Fundamental Algorithms and Implementation Techniques for Computing with Regular Chains

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How did regular chains emerge? (1/3)

- Let $\mathbf{K}$ be an algebraically closed field, say $\mathbb{C}$, and $\mathbf{k}$ be a subfield of $\mathbf{K}$, say $\mathbb{Q}$. Consider n variables $x_1 < \cdots < x_n$.

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- A subset $V \subset \mathbf{K}^n$ is a **(affine) variety over** $\mathbf{k}$ if there exists $F \subset \mathbf{k}[x_1, \cdots, x_n]$ such that $V = V(F)$ where

$$V(F) := \{z \in \mathbf{K}^n \mid f(z) = 0 \ \ (\forall\, f \in F)\}.$$

The variety V is **irreducible** if for all varieties $V_1, V_2 \subset \mathbf{K}^n$

$$V = V_1 \cup V_2 \quad \Rightarrow \quad V = V_1 \ \text{or} \ V = V_2.$$

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- **Theorem** (E. Lasker) *For each variety $V \subset \mathbf{K}^n$ there exist finitely many irreducible varieties $V_1, \ldots, V_e \subset \mathbf{K}^n$* such that

$$V = V_1 \cup \cdots \cup V_e.$$

  Moreover, if $V_i \not\subseteq V_j$ for $1 \le i < j \le e$ then $\{V_1, \ldots, V_e\}$ is unique. This is the **irreducible decomposition of V**.

How did regular chains emerge? (2/3)

- **Theorem** (J.F. Ritt) *Let* $V \subset \mathbf{K}^n$ *be an irreducible non-empty variety and let* $F \subset \mathbf{k}[x_1, \cdots, x_n]$ *s.t.* $V = V(F)$*. Then, one can compute a (reduced) triangular set* $T \subset \langle F \rangle$ *s.t.*

  $$(\forall\ g \in \langle F \rangle)\ \ \mathrm{prem}(g, T) = 0.$$

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- **Theorem** (W.T. Wu) *Let $V \subset \mathbf{K}^n$ be a variety and let $F \subset \mathbf{k}[x_1, \cdots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.*

$$(\forall\ g \in F)\ \ \mathrm{prem}(g, T) = 0.$$

  This leads to a factorization free *algorithm for decomposing varieties (but not into irreducible components).*

How did regular chains emerge? (3/3)

- **Example.** Applying the `charset` procedure to $F = \{x_2^2 - x_1, x_1 x_3^2 - 2x_2x_3 + 1, (x_2x_3 - 1)x_4^2 + x_2^2\}$ produces $T = F$. However $V(F) = \emptyset$. Indeed

$$x_1 x_3^2 - 2x_2x_3 + 1 \equiv (x_2x_3 - 1)^2 \mod x_2^2 - x_1.$$

Thus, the initial $(x_2x_3 - 1)$ is a zero-divisor modulo $\langle x_2^2 - x_1, x_1x_3^2 - 2x_2x_3 + 1\rangle$.

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- The notion of a regular chain (Lu Yang, Jingzhong Zhang 91) (Michael Kalkbrener 91) solves this difficulty: for any input $F \subseteq \mathbf{k}[x_1, \ldots, x_n]$ one can compute regular chains $T_1, \ldots, T_e$ such that a point $z \in \mathbf{K}^n$ is a zero of F if and only if z is a zero of one of the $T_1, \ldots, T_e$ (in some technical sense). (Dong Ming Wang 2000) (Marc Moreno Maza 2000)

Outline

- Regular chains
- Normal Forms
- Regular GCDs
- Regularity test
- The `RegularChains` library

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Part I: The Notion of a Regular Chain

- Regular chain, saturated ideal
- Algorithmic properties
- Zero-dimensional case (as many equations as variables)

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- T is a regular chain if for each $v \in \mathrm{mvar}(T)$ the initial of T_v is regular modulo $\mathrm{sat}(T_{<v})$ (Michael Kalkbrener 91).

Algorithmic Properties

- Let $p \in \mathbf{k}[x_1 < \cdots < x_n]$ and $T \subset \mathbf{k}[x_1 < \cdots < x_n]$ be a triangular set. If T is empty then, the *iterated resultant* of p w.r.t. T is $\mathrm{res}(T, p) = p$. Otherwise, writing $T = T_{<w} \cup T_w$

$$\mathrm{res}(T, p) = \begin{cases} p & \text{if } \deg(p, w) = 0 \\ \mathrm{res}(T_{<w}, \mathrm{res}(T_w, p, w)) & \text{otherwise} \end{cases}$$

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$$\{p \mid \mathrm{prem}(p, T) = 0\} \quad = \quad \mathrm{sat}(T)$$

(Philippe Aubry, Daniel Lazard, Marc Moreno Maza 97).

