# A MAPLE Package for Parametric Matrix Computations

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# Outline

- 1. Motivation
- 2. Regular Chains
- 3. Background
- 4. Rank
- 5. Zigzag Form
- 6. Future Work

Computer algebra systems are currently **unable** to handle case discussion for parametric matrices.



### Jordan canonical form of:

$$\begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha - 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{with} \quad \alpha \in \mathbb{C}$$

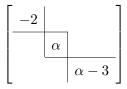


Parametric Matrix Computations

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#### Maple



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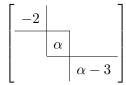
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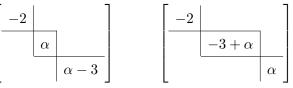
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#### Mathematica



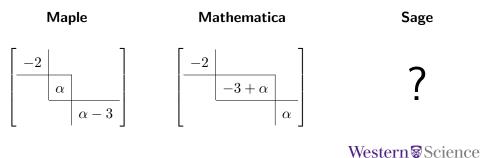




RMC, MMM, SET ()

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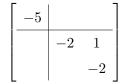
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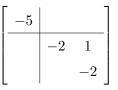
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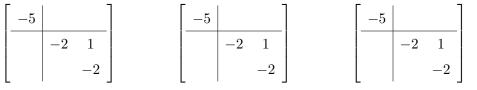
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# REGULAR HAINS



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# $\blacktriangleright$ Let $\mathbbm{K}$ be an algebraically closed of real closed field

- Let  $x_1 < \cdots < x_n$  be  $n \ge 1$  ordered variables
- $\mathbb{K}[x_1,\ldots,x_n]$  is the ring of polynomials in  $x_1,\ldots,x_n$
- For non-constant  $p \in \mathbb{K}[x_1, \dots, x_n]$ , mvar(p) denotes the greatest variable in p
- Leading coefficient of p w.r.t mvar(p) is called the initial denoted init(p)



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# Definition (Triangular Set)

A set T of non-constant polynomials in  $\mathbb{K}[x_1, \ldots, x_n]$  is called a **triangular set** if for all  $p, q \in T$  with  $p \neq q$ ,  $\operatorname{mvar}(p) \neq \operatorname{mvar}(q)$ .

#### Definition (Saturated Ideal)

The saturated ideal sat(T) of T is the ideal  $\langle T \rangle : h_T^{\infty}$ , where  $h_T$  is the product of the initials of the polynomials in T

### Definition (Regular Chain)

T is a **regular chain** if  $T=\varnothing$  or  $T=T'\cup\{t\}$  for  $t\in T$  with  $\mathrm{mvar}(t)$  maximum such that

- $\blacktriangleright$  T' is a regular chain
- $\operatorname{init}(t)$  is regular modulo  $\operatorname{sat}(T')$

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## **Definition (Constructible Set)**

A **constructible set** is the disjunction of systems of polynomial equations and inequations. Where the systems of equations are conjunctions of constraints.



#### Definition (Semi-algebraic System)

A semi-algebraic system of  $\mathbb{K}[x_1, \ldots, x_n]$  is any polynomial system S of the form

$$\left\{\begin{array}{c} f_1 = \dots = f_a = 0\\ g \neq 0\\ p_1 > 0, \dots, p_b > 0\\ q_1 \ge 0, \dots, q_c \ge 0 \end{array}\right\}$$

for  $f_1, \ldots, f_a, g, p_1, \ldots, p_b, q_1, \ldots, q_c \in \mathbb{K}[x_1, \ldots, x_n]$ .

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#### • Parameters $\alpha = \alpha_1, \ldots, \alpha_s$

- $\blacktriangleright$  K is an algebraically closed or real closed field
- $\mathbb{K}[\alpha]$  is the ring of polynomials in the parameters
- $\mathbb{K}(\alpha)$  is the quotient field of  $\mathbb{K}[\alpha]$
- S is a constructible (resp. semi-algebraic) set of  $\mathbb{K}^s$

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- intersection  $S_1 \cap S_2$
- union  $S_1 \cup S_2$
- set theoretic difference  $S_1 \setminus S_2$

<sup>1</sup>C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao. Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:326, 2013.

