Using the Aldor [6] implementation (by the second author) of the PARDI algorithm we found the signature manifolds for all canonical cubics in terms of the above invariants. These computations suggested a new set of invariants:

$$\begin{aligned} i_1 &= 10\,(6\, I_1 + 1), \\ i_2 &= 6\, I_2 + 126\, I_1 + 45\, I_3 - 10, \\ i_3 &= 10\,(9\, I_3 + 2). \end{aligned} \tag{19}$$

leading to exceptionally simple results listed in Section 4.

Remark 3. *None of the invariants is defined when $I_{1,1} \equiv 0$ (or equivalently $d \equiv 0$). This happens if and only if the inhomogenization of the Hessian:*

$$3\, f(f_{pq}^2 - f_{pp}\, f_{qq}) + 2\,(f_{pp}\, f_q^2 + 2\, f_p\, f_q\, f_{pq} + f_{qq}\, f_p^2)$$

is identically zero, and hence if and only if the ternary form can be transformed into a binary form (see [28] p234 for the remarks's on Hesse's "theorem").

4. CANONICAL FORMS

In order to match a cubic with its class we need to compute the signatures for each of the canonical forms listed in Theorems 1 and 2.

Example 3. *To compute the signature for $F = z(x^2 + y^2 + z^2)$ we first write its inhomogeneous version $f = p^2 + q^2 + 1$ and restrict invariants i_1, i_2 and i_3 to f. We obtain:*

$$\begin{aligned} i_1|_f &= 90\,\frac{(p^2+q^2+1)^2}{(p^2-3+q^2)^2}, \\ i_2|_f &= 270\,\frac{(p^2+q^2+1)^3}{(p^2-3+q^2)^3}, \\ i_3|_f &= 180\,\frac{(p^2+q^2+1)\,\left((p^2+q^2+3)^2 - 12\right)}{(p^2-3+q^2)^3} \end{aligned}$$

Elimination of p and q leads to the defining equations for the one-dimensional signature manifold:

$$i_1\,(i_3 - i_2) + 30\, i_2 = 0, \quad 10\, i_2^2 - i_1^3 = 0.$$

A cubic $\bar{F}$ can be transformed to F by a complex linear change of variables if and only if $i_1|_{\bar{f}}, i_2|_{\bar{f}}, i_3|_{\bar{f}}$, define a one-dimensional variety and satisfy the above equations.

We list below the signature manifolds for the canonical cubics. In order to match an arbitrary cubic f with its canonical form we need to restrict the three invariants i_1, i_2 and i_3 to f. In order do determine its class, one does not need to compute the signature of f itself, but just to check which of the following relations are satisfied. The latter computations are not difficult and were programed in Maple.

We recall also that the signature of f is the projection along the orbits of its prolonged graph to the cross-section $\mathcal{K}$. Since the graph of f is two-dimensional the absolute majority of cubics have two-dimensional signatures. The graphs of some cubics, however, are not transversal to the orbits, and thus their signatures are one- or zero-dimensional. According to Theorem 4 this indicates the increase in the dimension of the symmetry groups G_F. We will list the symmetry groups below, postponing details until Section 5.

Theorem 5. *Every homogeneous cubic $F(x, y, z)$ can be transformed by a linear change of variables to one and only one of the canonical forms listed below. In order to determine its canonical form, one need to compute $i_1|_f, i_2|_f, i_3|_f$ and to determine which of the relations listed below they satisfy. The symmetry group of F is isomorphic to the symmetry group of its canonical form under conjugation (3).*

Irreducible cubics.

