Triangular decomposition of semi-algebraic systems

Presented by Marc Moreno Maza[1]
joint work with
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Graduate Computational Algebraic Geometry Seminar
University of Illinois at Chicago
Ocotober 2, 2013

Plan

1. Solving systems of polynomial equations symbolically
2. Triangular decomposition of algebraic systems
3. Triangular decomposition of a semi-algebraic system
4. Algorithm
5. Complexity analysis
6. Benchmarks
7. Applications
8. Concluding remarks

What does solving polynomial systems symbolically mean?

The algebra text book says:

For $F \subset \mathbf{k}[x_1, \ldots, x_n]$ this is simply

- a *primary decomposition* of $\langle F \rangle$ or
- the *irreducible decomposition* of $V(F)$ (the zero set of F in $\overline{\mathbf{k}}^n$).

The computer algebra system does well:

For $F \subset \mathbf{k}[x_1, \ldots, x_n]$, with $\mathbf{k} = \mathbb{Z}/p\mathbb{Z}$ or $\mathbf{k} = \mathbb{Q}$,

- computing a *Gröbner basis* of $\langle F \rangle$ or
- computing a *triangular decomposition* of $V(F)$.

But most scientists and engineers need:

- For $F \subset \mathbb{Q}[x_1, \ldots, x_n]$, a useful description of the points of $V(F)$ whose coordinates are real.
- For $F \subset \mathbb{Q}[u_1, \ldots, u_d][x_1, \ldots, x_n]$, the real $(x_1, \ldots, x_n)$-solutions as a function of the real parameter $(u_1, \ldots, u_d)$.

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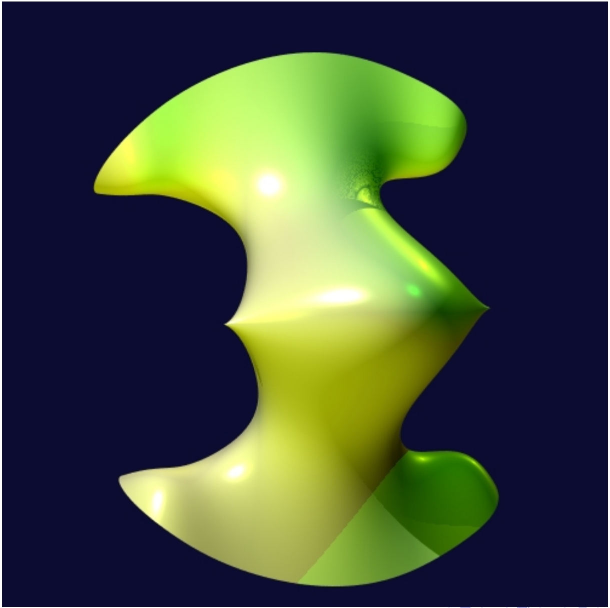
Computing th real points of an algebraic variety (1/2)



Computing th real points of an algebraic variety (2/2)

Applications Places System moreno

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$R := PolynomialRing([x, y, z]);\ F := [5*x\wedge 2 + 2*x*z\wedge 2 + 5*y\wedge 6 + 15*y\wedge 4 + 5*z\wedge 2 - 15*y\wedge 5 - 5*y\wedge 3$

$$polynomial\_ring$$

$$[5\,x^2 + 2\,x\,z^2 + 5\,y^6 + 15\,y^4 + 5\,z^2 - 15\,y^5 - 5\,y^3]$$

$RealTriangularize(F, R, output = record);$

$$\left\{\begin{array}{l} 5\,x^2 + 2\,z^2\,x + 5\,y^6 + 15\,y^4 - 5\,y^3 - 15\,y^5 + 5\,z^2 = 0 \\ 25\,y^6 - 75\,y^5 + 75\,y^4 - z^4 - 25\,y^3 + 25\,z^2 < 0 \end{array}\right.,$$

$$\left\{\begin{array}{c} 5\,x + z^2 = 0 \\ 25\,y^6 - 75\,y^5 + 75\,y^4 - 25\,y^3 - z^4 + 25\,z^2 = 0 \\ 64\,z^4 - 1600\,z^2 + 25 > 0 \\ z \neq 0 \\ z - 5 \neq 0 \\ z + 5 \neq 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x + 5 = 0 \\ y - 1 = 0 \\ z - 5 = 0 \end{array}\right.,$$

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Triangular Set

Definition

$T \subset \mathbf{k}[x_n > \cdots > x_1]$ is a *triangular set* if $T \cap \mathbf{k} = \emptyset$ and $\mathrm{mvar}(p) \neq \mathrm{mvar}(q)$ for all $p, q \in T$ with $p \neq q$.

Theorem (J.F. Ritt, 1932)

Let $V \subset \mathbf{K}^n$ be an irreducible *variety and $F \subset \mathbf{k}[x_1, \cdots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.*

$$(\forall\ g \in \langle \mathbf{F} \rangle)\ \ \mathrm{prem}(g, T) = 0.$$

Theorem (W.T. Wu, 1987)

Let $V \subset \mathbf{K}^n$ be a variety and let $F \subset \mathbf{k}[x_1, \cdots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

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Unfortunately, this procedure cannot decide whether $V = \emptyset$ holds or not.

