A Maple Package for Parametric Matrix Computations

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Outline

Parametric Matrix Computations in CAS

Computer algebra systems are currently **unable** to handle case discussion for parametric matrices.

Jordan Canonical Form: Maple, Mathematica, Sage

Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \textbf{with} \quad \alpha \in \mathbb{C}$$

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Maple

$$\left[\begin{array}{c|c|c} -2 & & \\ \hline & \alpha & \\ \hline & & \alpha - 3 \end{array}\right]$$

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Sage

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$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\alpha = -2} \begin{bmatrix} -2 & 0 & 1 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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Regular Chains

RegularChains

Multivariate Polynomials

- Let $\mathbb{K}$ be an algebraically closed of real closed field
- Let $x_1 < \cdots < x_n$ be $n \geq 1$ ordered variables
- $\mathbb{K}[x_1, \ldots, x_n]$ is the ring of polynomials in $x_1, \ldots, x_n$
- For non-constant $p \in \mathbb{K}[x_1, \ldots, x_n]$, $\mathrm{mvar}(p)$ denotes the greatest variable in p
- Leading coefficient of p w.r.t $\mathrm{mvar}(p)$ is called the **initial** denoted $\mathrm{init}(p)$

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Regular Chain

Definition (Triangular Set)

A set T of non-constant polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ is called a **triangular set** if for all p, $q \in T$ with $p \neq q$, $\mathrm{mvar}(p) \neq \mathrm{mvar}(q)$.

Definition (Saturated Ideal)

The **saturated ideal** $\mathrm{sat}(T)$ of T is the ideal $\langle T \rangle : h_T^{\infty}$, where h_T is the product of the initials of the polynomials in T

Definition (Regular Chain)

T is a **regular chain** if $T = \varnothing$ or $T = T' \cup \{t\}$ for $t \in T$ with $\mathrm{mvar}(t)$ maximum such that

- T' is a regular chain
- $\mathrm{init}(t)$ is regular modulo $\mathrm{sat}(T')$

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Constructible Set

Definition (Constructible Set)

A **constructible set** is the disjunction of systems of polynomial equations and inequations. Where the systems of equations are conjunctions of constraints.

Semi-algebraic Set

Definition (Semi-algebraic System)

A **semi-algebraic system** of $\mathbb{K}[x_1, \ldots, x_n]$ is any polynomial system S of the form

$$\left\{ \begin{array}{c} f_1 = \cdots = f_a = 0 \\ g \neq 0 \\ p_1 > 0, \ldots, p_b > 0 \\ q_1 \geq 0, \ldots, q_c \geq 0 \end{array} \right\}$$

for $f_1, \ldots, f_a, g, p_1, \ldots, p_b, q_1, \ldots, q_c \in \mathbb{K}[x_1, \ldots, x_n]$.

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Notation

- Parameters $\alpha = \alpha_1, \ldots, \alpha_s$
- $\mathbb{K}$ is an algebraically closed or real closed field
- $\mathbb{K}[\alpha]$ is the ring of polynomials in the parameters
- $\mathbb{K}(\alpha)$ is the quotient field of $\mathbb{K}[\alpha]$
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Proposition[1,2]

For two constructible (resp. semi-algebraic) sets $S_1, S_2 \subseteq \mathbb{K}^s$ one can compute a **triangular decomposition** of their

- intersection $S_1 \cap S_2$
- union $S_1 \cup S_2$
- set theoretic difference $S_1 \setminus S_2$

[1] **C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao.** Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

[2] **C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, W. Pan.** Comprehensive triangular decomposition. In Proc. of *CASC07. Volume 4770 of Lecture Notes in Computer Science.* (2007) 73101.

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Remark

Let $S \subseteq \mathbb{K}^s$ be a constructible (resp. semi-algebraic) set; for every $f(\alpha) \in \mathbb{K}[\alpha]$ a partition $(S_{\text{eq}}, S_{\text{neq}})$ of S can be computed by

$$\begin{aligned} S_{\text{eq}} &= S \cap V(f(\alpha)) \\ S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}} \,. \end{aligned}$$

- $f(\alpha)$ is **zero** everywhere on S_{eq}
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Parametric Matrix

Definition

A $m \times n$ **parametric matrix** takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for

$$f_{i,j} \in \mathbb{K}(\alpha) \quad 1 \leq i \leq m,\ 1 \leq j \leq n\,.$$

Require constructible (resp. semi-algebraic) set $S \subseteq \mathbb{K}^s$ such that denominator of every entry of $A(\alpha)$ is nonzero everywhere on S.

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Outline

Parametric Rank Algorithm (Complex Case)

Input:
- Parametric matrix $A(\alpha)$
- Constraints on the parameter values given in a set S by a polynomial system of the form

$$f_1(\alpha) = \cdots = f_a(\alpha) = 0, \; g(\alpha) \neq 0$$

Output:
- A list with elements of the form $[r_i, S_i]$ where S_i gives conditions on α such that the rank of $A(\alpha)$ is r_i
- S_i's form a partition of S

$$\bigcup_i S_i = S, \quad S_i \cap S_j = \varnothing \quad \forall i \neq j$$

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Parametric Rank Algorithm (Complex Case)

Step 1: Define polynomial system S' as **union** of equations of $A(\alpha)X = 0$ and S

Let

$$\mathcal{T} := \texttt{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$$

For $0 \leq r \leq n$, let C_r be the **constructible set** of $\mathbb{K}^s$ given by all regular systems $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$ such that $[T_j, h_j] \in \mathcal{T}$ and the number of polynomials of T_j of positive degree in (at least) one of the variables $x_1 < \cdots < x_n$ is exactly $n - r$.