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$$\begin{split} S_{\text{eq}} &= S \cap V(f(\alpha)) \\ S_{\text{neq}} &= S \setminus V(f(\alpha)) = S \setminus S_{\text{eq}} \,. \end{split}$$

- $f(\alpha)$  is **zero** everywhere on  $S_{eq}$
- $f(\alpha)$  is **nonzero** everywhere on  $S_{neq}$



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# Parametric Matrix

### Definition

A  $m \times n$  parametric matrix takes the form:

$$A(\alpha) = \begin{bmatrix} f_{1,1}(\alpha) & f_{1,2}(\alpha) & \cdots & f_{1,n}(\alpha) \\ f_{2,1}(\alpha) & f_{2,2}(\alpha) & \cdots & f_{2,n}(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,1}(\alpha) & f_{m,2}(\alpha) & \cdots & f_{m,n}(\alpha) \end{bmatrix}$$

for

$$f_{i,j} \in \mathbb{K}(\alpha) \quad 1 \le i \le m, \ 1 \le j \le n.$$

Require constructible (resp. semi-algebraic) set  $S \subseteq \mathbb{K}^s$  such that denominator of every entry of  $A(\alpha)$  is nonzero everywhere on S.



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Input:

- Parametric matrix  $A(\alpha)$
- $\blacktriangleright$  Constraints on the parameter values given in a set S by a polynomial system of the form

$$f_1(\alpha) = \dots = f_a(\alpha) = 0, \ g(\alpha) \neq 0$$

#### Output:

A list with elements of the form  $[r_i, S_i]$  where  $S_i$  gives conditions on  $\alpha$  such that the rank of  $A(\alpha)$  is  $r_i$ 

•  $S_i$ 's form a partition of S

$$\bigcup_{i} S_i = S, \quad S_i \cap S_j = \emptyset \quad \forall i \neq j$$

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# **Step 1:** Define polynomial system S' as **union** of equations of $A(\alpha)X = 0$ and S

Let

 $\mathcal{T} := \texttt{Triangularize}(S', \mathbb{K}[\alpha_1 < \cdots < \alpha_s < x_1 < \cdots < x_n])$ 

For  $0 \leq r \leq n$ , let  $C_r$  be the **constructible set** of  $\mathbb{K}^s$  given by all regular systems  $[T_j \cap \mathbb{K}[\alpha_1 < \cdots < \alpha_s], h_j]$  such that  $[T_j, h_j] \in \mathcal{T}$  and the number of polynomials of  $T_j$  of positive degree in (at least) one of the variables  $x_1 < \cdots < x_n$  is exactly n - r.

For r := n down to 1 do

 $C_r := \text{Difference}(C_r, C_{r-1} \cup \cdots \cup C_0)$ 

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Project out the  $x_i$  variables

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## $\mbox{Example}^3$ over $\mathbb C$

#### Example

Consider  $E\ddot{x} = A_1\dot{x} + A_2x + Bu$ 

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda, \mu \in \mathbb{C}$$

<sup>3</sup>M. I. García-Planas and J. Clotet. Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007. Western Science

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# $\mbox{Example}^3$ over $\mathbb C$

#### Example (cont.)

What is the rank of:

C =	-E	0	0	0	В	0	0	0	0	0
	$-A_1$	-E	0	0	0	В	0	0	0	0
	$A_2$	$-A_1$	-E	0	0	0	В	0	0	0
	0	$A_2$	$-A_1$	-E	0	0	0	В	0	0
	0	0	$A_2$	$-A_1$	0	0	0	0	В	0
	0	0	0	$A_2$	0	0	0	0	0	B

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### Example (cont.)

$$\operatorname{Rank}(C) = \begin{cases} 18 & \text{if} \quad \lambda \neq 0, \, \mu \neq 1/2 \\ 17 & \text{if} \quad \lambda \neq 0, \, \mu = 1/2 \\ 16 & \text{if} \quad \lambda = 0, \, \mu \neq -1 \\ 15 & \text{if} \quad \lambda = 0, \, \mu = -1 \end{cases}$$

<sup>3</sup>M. I. García-Planas and J. Clotet. Analyzing the set of uncontrollable second order generalized linear systems. *International Journal of Applied Mathematics and Informatics*, 2007. Western Science

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### Zigzag Form of Parametric Matrices





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Define a polynomial in  $\boldsymbol{x}$  with parameters  $\boldsymbol{\alpha}$  as

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \dots + f_{r-1}(\alpha)x^{r-1} + x^r$$

with  $f_0(\alpha), \ldots, f_{r-1}(\alpha) \in \mathbb{K}(\alpha)$ .

Require constructible (resp. semi-algebraic) set  $S \subseteq \mathbb{K}^k$  such that denominator of every coefficient is nonzero everywhere on S.