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Regular chain

Definition

Let $T \subset \mathbf{k}[x_n > \cdots > x_1]$ be a triangular set. For all $t \in T$ write $\mathrm{init}(t) := \mathrm{lc}(t, \mathrm{mvar}(t))$ and $h_T := \prod_{t \in T} \mathrm{init}(t)$. The *quasi-component* and *saturated ideal* of T are:

$$W(T) := V(T) \setminus V(h_T) \quad \text{and} \quad \mathrm{sat}(T) = \langle T \rangle : h_T^{\infty}$$

Theorem (F. Boulier, F. Lemaire and M.M.M. 2006)

We have: $\overline{W(T)} = V(\mathrm{sat}(T))$. *Moreover, if* $\mathrm{sat}(T) \neq \langle 1 \rangle$ *then* $\mathrm{sat}(T)$ *is strongly equi-dimensional.*

Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

T is a *regular chain* if $T = \emptyset$ or $T := T' \cup \{t\}$ with $\mathrm{mvar}(t)$ maximum s.t.

- T' is a regular chain,
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Regular chain: alternative definition

Regular chain: algorithmic properties

Definition

Let $T \subset \mathbf{k}[x_n > \cdots > x_1]$ be a triangular set and $p \in \mathbf{k}[x_n > \cdots > x_1]$. If T is empty then, the *iterated resultant* of p w.r.t. T is $\mathrm{res}(T, p) = p$. Otherwise, writing $T = T_{<w} \cup T_w$

$$\mathrm{res}(T, p) \quad = \quad \begin{cases} p & \text{if } \deg(p, w) = 0 \\ \mathrm{res}(T_{<w}, \mathrm{res}(T_w, p, w)) & \text{otherwise} \end{cases}$$

Theorem (P. Aubry, D. Lazard, M.M.M.)

T is a regular chain *iff*

$$\{p \mid \mathrm{prem}(p, T) = 0\} \quad = \quad \mathrm{sat}(T)$$

Theorem (L. Yang, J. Zhang 1991)

p is regular modulo $\mathrm{sat}(T)$ *iff*

$$\mathrm{res}(T, p) \neq 0$$

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Triangular decomposition of an algebraic variety

Kalkbrener triangular decomposition

Let $F \subset \mathbf{k}[\mathbf{x}]$. A family of regular chains $T_1, \ldots, T_e$ of $\mathbf{k}[\mathbf{x}]$ is called a Kalkbrener triangular decomposition of $V(F)$ if

$$V(F) = \cup_{i=1}^{e} V(\mathrm{sat}(T_i)).$$

Wu-Lazard triangular decomposition

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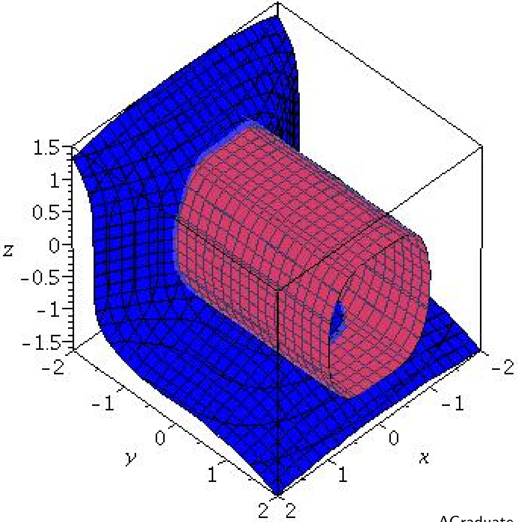
Triangularize applied to *sofa* and *cylinder* (1/2)

$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$



Triangularize applied to *sofa* and *cylinder* (2/2)

Applications Places System — Sun Feb 13, 10:36 PM — changbo

/home/changbo/src/maple/triade/mapleP4/lib/cylinder.mw - [Server 1] - Maple 14

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```
> R := PolynomialRing([z, y, x]): F := [x^2+y^3+z^5, x^4+z^2-1]: dec :=
  Triangularize(F, R): map(Display, dec, R);
```

$$\left[\begin{cases} (-2x^4+x^8+1)\,z+x^2+y^3=0 \\ y^6+2x^2y^3+10x^{12}-10x^8+x^{20}-5x^{16}+6x^4-1=0 \\ -2x^4+x^8+1\neq 0 \end{cases}\right]$$

```
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$$\begin{cases} z=0 \\ y-1=0 \\ x^2+1=0 \end{cases}, \begin{cases} z=0 \\ y^2-y+1=0 \\ x+1=0 \end{cases}, \begin{cases} z=0 \\ y^2-y+1=0 \\ x-1=0 \end{cases}, \begin{cases} z=0 \\ y+1=0 \\ x+1=0 \end{cases},$$

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Regular chain: specialization properties

Notation

Let $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ be a regular chain with $\mathbf{y} := \{\mathrm{mvar}(t) \mid t \in T\}$ and $\mathbf{u} := \mathbf{x} \setminus \mathbf{y} = u_1, \ldots, u_d$. Hence $\mathrm{sat}(T)$ has dimension d.