- Regular (Elliptic Curves):

  (1) $F \sim x^3 + axz^2 + z^3 - y^2z, \quad f \sim p^3 + a\,p + 1 - q^2$, non-equivalent for different values of a^3; $a \neq 0$ (else $F \sim (3)$), $a^3 \neq -27/4$ (else $F \sim (5)$), $|G_F| = 18 \times 3$, $\boxed{675\, i_1^3 + (10\, a)^3\, i_2^2 = 0.}$

  (2) $F \sim x^3 + xz^2 - y^2z, \quad f \sim p^3 + p - q^2$, $|G_F| = 36 \times 3$, $\boxed{i_2 = 0.}$

  (3) $F \sim x^3 + z^3 - y^2z, \quad f \sim p^3 + 1 - q^2$, $|G_F| = 54 \times 3$, $\boxed{i_1 = 0.}$

- Singular:

  (4) $F \sim x^3 - y^2z, \quad f \sim p^3 - q^2$, $G_F \sim$ 1-dimensional: $x \to x$, $y \to \alpha y$, $z \to \alpha^{-2}z$, $\boxed{i_1 = 0, \quad i_2 = 0.}$

  (5) $F \sim x^2(x + z) - y^2z, \quad f \sim p^2(p + 1) - q^2$ $|G_F| = 6 \times 3$, $\boxed{i_1^3 - 10\, i_2^2 = 0.}$

Reducible cubics:

- a linear and an irreducible quadratic factor:

  (6) $F \sim z(x^2 + yz), \quad f \sim (p^2 + q)$ $G_F \sim$ 2-dimensional noncommutative (affine) group, generated by: $x \to x + \alpha z$, $y \to -2\alpha x + y - \alpha^2 z$, $z \to z$, $x \to \beta x$, $y \to \beta^4 y$, $z \to \beta^{-2} z$, $\boxed{i_1 = 0,\ i_2 = 0,\ i_3 = 0.}$

  (7) $F \sim z(x^2 + y^2 + z^2), \quad f \sim p^2 + q^2 + 1$ $G_F \sim$ 1-dimensional (rotation in the xy) plane, $\boxed{i_1(i_3 - i_2) + 30\, i_2 = 0,\ 10\, i_2^2 - i_1^3 = 0.}$

- three linear factors:

  (8) non-coplaner $\Longleftrightarrow$ $F \sim xyz, \quad f \sim p\,q$; $G_F \sim$ 2-parameter: $x \to \alpha x$, $y \to \beta y$, $z \to \frac{1}{\alpha\beta} z$. $\boxed{i_1 = 90,\ i_2 = 270,\ i_3 = 180.}$

  (9) different, but coplaner $\Leftrightarrow$ $F \sim x\,y\,(x + y), \quad f \sim p\,q(p + q)$ $G_F \sim$ 3-dimensional direct product of arbitrary linear transformations $z \mapsto \alpha x + \beta y + \gamma z$ and a finite subgroup of order 18 of linear transformations on the (x, y)-plane, which preserve $xy(x+y)$.

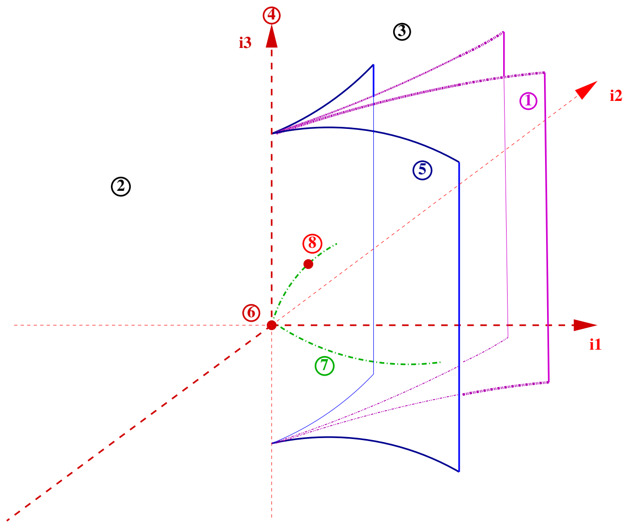
  (10) two repeated $\Leftrightarrow$ $F \sim x^2\, y, \quad f \sim p^2 q$ $G_F \sim$ 4-dimensional group of matrices generated by the transformations $x \to \alpha\, x, \quad y \to \frac{1}{\alpha^2} y, \quad z \to \beta\, x + \gamma\, y + \delta\, z.$

  (11) three repeated $\Leftrightarrow$ $F \sim x^3, \quad f \sim p^3.$ $G_F \sim$ 4-dimensional direct product of $GL(2, \mathbb{C})$ and cyclic group Z_3 (linear transformations on the variables y and z and multiplication of x by a cubic root of 1).

