Two New Modules of the RegularChains library in Maple: ParametricSystemTools and ConstructibleSetTools

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Introduction

▹ Solving systems of polynomial equations with parameters symbolically is in demand for an increasing number of applications such as dynamic systems and optimization.

▹ `ParametricSystemTools` is a new module of the RegularChains library, which implements comprehensive triangular decomposition (CTD), a new algorithmic approach for studying polynomial systems with parameters.

▹ In board terms, a CTD of a polynomial system F in parameters $\mathbf{u}$ and unknowns $\mathbf{x}$ over a field $\mathcal{K}$ ($\mathbb{Q}$ or $\mathbb{Z}_p$) consists of:

- a partition $\mathcal{C} = C_1, \ldots, C_e$ of the parameter space in $\mathbf{u}$,
- a well-behaved and generic decomposition of $Z_{C_i}(F)$, the zero-set of F arising from the parameter values $u \in C_i$, for all i.

Being "generic" and "well-behaved", this decomposition of $Z_{C_i}(F)$ allows us to study the geometrical properties of F at any parameter value u of C_i without solving F at $\mathbf{u} = u$.

▹ A constructible set is the zero set of a system of the form

$$\begin{cases} a_1 = 0, \\ \cdots \\ a_e \neq 0, \end{cases} \bigcup \begin{cases} b_1 = 0, \\ \cdots \\ b_f \neq 0, \end{cases} \bigcup \cdots$$

where a_i's and b_j's are polynomials.

▹ `ConstructibleSetTools` module of the RegularChains library in Maple, is the first complete computer algebra package providing constructible set as a type and supporting a rich collection of operations for doing algebraic geometry. Meanwhile, this module provides basic routines for solving polynomial systems with parameters.

A Glimpse of the User-Interface

```
> with(RegularChains);
[ChainTools, ConstructibleSetTools, DisplayPolynomialRing, Equations, ExtendedRegularGcd, Inequations, Initial, Inverse,
  IsRegular, MainDegree, MainVariable, MatrixCombine, MatrixTools, NormalForm, ParametricSystemTools, PolynomialRing,
  Rank, RegularGcd, RegularizeInitial, Separant, SparsePseudoRemainder, Tail, Triangularize]
> with(ConstructibleSetTools);
[Complement, ConstructibleSet, Difference, EmptyConstructibleSet, GeneralConstruct, Info, Intersection, IsContained, IsEmpty,
  MakePairwiseDisjoint, PolynomialMapImage, PolynomialMapPreimage, Projection, QuasiComponent, RefiningPartition,
  RegularSystem, RegularSystemDifference, RepresentingChain, RepresentingInequations, RepresentingRegularSystems, Union]
> with(ParametricSystemTools);
[BelongsTo, CoefficientsInParameters, ComprehensiveTriangularize, DefiningSet, DiscriminantSet,
  PreComprehensiveTriangularize, Specialize]
> R := PolynomialRing([x, y, s]);
                                R := polynomial_ring
> pctd, cells := ComprehensiveTriangularize([x*(y+1)-s, (x+1)*y-s], 1, R);
    pctd, cells := [regular_chain, regular_chain, regular_chain], [[constructible_set, [3, 2]], [constructible_set, [1]]]
> map(Equations, pctd, R);
                  [[x (y + 1) - s, y^2 + y - s], [x + 1, y + 1, s], [x, y, s]]
> Info(cells[1][1], R);
                                [[s], [1]]
> Info(cells[2][1], R);
                                [[ ], [s]]
```

An Example from Invariant Theory

▹ The Setting. While regarding u and v as parameters, the following polynomials g_1 and g_2 define two families of elliptic curves, respectively,

$$g_1 = x^3 + ux - y^2 + 1, \text{ and } g_2 = x^3 + vx - y^2 + 1.$$

▹ The Question. In invariant theory, a classical question is to ask whether there exists a linear rational map f between these two curves:

$$f : (x, y) \mapsto \left(\frac{A\,x + B\,y + C}{G\,x + H\,y + K}, \frac{D\,x + E\,y + F}{G\,x + H\,y + K} \right).$$