- *Recall that h_T is the product of the $\mathrm{init}(t)$, for $t \in T$.*
- *Denote by s_T the product of the $\mathrm{discrim}(\mathrm{t}, \mathrm{mvar}(t))$.*

Definition

We say that T *specializes well* at a point $u \in \mathbb{R}^d$ if $h_T(u) \neq 0$ and the triangular set $T(u)$ is a regular chain generating a radical ideal.

Theorem (X. Hou, B. Xia, L. Yang, 2001)

Define $BP_T := \mathrm{res}(T, h_T)\,\mathrm{res}(T, s_T)$, the border polynomial of T. Then

- *T specializes well at $u \in \mathbb{R}^d$ if and only if $BP_T(u) \neq 0$.*
- *For each connected component C of $BP_T(u) \neq 0$, the number of real solutions of $T(u)$ is constant for $u \in C$.*

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Border polynomial and specialization

Example (bad specialization of a regular chain)

$$T := \begin{cases} x_4 x_5{}^2 + 2x_5 + 1 \\ (x_1 + x_2) x_3{}^2 + x_3 + 1 \\ x_1{}^2 - 1. \end{cases} \qquad T_{x_2,x_4=-1,1} := \begin{cases} x_5{}^2 + 2x_5 + 1 \\ (x_1 - 1) x_3{}^2 + x_3 + 1 \\ x_1{}^2 - 1. \end{cases}$$

Example (border polynomial)

$$\mathrm{res}(\mathrm{dis}(t_2), t_1)\, \mathrm{res}(\mathrm{res}(\mathrm{dis}(t_3), t_2), t_1).\ \mathrm{res}(\mathrm{init}(t_2), t_1)\, \mathrm{res}(\mathrm{res}(\mathrm{init}(t_3), t_2), t_1).$$

For the above regular chain, it is

$$(4x_2 + 3)(4x_2 - 5)(x_2^2 - 1)(x_4 - 1)x_4$$

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Regular semi-algebraic system

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- *Let* $T \subset \mathbb{Q}[x_1 < \ldots < x_n]$ *be a regular chain with* $\mathbf{y} := \{\mathrm{mvar}(t) \mid t \in T\}$ *and* $\mathbf{u} := \mathbf{x} \setminus \mathbf{y} = u_1, \ldots, u_d$*.*
- *Let* P *be a finite set of polynomials, s.t. every* $f \in P$ *is regular modulo* $\mathrm{sat}(T)$*.*
- *Let* $\mathcal{Q}$ *be a quantifier-free formula of* $\mathbb{Q}[\mathbf{u}]$*.*

Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(i) $\mathcal{Q}$ defines a non-empty open semi-algebraic set S in $\mathbb{R}^d$,

(ii) the regular system $[T, P]$ specializes well at every point u of S

(iii) at each point u of S, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

Define

$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

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Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(*i*) $\mathcal{Q}$ defines a non-empty open semi-algebraic set S in $\mathbb{R}^d$,

(*ii*) the regular system $[T, P]$ specializes well at every point u of S

(*iii*) at each point u of S, the specialized system $[T(u), P(u)_>]$ has at least one real solution.

Define

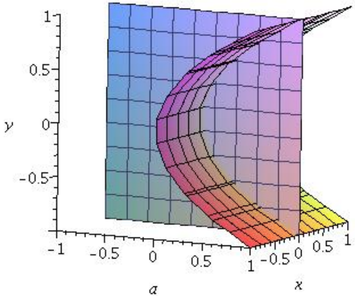
$$Z_{\mathbb{R}}(R) = \{(u, y) \mid \mathcal{Q}(u), t(u, y) = 0, p(u, y) > 0, \forall (t, p) \in T \times P\}.$$

Example

The system $[\mathcal{Q}, T, P_>]$, where

$$\mathcal{Q} := a > 0, \;\; T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases}, \;\; P_> := \{y > 0\}$$

is a regular semi-algebraic system.



Triangular decompositions of semi-algebraic systems (1/2)

Proposition

Let $R := [\mathcal{Q}, T, P_>]$ be a regular semi-algebraic system of $\mathbb{Q}[u_1, \ldots, u_d, \mathbf{y}]$. Then the zero set of R is a nonempty semi-algebraic set of dimension d.

Theorem

Every semi-algebraic system $\mathcal{S}$ can be decomposed as a finite union of regular semi-algebraic systems such that the union of their zero sets is the zero set of $\mathcal{S}$. We call it a (full) triangular decomposition of $\mathcal{S}$.

Triangular decompositions of semi-algebraic systems (2/2)

Notation

Let $\mathcal{S} = [F, N_{\geq}, P_{>}, H_{\neq}]$ be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Let c be the dimension of the constructible set of $\mathbb{C}^n$ corresponding to $\mathcal{S}$.