For $r := n$ down to 1 do

$$C_r \;:=\; \texttt{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$$

Project out the x_i variables

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Example[3] over $\mathbb{C}$

Example

Consider $E\ddot{x} = A_1\dot{x} + A_2 x + Bu$

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \lambda & 3\,\lambda & \lambda \\ 3\,\lambda + \mu & \lambda + \mu & \lambda + 3\,\mu \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda,\, \mu \in \mathbb{C}$$

[3] **M. I. García-Planas and J. Clotet.** Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007.

Example[3] over $\mathbb{C}$

Example (cont.)

What is the **rank** of:

$$C = \begin{bmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{bmatrix}$$

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Example[3] over $\mathbb{C}$

Example (cont.)

$$\mathrm{Rank}(C) = \begin{cases} 18 & \textbf{if} \quad \lambda \neq 0,\ \mu \neq 1/2 \\ 17 & \textbf{if} \quad \lambda \neq 0,\ \mu = 1/2 \\ 16 & \textbf{if} \quad \lambda = 0,\ \mu \neq -1 \\ 15 & \textbf{if} \quad \lambda = 0,\ \mu = -1 \end{cases}$$

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Zigzag Form of Parametric Matrices

Step 3: Step 4: Step 5: Zigzag Form

Parametric Polynomial

Define a polynomial in x with parameters α as

$$f(x;\alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with $f_0(\alpha), \ldots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$.

Require constructible (resp. semi-algebraic) set $S \subseteq \mathbb{K}^k$ such that denominator of every coefficient is nonzero everywhere on S.

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Companion Matrix

Frobenius companion matrix in x of a parametric polynomial

$$f(x;\alpha) = f_0(\alpha) + f_1(\alpha)x + \cdots + f_{r-1}(\alpha)x^{r-1} + x^r$$

takes the form

$$C_{f(x;\alpha)} = \begin{bmatrix} 0 & \cdots & 0 & -f_0(\alpha) \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -f_{r-2}(\alpha) \\ & & 1 & -f_{r-1}(\alpha) \end{bmatrix}$$

Off-diagonal Block

Let $b \in \{0, 1\}$. Define

$$B_b = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Zigzag Matrix[4]

$$\begin{bmatrix} C_{c_1(x;\alpha)} & B_{b_1} & & & & & & \\ & C^T_{c_2(x;\alpha)} & & & & & & \\ & B_{b_2} & C_{c_3(x;\alpha)} & B_{b_3} & & & & \\ & & & C^T_{c_4(x;\alpha)} & & & & \\ & & & & \ddots & & & \\ & & & & & C^T_{c_{d-2}(x;\alpha)} & & \\ & & & & & B_{b_{d-2}} & C_{c_{d-1}(x;\alpha)} & B_{b_{d-1}} \\ & & & & & & & C^T_{c_d(x;\alpha)} \end{bmatrix}$$

for d **even**

[4] **A. Storjohann.** An $\mathcal{O}(n^3)$ algorithm for the Frobenius normal form. In *Proceedings of ISSAC 1998*, pages 101-105. ACM, 1998.

Theorem

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For **every** *matrix* $A(\alpha) \in \mathbb{K}^{n \times n}[\alpha]$*, there exists a partition* $(S_1, \ldots, S_N)$ *of the input constructible (resp. semi-algebraic) set* S *such that for each* S_i*, there exists a matrix* $Z_i(\alpha) \sim A(\alpha)$ *in* **Zigzag form** *where the denominator of the entries of* $Z_i(\alpha)$ *are all nonzero everywhere on* S_i*.*

Algorithm

- The algorithm extends the work of **Storjohann**[4] on the computation of the Zigzag form.
- The algorithm only uses **similarity transformations** to obtain the Zigzag form.
- **Stages 1 and 3** of Storjohann's algorithm[4] are modified such that computation splits when searching for pivots.

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Semi-Algebraic Example over $\mathbb{R}$

Example

$$A(a,b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \textbf{with} \quad a \geq 0,\ b > 0,\ a \neq b$$

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Semi-Algebraic Example over $\mathbb{R}$

Example

$$A(a,b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \quad \textbf{with} \quad a \geq 0,\ b > 0,\ a \neq b$$

$$Z_1(a,b) = \begin{bmatrix} 0 & 0 & 12\,a \\ 1 & 0 & 18 \\ 0 & 1 & b \end{bmatrix} \quad \textbf{if} \quad a > 0,\ b > 0,\ a \neq b$$

$$Z_2(a,b) = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline & 0 & 1 \\ & 18 & b \end{array}\right] \quad \textbf{if} \quad a = 0,\ b > 0$$

Algebraic Example over $\mathbb{C}$

Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0 \\ -1/2 & \alpha - 2 & 1/2 \\ -2 & 3\,\alpha + 1 & -1 \end{bmatrix} \quad \textbf{with} \quad \alpha \in \mathbb{C}$$

Algebraic Example over $\mathbb{C}$

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$$Z_1(\alpha) = \begin{bmatrix} 0 & 0 & 4\,\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \textbf{if} \quad \alpha + 3 \neq 0$$

$$Z_2(\alpha) = \left[\begin{array}{cc|c} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{array}\right] \quad \textbf{if} \quad \alpha + 3 = 0$$

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Outline

Future Work

Research is currently being conducted to create a MAPLE package `ParametricMatrixTools` with methods to compute

- Rank ✓
- Frobenius (Rational) form
- Minimal polynomial
- Test for similarity
- Jordan form
- Weyr form[5]

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  - Parallel implementation
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Code & Examples

All current and future code and example worksheets will be available at
StevenThornton.ca/Code

Information on the RegularChains package of MAPLE is available at
RegularChains.org

Thank You!