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- **Unless k is finite, normalization blows up coefficients.**

Part II: Normal Forms

- Ideal membership, normal form computation
- The fast division trick
- FFT-based multiplication
- Fast Normal form computation

Ideal membership, normal form computation

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- A *descending* approach

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blows up intermediate expression

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- A *naive ascending* approach

$$\mathrm{rem}(\cdots, \mathrm{rem}(\mathrm{rem}(\mathrm{rem}(p, T_{x_1}), T_{x_2}), T_{x_1}), \cdots T_{x_1})$$

blows up algebraic complexity

The fast division trick (1/2)

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$$a(x) \; = \; q(x)\, b(x) + r(x)$$
- Replacing x by $1/x$ and multiplying the equation by x^n:
$$x^n\, a(1/x) \; = \; \left(x^{n-m} q(1/x)\right)\, \left(x^m\, b(1/x)\right) + x^{n-m+1}\, \left(x^{m-1}\, r(1/x)\right)$$
That is:
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- Computing $(\mathrm{rev}_m(b))^{-1} \mod x^{n-m+1}$ is a truncated inverse of a power series. (S. Cook, 1966) (H. T. Kung, 1974) and (M. Sieveking, 1972)

The fast division trick (2/2)

Input: $f \in \mathbb{A}[x]$ such that $f(0) = 1$ and $\ell \in \mathcal{N}$.

Output: $g \in \mathbb{A}[x]$ such that $f\,g \equiv 1 \mod x^\ell$

$g_0 := 1$
$r := \lceil \log_2(\ell) \rceil$
for $i = 1 \cdots r$ **repeat**
  $g_i := \left(2g_{i-1} - f\,{g_{i-1}}^2\right) \mod x^{2^i}$
return g_r

- This algorithm runs in $3\mathrm{M}(\ell) + 0(\ell)$ operations in $\mathbb{A}$.
- Improved versions run in $2\,\mathrm{M}(\ell) + O(\ell)$ operations in $\mathbb{A}$.
- Finally, the quotient q and the remainder r are computed in $3\,\mathrm{M}(n-m) + \mathrm{M}(\max(n-m, m)) + \mathcal{O}(n)$ operations in $\mathbb{A}$
- *Modern Computed Algebra* (Gathen Gerhard 99)

FFT-based multiplication

M(d) number of coefficient operations in degree less than d.

Classical Multiplication	$M(d) = 2d^2$
Karatsuba Multiplication	$M(d) = 9d^{1.59}$
FFT over appropriate ring	$M(d) = 9/2d \log d + 3\,d$

Input: $f, g \in \mathbf{k}[x]$ and ω a s-primitive root of unity for $s > \deg(f) + \deg(g)$ and s is a power of 2.

Output: the product fg

(1) Evaluate f and g at ω^i for $i = 0 \cdots s-1$

(2) Evaluate fg at ω^i for $i = 0 \cdots s-1$

(3) Interpolate and **return** fg

See (M.M.M. Yuzhen Xie 2009) for implementation techniques.

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- One can compute $A\,B \mod \langle T_1, \ldots, T_n\}$ in $O\tilde{~}(4^n \delta_{\mathbf{T}})$ operations in $\mathbf{k}$ (Xin Li, M.M.M., É. Schost 07).

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- One can compute $A\,B \mod \langle T_1, \ldots, T_n\}$ in $O\tilde{~}(4^n \delta_{\mathbf{T}})$ operations in $\mathbf{k}$ (Xin Li, M.M.M., É. Schost 07).
- Three key ideas: using the fast division trick and avoid mod $\langle T_1, \ldots, T_n\}$ as much as possible and reduce to multiplying polynomials over the base field $\mathbf{k}$ using FFT.

ModMul($A, B, \{T_1, \ldots, T_n\}$)
1 $D := AB$ computed in $\mathbf{k}[x_1, \ldots, x_n]$
2 **return** $\mathrm{NormalForm}_n(D, \{T_1, \ldots, T_n\})$

Fast Normal form computation (2/3)

$\underline{\mathrm{NormalForm}_1(A : R[x_1], \{T_1 : R[x_1]\})}$

$S_1 := \mathrm{Rev}(T_1)^{-1} \mod x_1^{\deg(A)-\deg(T_1)+1}$

$D := \mathrm{Rev}(A) S_1 \mod x_1^{\deg(A)-\deg(T_1)+1}$

$D := T_1 \, \mathrm{Rev}(D)$

return $A - D$

$\underline{\mathrm{NormalForm}_2(A : R[x_1, x_2], \{T_1 : R[x_1], T_2 : R[x_1, x_2]\})}$

$A := \mathrm{map}(\mathrm{NormalForm}_1, \mathrm{Coeffs}(A, x_2), \{T_1\})$

$S_2 := \mathrm{Rev}(T_2)^{-1} \mod T_1, x_2^{\deg(A,x_2)-\deg(T_2,x_2)+1}$

$D := \mathrm{Rev}(A) S_2 \mod x_2^{\deg(A,x_2)-\deg(T_2,x_2)+1}$

$D := \mathrm{map}(\mathrm{NormalForm}_1, \mathrm{Coeffs}(D, x_2), \{T_1\})$

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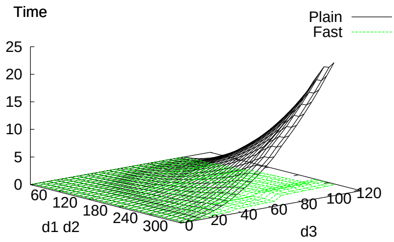
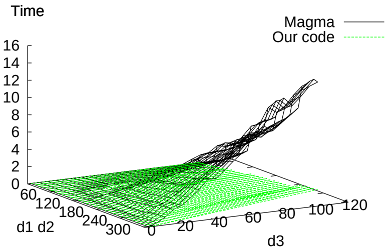
return $A - D$

Fast Normal form computation (3/3)

[left] comparison of classical (plain) and asymptotically fast strategies.