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Frobenius companion matrix in x of a parametric polynomial

$$f(x; \alpha) = f_0(\alpha) + f_1(\alpha)x + \dots + f_{r-1}(\alpha)x^{r-1} + x^r$$

takes the form

$$C_{f(x;\alpha)} = \begin{bmatrix} 0 & \cdots & 0 & -f_0(\alpha) \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -f_{r-2}(\alpha) \\ & & 1 & -f_{r-1}(\alpha) \end{bmatrix}$$

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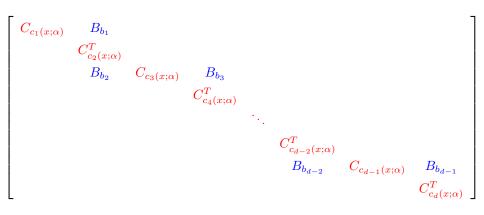
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Let  $b \in \{0, 1\}$ . Define

$$B_b = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$



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<sup>4</sup>A. Storjohann. An  $\mathcal{O}(n^3)$  algorithm for the Frobenius normal form. In Proceedings of ISSAC 1998, pages 101-105. ACM, 1998. Western  $\Im$  Science

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#### Theorem

For every matrix  $A(\alpha) \in \mathbb{K}^{n \times n}[\alpha]$ , there exists a partition  $(S_1, \ldots, S_N)$  of the input constructible (resp. semi-algebraic) set S such that for each  $S_i$ , there exists a matrix  $Z_i(\alpha) \sim A(\alpha)$  in **Zigzag form** where the denominator of the entries of  $Z_i(\alpha)$  are all nonzero everywhere on  $S_i$ .



- The algorithm extends the work of Storjohann<sup>4</sup> on the computation of the Zigzag form.
- The algorithm only uses similarity transformations to obtain the Zigzag form.
- Stages 1 and 3 of Storjohann's algorithm<sup>4</sup> are modified such that computation splits when searching for pivots.

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### Semi-Algebraic Example over ${\mathbb R}$

#### Example

$$A(a,b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix}$$

with 
$$a \ge 0, b > 0, a \ne b$$



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### Semi-Algebraic Example over ${\mathbb R}$

#### Example

$$A(a,b) = \begin{bmatrix} 0 & 0 & 2 \\ a & 0 & 3 \\ 0 & 6 & b \end{bmatrix} \text{ with } a \ge 0, b > 0, a \ne b$$
$$Z_1(a,b) = \begin{bmatrix} 0 & 0 & 12a \\ 1 & 0 & 18 \\ 0 & 1 & b \end{bmatrix} \text{ if } a > 0, b > 0, a \ne b$$
$$Z_2(a,b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 \\ 18 & b \end{bmatrix} \text{ if } a = 0, b > 0$$

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### Algebraic Example over $\mathbb C$

#### Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0\\ -1/2 & \alpha - 2 & 1/2\\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \quad \text{with} \quad \alpha \in \mathbb{C}$$



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### Algebraic Example over $\mathbb{C}$

#### Example

$$A(\alpha) = \begin{bmatrix} -1 & -\alpha - 1 & 0\\ -1/2 & \alpha - 2 & 1/2\\ -2 & 3\alpha + 1 & -1 \end{bmatrix} \text{ with } \alpha \in \mathbb{C}$$

$$Z_{1}(\alpha) = \begin{bmatrix} 0 & 0 & 4\alpha \\ 1 & 0 & 4(\alpha - 1) \\ 0 & 1 & \alpha - 4 \end{bmatrix} \quad \text{if} \quad \alpha + 3 \neq 0$$
$$Z_{2}(\alpha) = \begin{bmatrix} 0 & -4 & 1 \\ 1 & -4 & 0 \\ \hline & & -3 \end{bmatrix} \quad \text{if} \quad \alpha + 3 = 0$$

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### Outline

- 1. Motivation
- 2. Regular Chains
- 3. Background
- 4. Rank
- 5. Zigzag Form

### 6. Future Work



- ► Rank 🗸
- Frobenius (Rational) form
- Minimal polynomial
- Test for similarity
- Jordan form
- ▶ Weyr form<sup>5</sup>



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- Minimize unnecessary splitting
- Parallel implementation
- Cost analysis
- Matrix factorizations



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- Minimize unnecessary splitting
- Parallel implementation

#### Cost analysis

Matrix factorizations

- Speed improvements
  - Minimize unnecessary splitting
  - Parallel implementation
- Cost analysis
- Matrix factorizations

All current and future code and example worksheets will be available at StevenThornton.ca/Code

Information on the RegularChains package of  ${\rm MAPLE}$  is available at RegularChains.org