Definition

A finite set of regular semi-algebraic systems R_i is called a lazy triangular decomposition of $\mathcal{S}$ if

- for each i, $Z_{\mathbb{R}}(R_i) \subseteq Z_{\mathbb{R}}(\mathcal{S})$ holds, and
- there exists $G \subset \mathbb{Q}[\mathbf{x}]$ such that

$$Z_{\mathbb{R}}(\mathcal{S}) \setminus \left(\cup_{i=1}^{t} Z_{\mathbb{R}}(R_i)\right) \subseteq Z_{\mathbb{R}}(G),$$

where the complex zero set $V(G)$ has dimension less than c.

A detailed example

Original problem

Consider the following question (Brown, McCallum, ISSAC'05): when does $p(z) = z^3 + az + b$ have a non-real root $x + iy$ satisfying $xy < 1$.

The equivalent quantifier elimination problem

$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})[f = g = 0 \wedge y \neq 0 \wedge xy - 1 < 0]$, where

- $f = \mathrm{Re}(p(x + iy)) = x^3 - 3xy^2 + ax + b$
- $g = \mathrm{Im}(p(x + i))/y = 3x^2 - y^2 + a$

The semi-algebraic system to solve

$$\mathcal{S} := \begin{cases} f = 0, \\ g = 0, \\ y \neq 0, \\ xy - 1 < 0 \end{cases}$$

A lazy triangular decomposition

The command `LazyRealTriangularize`$([f, g, y \neq 0, xy-1 < 0], [y, x, b, a])$ returns the following:

$$\begin{cases} [\{t_1 = 0, t_2 = 0, 1 - xy > 0\}] & h_1 > 0, h_2 \neq 0 \\ \texttt{\%LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, \\ h_1 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & h_1 = 0 \\ \texttt{\%LazyRealTriangularize}([t_1 = 0, t_2 = 0, f = 0, \\ h_2 = 0, 1 - xy > 0, y \neq 0], [y, x, b, a]) & h_2 = 0 \\ [\,] & \text{otherwise} \end{cases}$$

where

$$t_1 = 8x^3 + 2ax - b, t_2 = 3x^2 - y^2 + a,$$
$$h_1 = 4a^3 + 27b^2,$$
$$h_2 = -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096.$$

A full triangular decomposition

Evaluate the output with the `value` command, which yields

$$\begin{cases} [\{t_1 = 0, t_2 = 0, 1 - xy > 0\}] & h_1 > 0, h_2 \neq 0 \\ [\,] & h_1 = 0 \\ [\{t_3 = 0, t_4 = 0, h_2 = 0\}] & h_2 = 0 \\ [\,] & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} t_3 &= (2a^3 + 32a + 18b^2)x - a^2b - 48b \\ t_4 &= xy + 1 \\ h_1 &= 4a^3 + 27b^2, \\ h_2 &= -4a^3b^2 - 27b^4 + 16a^4 + 512a^2 + 4096 \end{aligned}$$

Plan

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5 Complexity analysis

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8 Concluding remarks

Outline of the algorithm

Definition

Let $[T, P]$ be as before and $B \subset \mathbb{Q}[\mathbf{u}]$. We say that $[B_{\neq}, T, P_{>}]$ is a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$ if $[T, P]$ specializes well at every point of $B(\mathbf{u}) \neq 0$.

Computation in complex space

$$Z_{\mathbb{R}}(F, N_{\geq}, P_{>}, H_{\neq})$$
$$\downarrow$$
$$\bigcup Z_{\mathbb{R}}(B_{\neq}, T, P_{>})$$

Computation in real space

$$[B_{\neq}, T, P_{>}]$$
$$\downarrow$$
$$\mathcal{Q} := \exists \mathbf{y}\,(B(\mathbf{u}) \neq 0,\ T(\mathbf{u}, \mathbf{y}) = 0,\ P(\mathbf{u}, \mathbf{y}) > 0)$$
$$\downarrow$$
$$\text{output } [\mathcal{Q}, T, P_{>}],\ \text{where } \mathcal{Q} \neq \text{false}$$

Fingerprint polynomial set

Definition

Let $R := [B_{\neq}, T, P_{>}]$. Let $D \subset \mathbb{Q}[\mathbf{u}]$. Let dp and b be the product of D and B. We call D a *fingerprint polynomial set* (FPS) of R if:

(i) for all $\alpha \in \mathbb{R}^d$, $b \in B$: $dp(\alpha) \neq 0 \;\;\Rightarrow\;\; b(\alpha) \neq 0$,

(ii) for all $\alpha, \beta \in \mathbb{R}^d$ with $\alpha \neq \beta$ and $dp(\alpha) \neq 0$, $dp(\beta) \neq 0$, if for $p \in D$, $\mathrm{sign}(p(\alpha)) = \mathrm{sign}(p(\beta))$, then $R(\alpha)$ has real solutions iff $R(\beta)$ does.