[right] comparison with MAGMA.





- Asymptotically fast strategy dominates the classical one.
- Our fast implementation is better than Magma's one (the best known implementation).

Part III: Regular GCDs

- Plane curve intersection
- The notion of a regular GCD
- Subresultants
- Regular GCDs via subresultants
- Complexity estimates
- Experimental results

Plane curve intersection

A historical application of the resultant is to compute the intersection of two plane curves. Up to details, there are two steps:

- eliminate one variable by computing a resultant,
- compute a GCD modulo this resultant.

Example (From *Modern Computer Algebra*, Chapter 6)

. Let $P = \left(y^2+6\right)(x-1) - y\left(x^2+1\right)$ and
$Q = \left(x^2+6\right)(y-1) - x\left(y^2+1\right)$

- $\mathrm{res}(\mathrm{P}, \mathrm{Q}, \mathrm{y}) = 2\,\left(\mathrm{x}^2 - \mathrm{x} + 4\right)(\mathrm{x}-2)^2(\mathrm{x}-3)^2.$
- $\mathrm{gcd}(\mathrm{P}, \mathrm{Q}, \mathrm{x}-2=0) = (\mathrm{y}-2)(\mathrm{y}-3).$
- $\mathrm{gcd}(\mathrm{P}, \mathrm{Q}, \mathrm{x}-3=0) = (\mathrm{y}-2)(\mathrm{y}-3).$
- $\mathrm{gcd}(\mathrm{P}, \mathrm{Q}, \mathrm{x}^2-\mathrm{x}+4=0) = (2\,\mathrm{x}-1)\,\mathrm{y} - 7 - \mathrm{x}.$

Regular GCD

- Let $\mathbb{B}$ be a commutative ring with units. Let $P, Q \in \mathbb{B}[y]$ be non-constant with regular leading coefficients.
- $G \in \mathbb{B}[y]$ is a *regular GCD* of P, Q if we have:
  - (i) $\mathrm{lc}(G, y)$ is a regular element of $\mathbb{B}$,
  - (ii) $G \in \langle P, Q \rangle$ in $\mathbb{B}[y]$,
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- In practice $\mathbb{B} = \mathbf{k}[x_1, \ldots, x_n]/\mathrm{sat}(T)$, with T being a regular chain.

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- In practice $\mathbb{B} = \mathbf{k}[x_1, \ldots, x_n]/\mathrm{sat}(T)$, with T being a regular chain.

- Such a regular GCD may not exist. However one can compute $\mathcal{I}_i = \mathrm{sat}(T_i)$ and non-zero polynomials G_i such that

$$\sqrt{\mathcal{I}} = \cap_{i=0}^{e} \sqrt{\mathcal{I}_i} \quad \text{and} \quad G_i \quad \text{regular GCD of} \quad P, Q \bmod \mathcal{I}_i$$

Regularity test

- Regularity test is a fundamental operation:

$$\mathrm{Regularize}(p, \mathcal{I}) \quad \longmapsto \quad (\mathcal{I}_1, \ldots, \mathcal{I}_e)$$

  such that:

$$\sqrt{\mathcal{I}} = \cap_{i=0}^{e} \sqrt{\mathcal{I}_i} \;\text{ and }\; p \in \mathcal{I}_i \text{ or } p \text{ regular modulo } \mathcal{I}_i$$

- Regularity test reduces to regular GCD computation.

Related work

- This notion of a regular GCD was proposed in (M. M. 2000)
- In previous work (Kalkbrener 1993) and (Rioboo & M. M. 1995), other regular GCDs modulo regular chains were introduced, but with limitations.
- In other work (Wang 2000), (Yang etc. 1995) and (Jean Della Dora, Claire Dicrescenzo, Dominique Duval 85), related techniques are used to construct triangular decompositions.
- Regular GCDs modulo regular chains generalize GCDs over towers of field extensions for which specialized algorithms are available, (van Hoeij and Monagan 2002 & 2004).
- Asymptotically fast algorithms (when $\mathrm{sat}(T)$ is zero-dimensional and radical) appear in (Xavier Dahan, M. M. , Éric Schost, Yuzhen Xie, 2006)
- The next results appear in (Xin Li, M. M. , Wei Pan, 2009).

Subresultants (1/2)

- Let $P, Q \in (\mathbf{k}[x_1])[x_2]$ with $\deg(P, x_2) > \deg(Q, x_2)$.