Remark 4. *The first class is actually a one-parametric family of equivalence classes. The classes of $x^3 + a_1xz^2 + z^3 - y^2z$ and $x^3 + a_2xz^2 + z^3 - y^2z$ are different unless $a_1^3 = a_2^3$. In the latter case the transformation $x \to \omega x$, where $w = a_1/a_2$ is a cubic root of 1, maps the first cubic to the second one.*

Remark 5. *The last three classes (9), (10) and (11), in fact, depend on less than three variables. Invariants i_1, i_2, i_3 are not defined for such polynomials, neither the procedures for computing symmetry groups and linear transformations described in the next two section. This case should be studied with the invariants applicable for binary forms (see [28], [3]).*

The picture below shows the signatures of canonical cubics in the three-dimensional space parameterized by i_1, i_2, i_3.



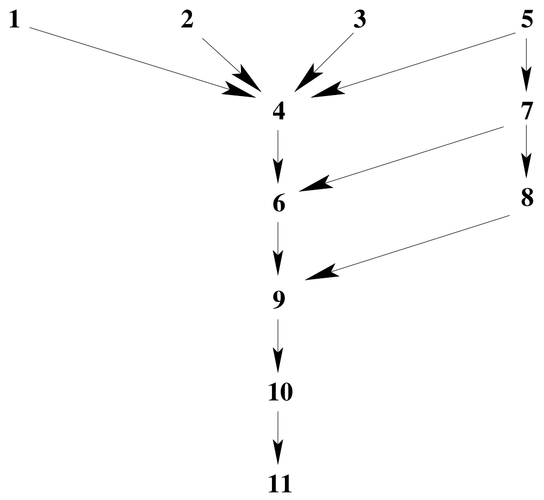
We notice that all low dimensional signatures are contained in some signatures of the higher dimension. This have a nice geometrical interpretation. As it has been mentioned in the introduction, the set of ternary homogeneous cubics form a ten-dimensional linear space W isomorphic to the symmetric tensor product $S^3\mathbb{C}^3$. The action (2) defines an irreducible representation on W. An inclusion of the signature of f into the signature of $\bar{f}$ indicates that the closure of the orbit of $\bar{f}$ contains the orbit of f.

EXAMPLE 4. *A cubic $x^3 + a\,xz^2 + z^3 - y^2z$ from the one parametric family of equivalence classes (1) is transformed to $x^3 + a\,\varepsilon^4 xz^2 + \varepsilon^6 z^3 - y^2z$ under the linear map*

$$x \to x, \quad y \to \frac{1}{\varepsilon}y, \quad z \to \varepsilon^2 z. \tag{20}$$

The latter cubic tends to the cubic $x^3 - y^2z$ of class (4) when $\varepsilon \to 0$. The signature of cubics of class (4), the i_3-axis, is included into the signature of $x^3 + a\,xz^2 + z^3 - y^2z$ for all a.

We summarize the inclusions of the signatures, and therefore the inclusions of one orbits in the closure of another, by the following diagram. Note that inclusions (9), (10), (11) of



binary forms are not seen from the signatures. The above picture can be compared with the analysis of the closures of the orbits under the action of the *special linear group* $SL(3,\mathbb{C})$ in Kraft [22].

The linear space of cubics is ten-dimensional, while the acting group $GL(3,\mathbb{C})$ is nine-dimensional, and thus we expect to have one invariant for this action. This invariant will be an absolute invariant in the sense of the classical invariant theory. This means that it depends on the coefficients of a cubic, not on the variables p, q . We notice that the ratio i_1^3/i_2^2 provides such invariant.

PROPOSITION 2. *The ratio i_1^3/i_2^2 is an absolute invarinat for the action (2), that is, it depends only on the coefficients of a cubic, and is constant for each of the equivalence classes, whenever it is defined.*

Observe that the ratio i_1^3/i_2^2 distinguishes equivalence classes in the set of non-singular irreducible cubics, but it does not distinguish the equivalence classes in the set of all cubics. For instance, it equals to 10 for classes (5), (7) and (8).