Open projection operator (Brown-McCalumn operator)

Let A be a squarefree basis in $\mathbb{Q}[u_1 < \cdots < u_d]$. Define

$$\mathrm{oproj}(A, u_d) := \bigcup_{f \in A} \mathrm{lc}(f, u_d) \cup \bigcup_{f \in A} \mathrm{discrim}(f, u_d) \cup \bigcup_{f,g \in A} \mathrm{res}(f, g, u_d).$$

Theorem

For $A \subset \mathbb{Q}[u_1, \ldots, u_d]$, let $\mathrm{oaf}(A) = \mathrm{der}(A, u_d) \cup \mathrm{oaf}(\mathrm{oproj}(\mathrm{der}(A, u_d), u_{d-1}))$. *If $R := [B_{\neq}, T, P_{>}]$ is a PRSAS, then,* $\mathrm{oaf}(B)$ *is a fingerprint polynomial*

Fingerprint polynomial set

Definition

Let $R := [B_{\neq}, T, P_{>}]$. Let $D \subset \mathbb{Q}[\mathbf{u}]$. Let dp and b be the product of D and B. We call D a *fingerprint polynomial set* (FPS) of R if:

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For $A \subset \mathbb{Q}[u_1, \ldots, u_d]$, let $\mathrm{oaf}(A) = \mathrm{der}(A, u_d) \cup \mathrm{oaf}(\mathrm{oproj}(\mathrm{der}(A, u_d), u_{d-1}))$. If $R := [B_{\neq}, T, P_{>}]$ is a PRSAS, then, $\mathrm{oaf}(B)$ is a fingerprint polynomial

A detailed example (1/3)

Applications Places System moreno

/home/moreno/src/Talks/ApplMath-UWO-2011/bp-fps.mw* - [Server 1] - Maple 14

nat Table Drawing Plot Spreadsheet Tools Window Help

RealRootIsolate, RefineBox, RefineListBox, RemoveRedundantComponents, RepresentingBox, RepresentingChain, RepresentingQuantifierFreeFormula, RepresentingRootIndex, VariableOrdering]

```
> F := [x^2+y+a, y^2+b*x+a];
```

$$F := [y + x^2 + a,\ y^2 + b\,x + a]$$

```
> R := PolynomialRing([y, x, b, a]);
```

$$R := polynomial\_ring$$

```
> dec := Triangularize(F, R);
  rc := dec[1];
```

$$dec := [regular\_chain]$$

$$rc := regular\_chain$$

```
> Display(rc, R);
```

$$\begin{cases} y + x^2 + a = 0 \\ x^4 + 2\,a\,x^2 + b\,x + a^2 + a = 0 \end{cases}$$

```
> rceqs := Info(rc, R);
```

$$rceqs := [y + x^2 + a,\ x^4 + 2\,a\,x^2 + b\,x + a^2 + a]$$

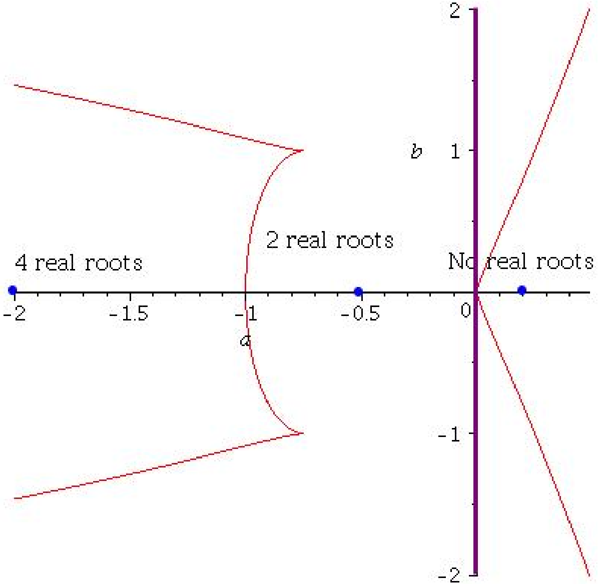
```
> bp := discrim(rceqs[2],x);
```

$$bp := 256\,b^2\,a^3 - 27\,b^4 + 288\,b^2\,a^2 + 256\,a^4 + 256\,a^3$$

```
> with(plots);
```

[animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, conformal,

A detailed example (2/3)



A detailed example (3/3)

Applications Places System moreno Fri 4 Feb,

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polynomial_ring

$$[x^2+y+a,\ y^2+b\,x+a]$$

$dec := Triangularize(F, R) : rc := dec[1] : Display(rc, R);$

$$\begin{cases} y+x^2+a=0 \\ x^4+2\,a\,x^2+b\,x+a^2+a=0 \end{cases}$$

$LazyRealTriangularize(F, R, output = record);$

$$\begin{cases} y+x^2+a=0 \\ x^4+2\,a\,x^2+b\,x+a^2+a=0 \\ 256\,b^2\,a^3-27\,b^4+288\,b^2\,a^2+256\,a^4+256\,a^3<0 \\ a\neq 0 \end{cases},$$