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- (Chee K. Yap 1993) (Lionel Ducos 1997) (M'hammed El Kahoui, 2003)

Subresultants (2/2)

- **Example.**
  The Chain of $P = X_2^4 + X_1X_2 + 1$ and $Q = 4X_2^3 + X_1$ in $(\mathbb{Q}[X_1])[X_2]$ produces the following sequence of polynomials:

$$\begin{aligned}
S_4 &= X_2^4 + X_1X_2 + 1 \\
S_3 &= 4X_2^3 + X_1 \\
S_2 &= -4(3X_1X_2 + 4) \\
S_1 &= -12X_1(3X_1X_2 + 4) \\
S_0 &= -27X_1^4 + 256
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- Let Φ be a homomorphism from $\mathbf{k}[x_1, x_2]$ to $\mathbf{K}[x_2]$. Assume $\Phi(a) \neq 0$ where $a = \mathrm{lc}(P, X_2)$. Then we have the specialization property of subresultants:

$$\Phi(sres_i(P, Q)) = \Phi(a)^{n-k} sres_i(\Phi(P), \Phi(Q))$$

  where $n = \deg(Q, x_2)$ and $k = \deg(\Phi(Q), x_2)$.

Regular GCDs (1/6)

- Let $P, Q \in \mathbf{k}[\mathbf{x}][y]$ with $\mathrm{mvar}(P) = \mathrm{mvar}(Q) = y$.
- Define $R = \mathrm{res}(P, Q, y)$.

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- Define $R = \mathrm{res}(P, Q, y)$.
- Let $T \subset \mathbf{k}[x_1, \ldots, x_n]$ be a regular chain such that
  - $R \in \mathrm{sat}(T)$,
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- $\mathbb{A} = \mathbf{k}[x_1, \ldots, x_n]$ and $\mathbb{B} = \mathbf{k}[x_1, \ldots, x_n]/\mathrm{sat}(T)$.
- For $0 \leq j \leq \mathrm{mdeg}(Q)$, we write S_j for the j-th subresultant of P, Q in $\mathbb{A}[y]$.

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- For $0 \leq j \leq \mathrm{mdeg}(Q)$, we write S_j for the j-th subresultant of P, Q in $\mathbb{A}[y]$.
- Recall that S_d regular GCD of P, Q modulo $\mathrm{sat}(T)$ means
  (*i*) $\mathrm{lc}(S_d, y)$ is a regular element of $\mathbb{B}$,
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  (*iii*) $\deg(S_d, y) > 0 \;\Rightarrow\; \mathrm{prem}(P, S_d, y) = \mathrm{prem}(Q, S_d, y) = 0$.

Regular GCDs (2/6)

- Let $1 \leq d \leq q$ such that $S_j \in \mathrm{sat}(T)$ for all $0 \leq j < d$.

Lemma

If $\mathrm{lc}(S_d, y)$ is regular modulo $\mathrm{sat}(T)$, then S_d is non-defective over $\mathbf{k}[\mathbf{x}]$.

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- Consequently, S_d is the last nonzero subresultant over $\mathbb{B}$, and it is also non-defective over $\mathbb{B}$.

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- Consequently, S_d is the last nonzero subresultant over $\mathbb{B}$, and it is also non-defective over $\mathbb{B}$.
- If $\mathrm{lc}(S_d, x_n)$ is not regular modulo $\mathrm{sat}(T)$ then S_d may be defective over $\mathbb{B}$.

Regular GCDs (3/6)

- Let $1 \leq d \leq q$ such that $S_j \in \mathrm{sat}(T)$ for all $0 \leq j < d$.

Lemma

If $\mathrm{lc}(S_d, y)$ *is in* $\mathrm{sat}(T)$*, then* S_d *is nilpotent modulo* $\mathrm{sat}(T)$*.*

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If $\mathrm{lc}(S_d, y)$ *is in* $\mathrm{sat}(T)$, *then* S_d *is nilpotent modulo* $\mathrm{sat}(T)$.

- Up to sufficient splitting of $\mathrm{sat}(T)$, S_d will vanish on all the components of $\mathrm{sat}(T)$.

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- Up to sufficient splitting of $\mathrm{sat}(T)$, S_d will vanish on all the components of $\mathrm{sat}(T)$.
- The above two lemmas completely characterize the last non-zero subresultant of P and Q over $\mathbb{B}$.

Regular GCDs (4/6)

Example

- Consider P and Q in $\mathbb{Q}[x_1, x_2][y]$:

$$P = x_2^2 y^2 - x_1^4 \ \text{and} \ Q = x_1^2 y^2 - x_2^4.$$

- We have:

$$S_1 = x_1^6 - x_2^6 \ \text{and} \ R = (x_1^6 - x_2^6)^2.$$

- Let $T = \{R\}$. Then we observe:
  - The last subresultant of P, Q modulo $\mathrm{sat}(T)$ is S_1, which is a defective one.
  - S_1 is nilpotent modulo $\mathrm{sat}(T)$.
- P and Q do not admit a regular GCD over $\mathbb{Q}[x_1, x_2]/\mathrm{sat}(T)$.

Regular GCDs (5/6)

- Let $1 \leq d \leq q$ such that $S_j \in \mathrm{sat}(T)$ for all $0 \leq j < d$.