▹ The Model. The above question is equivalent to ask for which value (u, v) the following equation always holds:

$$g_1(x, y) - (G\,x + H\,y + K)^3 g_2(f(x, y)) = 0. \qquad (1)$$

For simplicity, assume that the origin is mapped to the origin which sets $C = F = 0$. Collecting the coefficients of (1) in x and y, we have the following polynomial equations:

$$(\star) \begin{cases} K^3 - 1 = 0 \\ (vA + 3G)K^2 - u = 0 \\ (vB + 3H)K^2 = 0 \\ A^3 + G^3 + vAG^2 - D^2G - 1 = 0 \\ B^3 + H^3 + vBH^2 - E^2H = 0 \\ (E^2 - 2vBH - 3H^2)K - 1 = 0 \\ (D^2 - 2vAG - 3G^2)K = 0 \\ 3A^2B + 3G^2H + vBG^2 + 2vAGH - 2GDE - HD^2 = 0 \\ 3AB^2 + 3GH^2 + vAH^2 + 2vBGH - 2HDE - GE^2 = 0 \\ (3GH + vBG + vAH - DE)K = 0. \end{cases}$$

▹ The Resolution. A CTD of the system $(\star)$ is

```
[regular_chain,regular_chain,regular_chain,regular_chain,regular_chain,regular_chain,
 regular_chain,regular_chain,regular_chain,regular_chain,regular_chain],
[[constructible_set, [1, 2, 3, 10, 11]],
 [constructible_set, [4, 5, 6, 7, 8, 9]],
 [constructible_set, [1, 2, 3]]]
```

There are three constructible sets in the output given by:

$$\begin{cases} C_1 : u^3 = v^3 = 9, \\ C_2 : u \;\; = v \;\; = 0, \\ C_3 : u^3 = v^3, \;\; u \neq 0, \;\; v^3 \neq 9. \end{cases}$$

The union of the C_i's is the answer to our question; taking the union produces a single component, namely $C : u^3 = v^3$.

An Algorithmic Realization of the Chevalley Theorem

▹ The Chevalley Theorem states that the rational map image of a constructible set is still constructible, which is a classical result in elimination theory.

▹ Let $\pi : \mathcal{K}^n \to \mathcal{K}^m$ be a rational map given by rational functions

$$\frac{g_i(x_1, \ldots, x_n)}{h_i(x_1, \ldots, x_n)} \quad \text{for } i = 1 \cdots m.$$

Let W be a constructible set in $\mathcal{K}^n$; for simplicity assume that W is given by $t_1(x_1, \ldots, x_n) = \cdots = t_e(x_1, \ldots, x_n) = 0$ and $\ell(x_1, \ldots, x_n) \neq 0$, where $t_1, \ldots, t_e, \ell$ are polynomials.

▹ Our new modules can compute the image $\pi(W)$ as follows. Consider new variables $y_1, \ldots, y_m$ and the polynomials

$$\begin{cases} f_1 &= h_1 y_1 - g_1, \\ & \cdots \\ f_m &= h_m y_m - g_m. \end{cases}$$

Construct the polynomial system

$$F \;:\; \{\, f_1 = \cdots = f_m = t_1 = \cdots = t_e = 0, (h_1 \cdots h_m \ell) \neq 0 \},$$

where y_i's are regarded as parameters and x_j's as unknowns.

▹ Compute a CTD of F and let $C_1, \ldots, C_e$ be the corresponding partition of the parameter space in y_i's. Then the constructible set $\pi(W)$ is the union of the C_i's with $Z_{C_i}(F) \neq \emptyset$.

Conclusions and References

▹ The RegularChains library in Maple provides new software tools for studying parametric systems and manipulating constructible sets.

▹ We are currently working on a new module for manipulating semi-algebraic sets ($=, \neq, >$), which will enhance the library for solving systems over the real number field.

▹ Some References:

- Changbo Chen, Oleg Golubitsky, François Lemaire, Marc Moreno Maza, and Wei Pan. Comprehensive Triangular Decomposition, 2007.
- François Lemaire, Marc Moreno Maza, and Yuzhen Xie. The RegularChains library in Maple 10, 2005.
- Joyce O'Halloran and Michael Schilmoeller. Gröbner bases for constructible sets, 2002.