Design of nonseparable bidimensional wavelets and filter banks using Gröbner bases techniques and triangualr systems

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Summary

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2. Generating the system and reformulation
3. Solving the problem
4. Experimentation
5. Conclusion and results

Statement of the problem

$$W = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad H = \left| \begin{matrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{matrix} \right.$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad u = \left| \begin{matrix} 1 \\ z_1 \\ z_2 \\ z_1 z_2 \end{matrix} \right.$$

$$D(z_1, z_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 z_2 \end{bmatrix}$$

$$R_i = \begin{bmatrix} cos\alpha_i & -sin\alpha_i & 0 & 0 \\ sin\alpha_i & cos\alpha_i & 0 & 0 \\ 0 & 0 & cos\beta_i & -sin\beta_i \\ 0 & 0 & sin\beta_i & cos\beta_i \end{bmatrix}$$

Define

(1) $H = A_0 A_1 \cdots A_K u$

(2) $A_0 = R_0 WP,\ A_i = D(z_1, z_2)PWR_iWP$

The problem : solve the Vetterli and Kovacevic equation for filter banks.

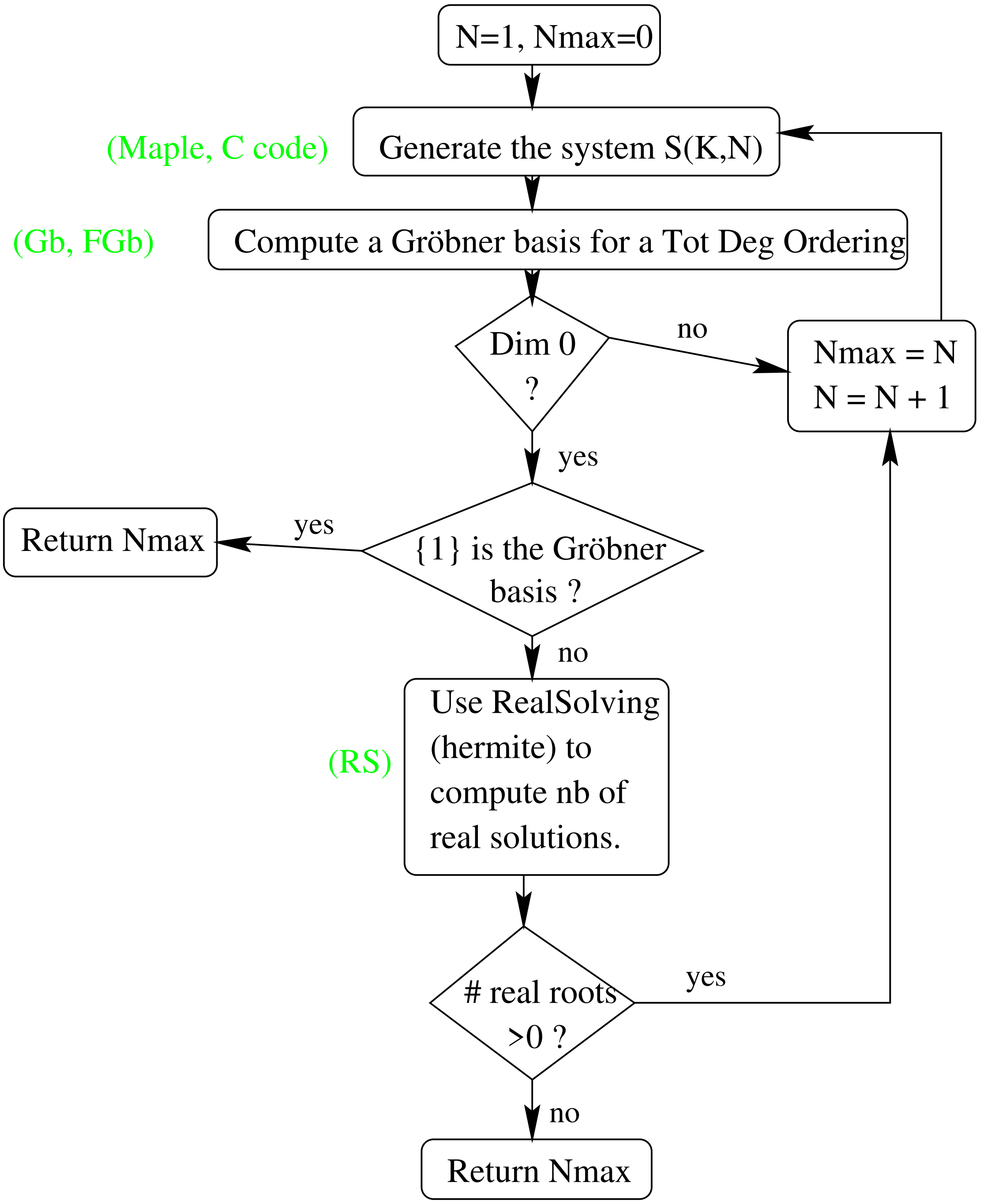
Given an integer K,

task 1 compute the biggest N such that $S(K, N)$ has real solutions where

$$S(K,N) = \left\{ \frac{\partial^k H_i}{\partial^{k_1} z_1 \partial^{k_2} z_2}(+/-1, +/-1) = 0 \right.$$
$$i = 1,2,3,4 \ \ 0 \le k = k_1 + k_2 \le N$$

task 2 solve $S(K, N_{\max})$

Computing the maximum flattness $N_{\max}$



First Results

- This idealistic algorithm gives N_{max} for $K \leq 4$.

- Generating $S(N, K)$ in Maple is very hard ($K < 7$ only).
  $\Longrightarrow$ need better techniques : use of complex expon., taylor series and very efficient C code (add in Gb)