Proposition

Assume

- $\mathrm{lc}(S_d, y)$ *is regular modulo* $\mathrm{sat}(T)$,
- $\mathrm{sat}(T)$ *is radical.*

Then, S_d *is a regular GCD of* P, Q *modulo* $\mathrm{sat}(T)$.

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Proposition

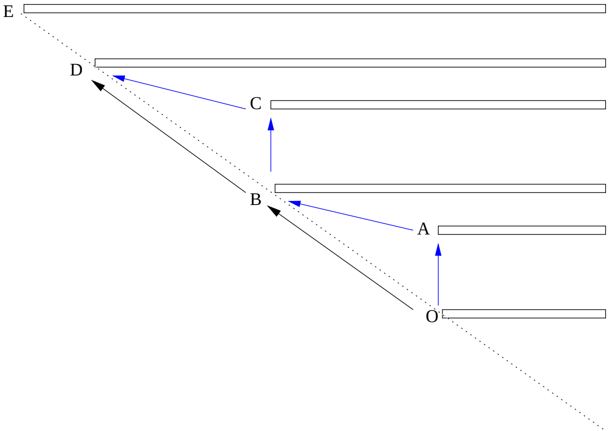
Assume

- $\mathrm{lc}(S_d, y)$ *is regular modulo* $\mathrm{sat}(T)$*,*
- *for all* $d < k \leq q$*,* $\mathrm{coeff}(S_k, y^k)$ *is either* 0 *or regular modulo* $\mathrm{sat}(T)$*.*

Then, S_d *is a regular GCD of* P, Q *modulo* $\mathrm{sat}(T)$*.*

Regular GCDs (6/6)

- Assume that the subresultants S_j for $1 \leq j < q$ are computed.
- Then one can compute a regular GCD of P, Q modulo $\mathrm{sat}(T)$ by performing a bottom-up search.



Implementation and Complexity Estimates (1/2)

We assume that the the base field **k** supports FFT.

- Recall $P, Q \in \mathbf{k}[x_1, \ldots, x_n][y]$. Let $x_{n+1} := y$.

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- To do so, we need bounds. We consider the Sylvester Matrix. Define $d_i := \max(\deg(P, x_i), \deg(Q, x_i))$. We have

$$\deg(R, x_i) \leq b_i := 2 d_i d_{n+1}.$$

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$$\deg(R, x_i) \leq b_i := 2 d_i d_{n+1}.$$
- $B := (b_1 + 1) \cdots (b_n + 1)$ is the number of points at which we need to evaluate P, Q.

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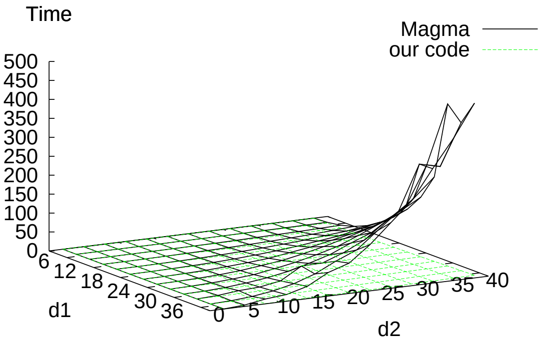
- Regularity tests (and normal forms) also fit these bounds.
- If a regular GCD is expected to have degree 1 in y all computations fit in $O\tilde{~}(d_{n+1} B)$.

Generic Bivariate Systems

- "our code" means `BivariateModularTriangularize` in MAPLE 13.
- Random generic input systems, thus equiprojectable.
- For the largest examples (having about 5700 solutions), the ratio is about 460/7 in our favor.



Non-generic Bivariate Systems

- Examples designed to enforce many "splittings" (many equiprojectable components).
- For the largest examples, the ratio is 5260/80, in our favor.

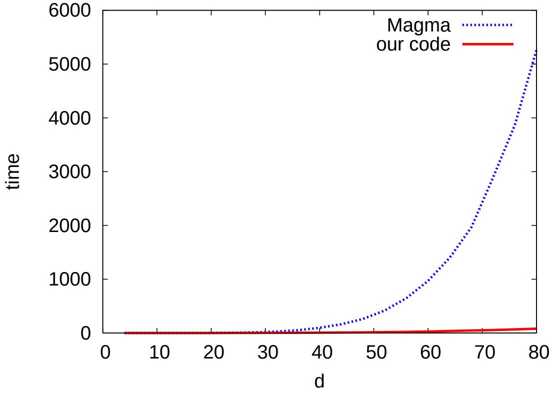


Figure: Non-generic bivariate systems: MAGMA vs. us.

Generic Trivariate Systems

- MAPLE means the experimental and fast version of `Triangularize` to be integrated in MAPLE 14.