5. THE SYMMETRY GROUPS

According to Definition 2 the *symmetry group* of a cubic F is the subgroup $G_F \subset GL(3,\mathbb{C})$ consisting of all linear transformations that map F to itself. It coincides with the group $G_f \subset GL(3,\mathbb{C})$ which maps inhomogeneous version f of F to itself under transformation (9). Due to Theorem 4 the dimension of the signature $\mathcal{C}_f$ determines the dimension of the symmetry group: $\dim G_f = 2 - \dim \mathcal{C}_f$. Since it is useful for computations of the symmetry groups, we sketch the main ideas underlining the proof of Theorem 4. We start with the following simple proposition:

PROPOSITION 3. *Assume $\mathbf{u}^{(3)}$ and $\bar{\mathbf{u}}^{(3)}$ are two points on the prolonged graph $\Gamma_f^{(3)} \subset J^3$, which lie on the same orbit. That means there exists a matrix $g \in GL(3,\mathbb{C})$, such that $g \cdot \mathbf{u}^{(3)} = \bar{\mathbf{u}}^{(3)}$. Then $g \in G_f$.*

Proof. The graphs $\Gamma_f^{(3)}$ and $g \cdot \Gamma_f^{(3)}$ are two prolonged graphs that pass through the point $\bar{\mathbf{u}}^{(3)}$, but there is only one such graph through each point. Thus $g \cdot \Gamma_f^{(3)} = \Gamma_f^{(3)}$, and hence $f = g \cdot f$. $\square$

It follows that in order to find G_F one needs to study the intersection of the prolonged graph $\Gamma_f^{(3)}$ with the orbit $\mathcal{O}_{\mathbf{u}}^{(3)}$ of $\mathbf{u}^{(3)}$. In fact, since the isotropy group of each point of J^3 consists of three group elements, the points of the intersection $\Gamma_f^{(3)} \cap \mathcal{O}_{\mathbf{u}}^{(3)}$ are in one-to-three correspondence with the elements of the symmetry group. In order to determine this intersection, we recall that invariants i_1, i_2, i_3 define a projection π of J^3 onto the cross-section $\mathcal{K}$. Their restriction $i_1|_f, i_2|_f, i_3|_f$ to $\Gamma_f^{(3)}$ define a projection $\pi|_f \colon \Gamma_f^{(3)} \to \mathcal{K}$, whose image is the signature manifold of f.

DEFINITION 4. *A pair of coordinates (p, q) is called a generic point of a cubic f, if the corresponding point $\mathbf{u}^{(3)}$ on $\Gamma_f^{(3)}$, computed by formulae (10), projects to a generic point c on the signature manifold $\mathcal{C}_f$. That means that the point c is a non-singular of the variety $\mathcal{C}_f$ and it does not belong to any lower-dimensional signature manifold.*

A randomly chosen point will be generic with probability one. Let us choose a generic point (p_0, q_0), and compute the corresponding point $\mathbf{u}^{(3)} \in \Gamma_f^{(3)}$ and its projection $c = \pi|_f(\mathbf{u}^{(3)}) \in \mathcal{C}_f$. Since $\pi|_f$ defines the projection along the orbits, then the preimage $\pi|_f^{-1}(c) = \Gamma_f^{(3)} \cap \mathcal{O}_{\mathbf{u}^{(3)}}$ is the desired intersection. Moreover,

$$\dim G_f = \dim \pi|_f^{-1}(c) = \dim \Gamma_f^{(3)} - \dim \mathcal{C}_f = 2 - \dim \mathcal{C}_f.$$