$$\begin{cases} y+x^2+a=0 \\ x^4+2\,a\,x^2+b\,x+a^2+a=0 \\ 256\,b^2\,a^3-27\,b^4+288\,b^2\,a^2+256\,a^4+256\,a^3>0 \\ a<0 \end{cases}, \%LazyRealTriangularize\big([a=0,\ y+x^2+a$$

$$=0,\ y^2+b\,x+a=0,\ x^4+2\,a\,x^2+b\,x+a^2+a=0],\ polynomial\_ring,\ output=record\big),$$

$$\%LazyRealTriangularize\big([y+x^2+a=0,\ y^2+b\,x+a=0,\ 256\,b^2\,a^3-27\,b^4+288\,b^2\,a^2+256\,a^4+256\,a^3$$

$$=0,\ x^4+2\,a\,x^2+b\,x+a^2+a=0],\ polynomial\_ring,\ output=record\big)$$

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LazyRealTriangularize for a system of equations

Algorithm 1: LazyRealTriangularize($\mathcal{S}$)

Input: a semi-algebraic system $\mathcal{S} = [F, \emptyset, \emptyset, \emptyset]$
Output: a lazy triangular decomposition of $\mathcal{S}$
$\mathcal{T} :=$ Triangularize(F)
for $T_i \in \mathcal{T}$ **do**
  $Bp_i :=$ BorderPolynomial($T_i, \emptyset$)
  solve $\exists \mathbf{y}(Bp_i(\mathbf{u}) \neq 0, T_i(\mathbf{u}, \mathbf{y}) = 0)$,
  and let $\mathcal{Q}_i$ be the resulting quantifier-free formula
  if $\mathcal{Q}_i \neq$ *false* **then** output $[\mathcal{Q}_i, T_i, \emptyset]$

Complexity results (1/2)

Assumptions

($\mathbf{H_0}$) $V(F)$ is equidimensional of dimension d,

($\mathbf{H_1}$) $x_1, \ldots, x_d$ are algebraically independent modulo each associated prime ideal of the ideal generated by F in $\mathbb{Q}[\mathbf{x}]$,

($\mathbf{H_2}$) F consists of $m := n - d$ polynomials, $f_1, \ldots, f_m$.

Geometrical formulation

Hypotheses ($\mathbf{H_0}$) and ($\mathbf{H_1}$) are equivalent to the existence of regular chains $T_1, \ldots, T_e$ of $\mathbb{Q}[x_1, \ldots, x_n]$ such that

- $x_1, \ldots, x_d$ are free w.r.t. each T_i
- $V(F) = V(\mathrm{sat}(T_1)) \cup \ldots \cup V(\mathrm{sat}(T_e))$.

Complexity results (2/2)

Notation

Let n, m, δ, $\hbar$ be respectively the number of variables, number of polynomials, maximum total degree and height of polynomials in F.

Proposition

Within $m^{O(1)}(\delta^{O(n^2)})^{d+1} + \delta^{O(m^4)O(n)}$ operations in $\mathbb{Q}$, one can compute a Kalkbrener triangular decomposition $E_1, \ldots, E_e$ of $V(F)$, where each polynomial of each E_i

- has total degree upper bounded by $O(\delta^{2m})$,
- has height upper bounded by $O(\delta^{2m}(m\hbar + dm\log(\delta) + n\log(n)))$.

From which, a lazy triangular decomposition of F can be computed in $\left(\delta^{n^2} n 4^n\right)^{O(n^2)} \hbar^{O(1)}$ bit operations.

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Notations

Table 1 Notions for Tables 2 and 3

symbol	meaning
#e	number of equations in the system
#v	number of variables in the equations
d	max total degree of the equations
G	Groebner:-Basis (with plex order) in MAPLE 13
T	Triangularize in REGULARCHAINS library of MAPLE
LR	lazy RealTriangularize implemented in MAPLE
R	complete RealTriangularize implemented in MAPLE
Q	QEPCAD B
$> 1h$	the examples cannot be solved in 1 hour
FAIL	QEPCAD B failed due to prime list exhausted

Timings for algebraic varieties

Table 2 Timings for algebraic varieties

system	#v/#e/d	G	T	LR
Hairer-2-BGK	13/ 11/ 4	25	1.924	2.396
Collins-jsc02	5/ 4/ 3	876	0.296	0.820
Leykin-1	8/ 6/ 4	103	3.684	3.924
8-3-config-Li	12/ 7/ 2	109	5.440	6.360
Lichtblau	3/ 2/ 11	126	1.548	11
Cinquin-3-3	4/ 3/ 4	64	0.744	2.016
Cinquin-3-4	4/ 3/ 5	$> 1h$	10	22
DonatiTraverso-rev	4/ 3/ 8	154	7.100	7.548
Cheaters-homotopy-1	7/ 3/ 7	3527	174	$> 1h$
hereman-8.8	8/ 6/ 6	$> 1h$	33	62
L	12/ 4/ 3	$> 1h$	0.468	0.676
dgp6	17/19/ 2	27	60	63
dgp29	5/ 4/ 15	84	0.008	0.016