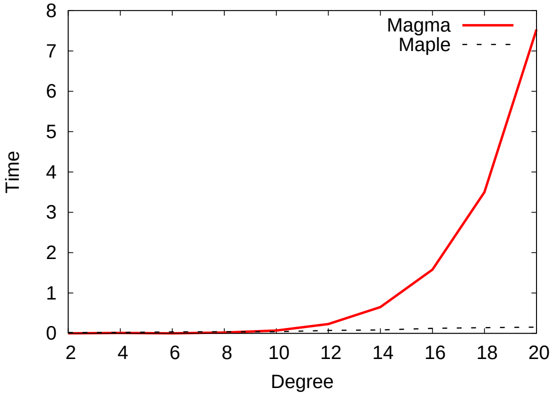


Figure: *Generic dense 3-variable.*

Part IV: Regularity test

- Testing regularity
- Experimental results

Regularity Test

For T 0-dim, auto-reduced and with $h_T = 1$ this procedure returns $T^1, \ldots, T^e$ such that Q is either zero or invertible modulo T^i.

RegularizeDim0(Q, T) ==

(0) **if** $Q \in \mathbf{k}$ **then return** $[T]$

(1) $\quad$ $Results := []$; $v := \mathrm{mvar}(Q)$

(2) $\quad$ $R := \mathrm{res}(Q, T_v, v)$

(3) $\quad$ **for** $D \in$ RegularizeDim0$(R, T_{<v})$ **do**

(4) $\quad\quad$ $s :=$ NormalForm(R, D)

(5) $\quad\quad$ **if** $s \neq 0$ **then**

(7) $\quad\quad\quad$ $Results := \{\{D \cup \{T_v\} \cup T_{>v}\}\} \cup Results$

(8) $\quad\quad$ **else for** $(g, E) \in$ RegularGcd(Q, T_v, D) **do**

(9) $\quad\quad\quad$ $g :=$ NormalForm(g, E)

(11) $\quad\quad\quad$ $Results := \{\{E \cup \{g\} \cup T_{>v}\}\} \cup Results$

(12) $\quad\quad\quad$ $c :=$ NormalForm$(\mathrm{quo}(T_v, g), E)$

(13) $\quad\quad\quad$ **if** $\deg(c, v) > 0$ **then**

(14) $\quad\quad\quad\quad$ $Results :=$ RegularizeDim0$(q, E \cup c \cup T_{>}v) \cup Results$

(15) **return** $Results$

Regularity Test (= Saturation)

d_1	d_2	d_3	Regularize	Fast Regularize	Magma
2	2	3	0.032	0.004	0.010
3	4	6	0.160	0.016	0.020
4	6	9	0.404	0.024	0.060
5	8	12	>100	0.129	0.330
6	10	15	>100	0.272	1.300
7	12	18	>100	0.704	5.100
8	14	21	>100	1.276	14.530
9	16	24	>100	5.836	40.770
10	18	27	>100	9.332	107.280
11	20	30	>100	15.904	229.950
12	22	33	>100	33.146	493.490

Table: Generic dense 3-variable.

- In the non-generic case, both gaps are even larger.
- "Fast Regularize" means `RegularizeDim0` in MAPLE 13.

Conclusions

- Modular methods help reducing expression swell and algebraic complexity.
- Modular methods create opportunities for fast arithmetic and parallelism.
- Fast arithmetic reduces algebraic complexity further.
- Performance improvements can come also from other factors: avoiding re-computations, controlling memory traffic
- Controlling expression swell may require to understand the structure of the computed objects.

Xie Xie! Thank You!

Positive Dimensional Regular Chains

- Let $\mathbf{u} = u_1, \ldots, u_d$ be parameters, $\mathbf{y} = y_1, \ldots, y_m$ be unknowns, Π_U be the projection from $\mathbf{K}^{m+d}$ to $\mathbf{K}^d$.

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- A regular chain $T \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$ *specializes well* at $u \in \mathbf{K}^d$ if $T(u)$ is a regular chain in $\mathbf{K}[\mathbf{y}]$ and $\mathrm{rank}(T(u)) = \mathrm{rank}(T_{>U_d})$.

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- Let $T \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$ be a reg. chain and $u \in \Pi_U(\mathbf{W}(T \cap \mathbf{k}[U]))$. T specializes well at $u \iff \mathrm{res}(h_{T_{>U_d}}, T_{>U_d}) \neq 0$ at $\mathbf{u} = u$

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- Replacing *regular chain* by squarefree reg. ch. in char. 0 and $h_{T_{>U_d}}$ by $\mathrm{Sep}_{T_{>U_d}}$ one obtains the *border polynomial of T*.

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- Replacing *regular chain* by squarefree reg. ch. in char. 0 and $h_{T_{>U_d}}$ by $\mathrm{Sep}_{T_{>U_d}}$ one obtains the *border polynomial of T*.