If the signature manifold has a non maximal dimension, then a cubic has a continuous group of symmetries of dimension one or two. The corresponding linear transformations, listed in Theorem 5, can be easily found using Lie's infinitesimal methods [26], proposition 2.6. When a signature manifold of f has the maximal dimension, two, the corresponding cubic has a finite group of symmetries which can be explicitly found by solving the equations:

$$\begin{aligned} i_1|_f(p,q) = i_1|_f(P,Q), \qquad & i_2|_f(p,q) = i_2|_f(P,Q), \\ i_3|_f(p,q) \;\;=\;\; & i_3|_f(P,Q) \end{aligned} \tag{21}$$

for P and Q in terms of p and q. Indeed, the solution of this system will determine all points on Γ_f which are projected under π_f to the same point on the signature $\mathcal{C}_f$. Remarkably all the solutions of the equations (21) are linear fractional (see [28] chapter 8, p. 190 and [3]):

$$\begin{aligned} P &= \frac{\alpha\, p+\beta\, q+\lambda}{a\, p+b\, q+\eta}, \\ Q &= \frac{\gamma\, p+\delta\, q+\mu}{a\, p+b\, q+\eta}. \end{aligned}$$

These symmetries are called *projective*. Each projective symmetry gives rise to three genuine symmetries of the form $f(p,q)$. In practice however the equations (21) are difficult to solve. Nevertheless, fixing a specific generic point (p_0, q_0), one can find the number of the solutions using a well known algebraic result ([9], Proposition 8, ch. 5, § 3) and the Gröbner basis computation (see [10], ch. 2,§ 2). The number of the solutions determines the cardinality of the symmetry group of f. We conclude this section with a simple corollary from Theorems 5 and formula (3).

COROLLARY 1. *A homogeneous cubic F in three variables is irreducible if and only if $G_F \cap SL(3,\mathbb{C})$ is discrete. A cubic F is a product of a linear factor and an irreducible quadratic factor if and only if its $G_F \cap SL(3,\mathbb{C})$ is one-dimensional. A cubic F is a product of three different linear factors if and only if its $G_F \cap SL(3,\mathbb{C})$ is two-dimensional.*

6. TRANSFORMING A CUBIC TO ITS CANONICAL FORM

Given an arbitrary cubic polynomial F we can match it with one of the canonical forms listed in Theorem 5 by computing invariants $i_1|_f, i_2|_f, i_3|_f$ and determining which of the relations listed in Theorem 5 they satisfy. From Theorem 3 we know that F can be mapped to the corresponding canonical form $\bar{F}$ by a linear change of variables g. We would like to compute g explicitly. We first note that we should not expect a unique answer. Indeed, if $\bar{F} = g \cdot F$ and $h \in G_F$, then also $\bar{F} = gh \cdot F$. In fact, the entire left coset gG_F will map F to $\bar{F}$. Thus the number of possible transformations depends on the size of the symmetry group of F. In the finite case one can find all such transformations by solving the system of equations:

$$\begin{aligned} i_1|_f(p,q) = i_1|_{\bar{f}}(P,Q), \qquad & i_2|_f(p,q) = i_2|_{\bar{f}}(P,Q), \\ i_3|_f(p,q) \quad = \quad & i_3|_{\bar{f}}(P,Q) \end{aligned} \tag{22}$$

for P, Q in terms of p, q, where f and $\bar{f}$ are inhomogeneous versions F and $\bar{F}$ respectively (see [28] Chapter 8). This computation, however, is too difficult to be performed in practice and non-applicable in the infinite case, therefore, we propose an alternative more practical algorithm.