Timings for semi-algebraic systems

Table 3 Timings for semi-algebraic systems

system	#v/#e/d	T	LR	R	Q
BM05-1	4/ 2/ 3	0.008	0.208	0.568	86
BM05-2	4/ 2/ 4	0.040	2.284	$> 1h$	FAIL
Solotareff-4b	5/ 4/ 3	0.640	2.248	924	$> 1h$
Solotareff-4a	5/ 4/ 3	0.424	1.228	8.216	FAIL
putnam	6/ 4/ 2	0.044	0.108	0.948	$> 1h$
MPV89	6/ 3/ 4	0.016	0.496	2.544	$> 1h$
IBVP	8/ 5/ 2	0.272	0.560	12	$> 1h$
Lafferriere37	3/ 3/ 4	0.056	0.184	0.180	10
Xia	6/ 3/ 4	0.164	2.192	230.198	$> 1h$
SEIT	11/ 4/3	0.400	33.914	$> 1h$	$> 1h$
p3p-isosceles	7/ 3/ 3	1.348	$> 1h$	$> 1h$	$> 1h$
p3p	8/ 3/ 3	210	$> 1h$	$> 1h$	FAIL
Ellipse	6/ 1/ 3	0.012	0.904	$> 1h$	$> 1h$

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Laurent's model for the mad cow disease (1/4)

The dynamical system ruling the transformation

The normal form PrP^C is harmless, while the infectious form PrP^{S_c} catalyzes a transformation from the normal form to the infectious one.

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & k_1 - k_2 x - ax\frac{(1+by^n)}{1+cy^n} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & ax\frac{(1+by^n)}{1+cy^n} - k_4 y \end{array} \right.$$

where $x = \left[PrP^C\right]$, $y = \left[PrP^{S_c}\right]$ and where b, c, n, a, k_4, k_1 are biological constants which can be set as follows:

$$b = 2, \quad c = 1/20, \quad n = 4, \quad a = 1/10, \quad k_4 = 50 \ \text{ and } \ k_1 = 800.$$

The dynamical system to study

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{2(x+2xy^4-500y-25y^5)}{20+y^4} \end{array} \right.$$

Laurent's model for the mad cow disease (2/4)

The semi-algebraic system to be solved

$$\mathcal{S} := \left\{ \begin{array}{rcl} 16000 + 800y^4 - 20k_2x - k_2xy^4 - 2x - 4xy^4 & = & 0 \\ 2(x + 2xy^4 - 500y - 25y^5) & = & 0 \\ k_2 & > & 0 \end{array} \right.$$

Computations (1/5)

LazyRealTriangularize to this system, yields the following regular semi-algebraic system (and unevaluated recursive calls)

$$\left\{ \begin{array}{r} (2y^4+1)x - 500y - 25y^5 = 0 \\ (k_2+4)y^5 - 64y^4 + (20k_2+2)y - 32 = 0 \\ (k_2 > 0) \ \wedge \ (R_1 \neq 0) \end{array} \right.$$

where

$$R_1 = 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 - 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056.$$

Laurent's model for the mad cow disease (3/4)

Computations (2/5)

Through the computation of sample points, we easily obtain the following observation. Whenever $R_1 > 0$ holds, the system has 1 equilibrium, while $R_2 < 0$ implies that the system has 3 equilibria.

Computations (3/5)

Now we study the stability of those equilibria. To this end, we consider the two Hurwitz determinants.
Adding to $\mathcal{S}$ the constraints $\{\Delta_1 > 0, a_2 > 0\}$

$$\Delta_1 = 54y^8 + 40k_2y^4 + 2082y^4 - 312xy^3 + 20040 + k_2y^8 + 400k_2,$$
$$a_2 = 20000k_2 + 2000 + 50k_2y^8 + 200y^8 + 2000k_2y^4 - 312k_2xy^3 + 4100y^4.$$

we obtain a new semi-algebraic system $\mathcal{S}'$.

Laurent's model for the mad cow disease (4/4)

Computations (4/5)

Applying LazyRealTriangularize to $\mathcal{S}'$ in conjunction with sample point computations brings the following conclusion. If $R_1 > 0$, then the system has 1 asymptotically stable hyperbolic equilibria.

Computations (5/5)

If $R_1 < 0$ and $R_2 \neq 0$, then System has 2 asymptotically equilibria, where R_2 is given by:

$$
\begin{aligned}
R_2 = {} & 10004737927168 k_2^9 + 624166300700672 k_2^8 + 7000539052537600 k_2^7 \\
& + 45135589467012800 k_2^6 - 840351411856453750 k_2^5 - 5009800435224846875 k_2^4 \\
& - 2738816898945500000 k_2^3 - 867520926669600000 k_2^2 \\
& + 102960917356800000000 k_2 + 59325460641024000000000.
\end{aligned}
$$

To further investigate the number of asymptotically stable hyperbolic equilibria on the hypersurface $R_2 = 0$ and the equilibria when $R_1 = 0$, one can apply SamplePoints on $\mathcal{S}'$, which produces 14 points.