Related Work

On a projection theorem of quasi-varieties in elimination theory (Wen-Tsün Wu 90). (Xiao-Shan Gao, Shang-Ching Chou 92) (Dongming Wang 00 & 01) (Lu Yang, Xiaorong Hou, Bican Xia 01) (Xiao-Shan Gao, Ding-Kang Wang 03) (Changbo Chen, Oleg Golubitsky, François Lemaire, Marc Moreno Maza, Wei Pan 07)

Equiprojectable Decomposition (1/2)

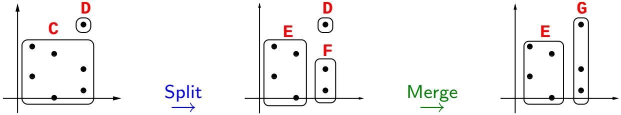
$$C \left| \begin{array}{l} C_2 = y^2 + 6yx^2 + 2y + x \\ C_1 = x^3 + 6x^2 + 5x + 2 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$\downarrow$ Split C : GCD $\downarrow$

$$E \left| \begin{array}{l} {C_2}' = y^2 + x \\ {C_1}' = x^2 + 5 \end{array} \right. , \quad F \left| \begin{array}{l} C_2'' = y^2 + y + 1 \\ C_1'' = x + 6 \end{array} \right. , \quad D \left| \begin{array}{l} D_2 = y + 6 \\ D_1 = x + 6 \end{array} \right.$$

$\downarrow$ Merge F and D : CRT $\downarrow$

$$E \left| \begin{array}{l} C_2' = y^2 + x \\ C_1' = x^2 + 5 \end{array} \right. , \quad G \left| \begin{array}{l} G_2 = y^3 + 6 \\ G_1 = x + 6 \end{array} \right.$$



Equiprojectable Decomposition (2/2)

$$\begin{array}{ccc}
F \subset \mathbb{Q}[x_1, \ldots, x_n] \\ p \text{ prime}, \ p > A(F) & \longrightarrow & Eq.D._{\mathbb{Q}}(F) \\
\Big\downarrow \text{Triang.}(F \mod p) & & \Big\uparrow \textit{Lifting} \\
\{T^1, \ldots, T^s\} \\ \subset \mathbb{F}_p[x_1, \ldots, x_n] & \xrightarrow{\textbf{Split} + \textbf{Merge}} & Eq.D._{\mathbb{F}_p}(F)
\end{array}$$

- $A(F) := 2n^2 d^{2n+1}(3h + 7\log(n+1) + 5n\log d + 10)$ where h and d upper bound coeff. sizes and total degrees for $f \in F$. Assumes F square and generates a 0-dimensional radical ideal.

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- $A(F) := 2n^2 d^{2n+1}(3h + 7\log(n+1) + 5n\log d + 10)$ where h and d upper bound coeff. sizes and total degrees for $f \in F$. Assumes F square and generates a 0-dimensional radical ideal.
- If $p \not| A(F)$, the equiprojectable decomposition specializes well mod p.
- In practice we choose p much smaller with a probability of success, i.e. $> 99\%$ with $p \approx ln(A(F))$ (Xavier Dahan, M. M. M., Éric Schost, Wenyuan Wu, Yuzhen Xie 05).

Incremental Solving

- Let $F \subset \mathbf{k}[\mathbf{x}]$, $f \in \mathbf{k}[\mathbf{x}]$, $T, T^m \ldots, T^e \subset \mathbf{k}[\mathbf{x}]$ reg. chains. Assume we have *solved* F as $V(F) = W(T^i) \cup \cdots \cup W(T^e)$.

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- Assume that we have an operation $(f, T) \longmapsto \mathbf{Intersect}(f, T) = (C_1, \ldots, C_d)$ such that

$$V(f) \cap W(T) \;\subseteq\; \cup_i W(C_i) \;\subseteq\; V(f) \cap \overline{W(T)}.$$

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Related Work

(D. Lazard 91) proposes the principle. (M. M. M. 00) introduces regular GCDs and gives a complete incremental algorithm which, in addition, generates components by decreasing order of dimension.

The notion of a Regular GCD

- Let $P, Q, G \in \mathbf{k}[x_1 < \cdots < x_n][y]$ and $T \subset \mathbf{k}[x_1 < \cdots < x_n]$ reg. chain. G is a *regular GCD* of P, Q modulo $\mathrm{sat}(T)$ if
  - (*i*) $\mathrm{lc}(G, y)$ is a regular modulo $\mathrm{sat}(T)$,
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  - (*iii*) $\deg_y(G) > 0 \Rightarrow \mathrm{prem}_y(P, G), \mathrm{prem}_y(Q, G) \in \mathrm{sat}(T)$.

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- If both $T \cup P$ and $T \cup Q$ are regular chains and if G is a GCD of P, Q modulo $\mathrm{sat}(T)$ with $\deg_y(G) > 0$ then we have

$$\begin{array}{rl} W(T \cup P) \;\cap\; V(Q) \subseteq & W(T \cup G) \cup \\ & W(T \cup P) \;\cap\; V(Q, h_G) \;\subseteq\; \overline{W(T \cup P)} \;\cap\; V(Q). \end{array}$$

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$$

- One can compute $T^1, \ldots, T^e$ and $G_1, \ldots, G_e$ such that G_i is a reg. GCD of P, Q mod $\mathrm{sat}(T_i)$ and $\sqrt{\mathrm{sat}(T)} = \cap_{i=0}^{e} \sqrt{\mathrm{sat}(T^i)}$.