Given an inhomogeneous version f of F one can compute the elements of the matrix g (see (4)) using formulae (14, 17, 18). The entries of the obtained matrix are some functions of (p,q) and therefore we denote the matrix by $g|_f(p,q)$. Let us choose specific numeric values (p_0, q_0) that are generic in the sense of Definition 4 and denote a corresponding matrix with constant entries as g. By construction, g brings $\mathbf{u}^{(3)}$ to a point on the signature $\mathcal{C}_f$: $g \cdot \mathbf{u}^{(3)} = c \in \mathcal{C}_f$. Let $\bar{f}$ be equivalent to f, then $\mathcal{C}_{\bar{f}} = \mathcal{C}_f$, and so we can find a point (P_0, Q_0) such that the corresponding point $\bar{\mathbf{u}}^{(3)} \in \Gamma^{(3)}_{\bar{f}}$ is projected to the same point c on the signature manifold. This can be done by solving for P, Q the system (22) with $p = p_0, q = q_0$. Although the system we consider has many, or even infinitely many solutions, depending on the size of the symmetry group G_f, we are interested in any one of the solutions (P_0, Q_0). The corresponding point $\bar{\mathbf{u}}^{(3)} \in \Gamma^{(3)}_{\bar{f}}$ is projected to the same point $c \in \mathcal{C}_f$ as the point $\mathbf{u}^{(3)}$. Using again formulae (14, 17, and 18), we compute the matrix $\bar{g}|_{\bar{f}}(p,q)$, that corresponds to $\bar{f}$ and evaluate it at the point (P_0, Q_0) to obtain the matrix $\bar{g}$. The matrix $\bar{g}^{-1}\, g$ is the desired linear transformation.

This procedure can be summarized as follows.

1) Given a homogeneous cubic $F(x,y,z)$ compute its inhomogeneous version $f(p,q) = F(p,q,1)$ and the corresponding differential invariants $i_1|_f, i_2|_f, i_3|_f$ (19).

2) Determine the canonical form $\bar{F}$, by determining which of the relations listed in Theorem 5 are satisfied and compute the corresponding invariants $i_1|_{\bar{f}}, i_2|_{\bar{f}}, i_3|_{\bar{f}}$.

3) Choose a generic point (p_0, q_0) of f and find a corresponding point (P_0, Q_0) for $\bar{f}$ as a solution of equation (22).

4) Compute matrices $g = g|_f(p_0, q_0)$ and $\bar{g} = g_{\bar{f}}(P_0, Q_0)$ using (14, 17, and 18).

5) The matrix $g_0 = \bar{g}^{-1}\, g$ is the desired linear transformation, such that $\bar{F}(g_0 \cdot \mathbf{x}) = F(\mathbf{x})$.

7. CONCLUSIONS

In this paper we conduct a careful study of the equivalence classes of ternary cubics under general complex linear changes variables via moving frame method. The main ideas of such application appeared first in Olver [28].

The new contribution of this paper is the computation of the signature manifolds for each of the equivalence classes and a practically feasible algorithm that matches a cubic with its canonical form, producing explicitly a required linear transformation. The implementation is made possible by triangular decomposition methods. We also make an interesting observation of the correspondence between the geometry of signature manifolds and the geometry of the orbits in the linear space of ternary cubics under the action of $GL(3,\mathbb{C})$.

There is no theoretical difficulty in applying the same methods to the polynomials of higher degree or higher number of variables. For ternary forms of *any* degree n the problem can be solved at the jet space of order $\min\{n, 6\}$ and the invariants for cubics can be reused with addition of new invariants of higher orders. The rational invariants of order four were computed in [20]. However, the computations become more challenging and we are unaware of the complete classifications results for ternary forms of higher degrees.

The advantage of the moving frame method lies in its generality: it is applicable to any equivalence problems under an arbitrary finite-dimensional group of transformation. In particular, instead of the general linear group one can consider its subgroups, such as the special linear group or the orthogonal group. In this paper the dehomogenization of the cubic $F(x, y, z)$ leads us to the linear fractional action (8) on polynomials in two variables of degree less or equal to three. One can, more generally, consider nonzero weight transformation rules $f \to \bar{f}$:

$$\bar{f} = (\det g)^k\,(ap + bq + \eta)^3\, f\left(\frac{\alpha p + \beta q + \lambda}{ap + bq + \eta}, \frac{\gamma p + \delta q + \mu}{ap + bq + \eta}\right).$$

The affine action is also of interest.

Cartan's method of equivalence was formulated in the category of smooth manifolds. Hence, its direct applicability is restricted to polynomials over complex or real numbers. It would be a worthwhile and interesting project to reformulate the method of moving frames in the algebraic-geometry language, so that it can be applied to the problem of the equivalence and symmetry of algebraic varieties over fields of arbitrary characteristics.

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