Program verification: an example from Lafferriere (1/4)

Reachability computation

This problem reduces to determine the set

$$\{(y_1, y_2) \in \mathbb{R}^2 \mid (\exists a \in \mathbb{R})(\exists z \in \mathbb{R})\, (0 \leq a) \wedge (z \geq 1) \wedge (h_1 = 0) \wedge (h_2 = 0)\}$$

where

$$h_1 = 3\, y_1 - 2\, a(-z^4 + z) \;\; \text{and} \;\; h_2 = 2\, y_2 z^2 - a(z^4 - 1).$$

The semi-algebraic system to be solved

One wishes to compute the projection of the semi-algebraic set defined by

$$(0 \leq a) \wedge (z \geq 1) \wedge (h_1 = 0) \wedge (h_2 = 0)$$

onto the (y_1, y_2)-plane.
For the variable ordering $a > z > y_1 > y_2$. we obtain the five following regular semi-algebraic systems R_1 to R_5

Program verification: an example from Lafferriere (2/4)

The triangular decomposition (1/3)

$$R_2^T = \begin{cases} a \\ y_1 \\ y_2 \end{cases} \qquad R_3^T = \begin{cases} z-1 \\ y_1 \\ y_2 \end{cases} \qquad R_4^T = \begin{cases} a \\ z-1 \\ y_1 \\ y_2 \end{cases}$$

$$R_2^P = \{\ z > 1 \qquad R_3^P = \{\ 0 < a$$

The projection on the (y_1, y_2)-plane of $Z_{\mathbb{R}}(R_2) \cup Z_{\mathbb{R}}(R_3) \cup Z_{\mathbb{R}}(R_4)$ is clearly equal to the $(y_1, y_2) = (0, 0)$ point.

Program verification: an example from Lafferriere (3/4)

The triangular decomposition (2/3)

$$R_1^T = \begin{cases} (z^4 - 1)\, a - 2\, z^2 y_2 \\ 4\, y_2\, z^5 + 4\, y_2\, z^4 + (3\, y_1 + 4\, y_2)\, z^3 + 3\, y_1\, z^2 + 3\, y_1\, z + 3\, y_1 \end{cases}$$

$$R_1^{\mathcal{Q}} = \begin{cases} (y_1 + y_2 < 0) \wedge (y_1 < 0) \wedge (0 < y_2) \\ 3y_1^5 - 6y_2y_1^4 - 63y_2^2y_1^3 + 192y_2^3y_1^2 + 112y_2^4y_1 + 16y_2^5 \neq 0 \end{cases}$$

$$R_1^P = \{\ z > 1$$

The projection on the (y_1, y_2)-plane of $Z_{\mathbb{R}}(R_1)$ is given by $Z_{\mathbb{R}}(R_1^{\mathcal{Q}})$.

Program verification: an example from Lafferriere (4/4)

The triangular decomposition (3/3)

$$R_5^T = \begin{cases} \left(z^4 - 1\right) a - 2\, z^2 y_2 \\ t_z \\ 3\, {y_1}^5 - 6\, y_2\, {y_1}^4 - 63\, {y_2}^2 {y_1}^3 + 192\, {y_2}^3 {y_1}^2 + 112\, {y_2}^4 y_1 + 16\, {y_2}^5 \end{cases}$$

$$R_5^Q = \{\ 0 < y_2 \qquad R_5^P = \{\ z > 1$$

where t_z is a large polynomial of degree 4 in z.
The polynomial with main variable y_1, say t_{y_1} is delineable above $0 < y_2$.
Using a sample point we check that t_{y_1} admits a single real root.

Conclusion

It follows that the projection on the (y_1, y_2)-plane of $Z_{\mathbb{R}}(R_5)$ is given by:

$$(0 < y_2) \wedge (3\, {y_1}^5 - 6\, y_2\, {y_1}^4 - 63\, {y_2}^2 {y_1}^3 + 192\, {y_2}^3 {y_1}^2 + 112\, {y_2}^4 y_1 + 16\, {y_2}^5).$$

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8. Concluding remarks

Conclusion

- We have proposed adaptations of the notions of regular chains and triangular decompositions in order to solve semi-algebraic systems symbolically.
- We have shown that any such system can be decomposed into finitely many *regular semi-algebraic systems*.
- We propose two specifications of such a decomposition and present corresponding algorithms:
- Under some assumptions, one type of decomposition (`LazyRealTriangularize`) can be computed in singly exponential time w.r.t. the number of variables.
- We have implemented both types of decompositions and reported on comparative benchmarks.
- Our experimental results suggest that these approaches are promising.

Recent work

- We have obtained geometrical invariants for the notion of border polynomial.
- We have improved the performances of our algorithms by avoiding unnecessary recursive calls
- We have developed an incremental algorithms for decomposing semi-algebraic systems
- We have procedures for performing set theoretical operations on semi-algebraic sets.
- As a consequence we can produce decomposition free of redundant components.

Thank you!