Regularity test

- Regularity test is a fundamental operation:

$$\mathrm{Regularize}(p, \mathcal{I}) \quad \longmapsto \quad (\mathcal{I}_1, \ldots, \mathcal{I}_e)$$

  such that:

$$\sqrt{\mathcal{I}} = \cap_{i=0}^{e} \sqrt{\mathcal{I}_i} \ \text{ and } \ p \in \mathcal{I}_i \text{ or } p \text{ regular modulo } \mathcal{I}_i$$

- Regularity test reduces to regular GCD computation.

Related work

- This notion of a regular GCD was proposed in (M. M. 2000)
- In previous work (Kalkbrener 1993) and (Rioboo & M. M. 1995), other regular GCDs modulo regular chains were introduced, but with limitations.
- In other work (Wang 2000), (Yang etc. 1995) and (Jean Della Dora, Claire Dicrescenzo, Dominique Duval 85), related techniques are used to construct triangular decompositions.
- Regular GCDs modulo regular chains generalize GCDs over towers of field extentions for which specialized algorithms are available, (van Hoeij and Monagan 2002 & 2004).
- Asymptotically fast algorithms (when $\mathrm{sat}(T)$ is zero-dimensional and radical) appear in (Xavier Dahan, M. M. , Éric Schost, Yuzhen Xie, 2006)
- The next results appear in (Xin Li, M. M. , Wei Pan, 2009).

Regular GCDs: Bottom-up or Top-down?

- Let $P, Q \in \mathbf{k}[x_1 < \cdots < x_n][y]$ and $T \subset \mathbf{k}[x_1 < \cdots < x_n]$ reg. chain. How to compute a *regular GCD* of P, Q mod $\mathrm{sat}(T)$?
- (M. Kalkbrener 91) uses Pseudo-Remainder Sequences. Inefficient!

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- (M. M. M. and R. Rioboo 95) assume $\mathrm{sat}(T)$ radical + 0-dimensional and use the subresultant chain of P, Q directly in $\mathbf{k}[x_1 < \cdots < x_n][y]$ mod $\mathrm{sat}(T)$. Better but removing the assumptions removes the efficiency.

Subresultants (3/3

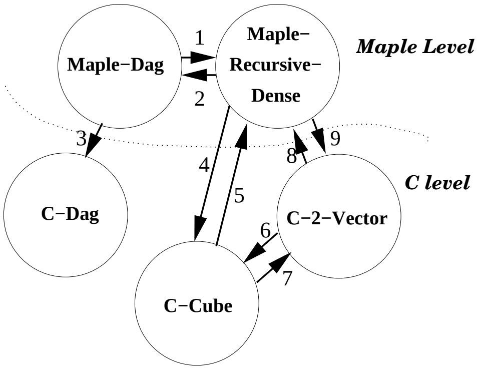
- Let $P, Q \in \mathbb{B}[y]$ with $p = \deg(P) \geq \deg(Q) = q > 0$.
- For $0 \leq d < q$ let $S_d = S_d(P, Q)$ be the d-th *subresultant* of P and Q. Let $s_d = \mathrm{coeff}(S_d, x^d)$. If $s_d = 0$ we say S_d is defective, otherwise we say S_d is non-defective.
- Let $d = q-1, \ldots, 1$. Assume S_d, S_{d-1} nonzero, with resp. degrees d and e. Assume s_d regular in $\mathbb{B}$. Then we have
$$\mathrm{lc}(S_{d-1})^{d-e-1} S_{d-1} = {s_d}^{d-e-1} S_e.$$
- Moreover, there exists $C_d \in \mathbb{B}[X]$ such that we have:
$$(-1)^{d-1}\mathrm{lc}(S_{d-1}) s_e S_d + C_d S_{d-1} = {s_d}^2 S_{e-1}.$$

  In addition $S_{d-2} = S_{d-3} = \cdots = S_{e+1} = 0$ also holds.
- (Yap 1993) (Ducos 1997) (El Kahoui, 2003)

The RegularChains library in MAPLE

- 80,000 lines of MAPLE code, 36,000 lines of C code, 121 Commands, 6 modules ChainTools, MatrixTools, ConstructibleSetTools. ParametricSystemTools, SemiAlgebraicSetTools , FastArithmeticTools .
- Main new commnands in MAPLE 13: IsPrimitive, ComplexRootClassification, RealRootClassification, RealRootIsolate RealRootCounting, BorderPolynomial, + those of FastArithmeticTools (see demo).
- Current contributors: Changbo Chen, Francçois Lemaire, Liyun Li, Xin Li, M.M.M., Wei Pan, Bican Xia, Rong Xiao, Yuzhen Xie.

The MODPN library



- *C-Dag* for straight-line program.
- *C-Cube* for FFT-based computations.
- *C-2-Vector* for compact dense representation.
- *Maple-Dag* for calling `RegularChains` library.
- *Maple-Recursive-Dense* for calling RECDEN library.

Generic Bivariate Systems

- "our code" means `BivariateModularTriangularize` in MAPLE 13.
- Random generic input systems, thus equiprojectable.
- For the largest examples (having about 5700 solutions), the ratio is about 460/7 in our favor.