  $\Longrightarrow$ we can now generate $S(K, N)$ for $K \leq 10,\ N \leq 6$.

- Input systems of the Gröbner basis computation are **Huge** (for instance 60 Meg for $K = 8, N = 6$)

- Except for $K = 3$, $S(K, N_{\max})$ is not a zero dimensional system.

- In general $S(K, N_{\max} + 1)$ are not too difficult to compute but $S(K, N_{\max})$ is very difficult (even DRL ordering, mod p) ($K \leq 3$ Lex Order only).

Change of variables

We use incremental Gröbner basis computations:

K given, begin with $N = 2$, compute $S(K,2)$ and

$G = Groebner(S(K,2))$

For $i = 3$ to $N_{\max}$ do

Compute $S = S(K,i) \backslash S(K,i-1)$

$X_i = e^{\alpha_i}, x_i = \frac{1}{X_i}, \imath$ (complex number)

Reduce S by G with the following modifications:

Reduce in Complex Integers $\imath^2 + 1 = 0$

Reduce with “inverse” variables $x_i X_i - 1 = 0$

$\Longrightarrow$ System S is now smaller.

Other substitutions

$$
\begin{aligned}
\alpha_1 &= a_1 - \tfrac{a_2 - b_2}{2} \\
\beta_1 &= \tfrac{a_2 + b_2}{2} \\
\alpha_k &= \tfrac{a_k - b_k}{2} - \tfrac{a_{k+1} - b_{k+1}}{2} \text{ for } k = 2 \ldots K-1 \\
\beta_k &= \tfrac{a_{k+1} + b_{k+1}}{2} - \tfrac{a_k + b_k}{2} \text{ for } k = 2 \ldots K-1 \\
\alpha_K &= \tfrac{a_K - b_K}{2} \\
\beta_K &= b_1 - \tfrac{a_K + b_K}{2}
\end{aligned}
$$

$\Longrightarrow$ equations are now very simple (integer coeffs)

$\Longrightarrow$ $S(N, K)$ splits into two **identical** sub-systems : one for the a_k and one for b_k. In each sub-system: K variables instead of $4K$

- $a_1 = -\frac{\pi}{4}$

$$\text{Let } A_1 = \frac{1}{2}, \quad A_k = e^{i(a_k - a_1)} \text{ for } k = 2 \ldots K$$

The reformulated system

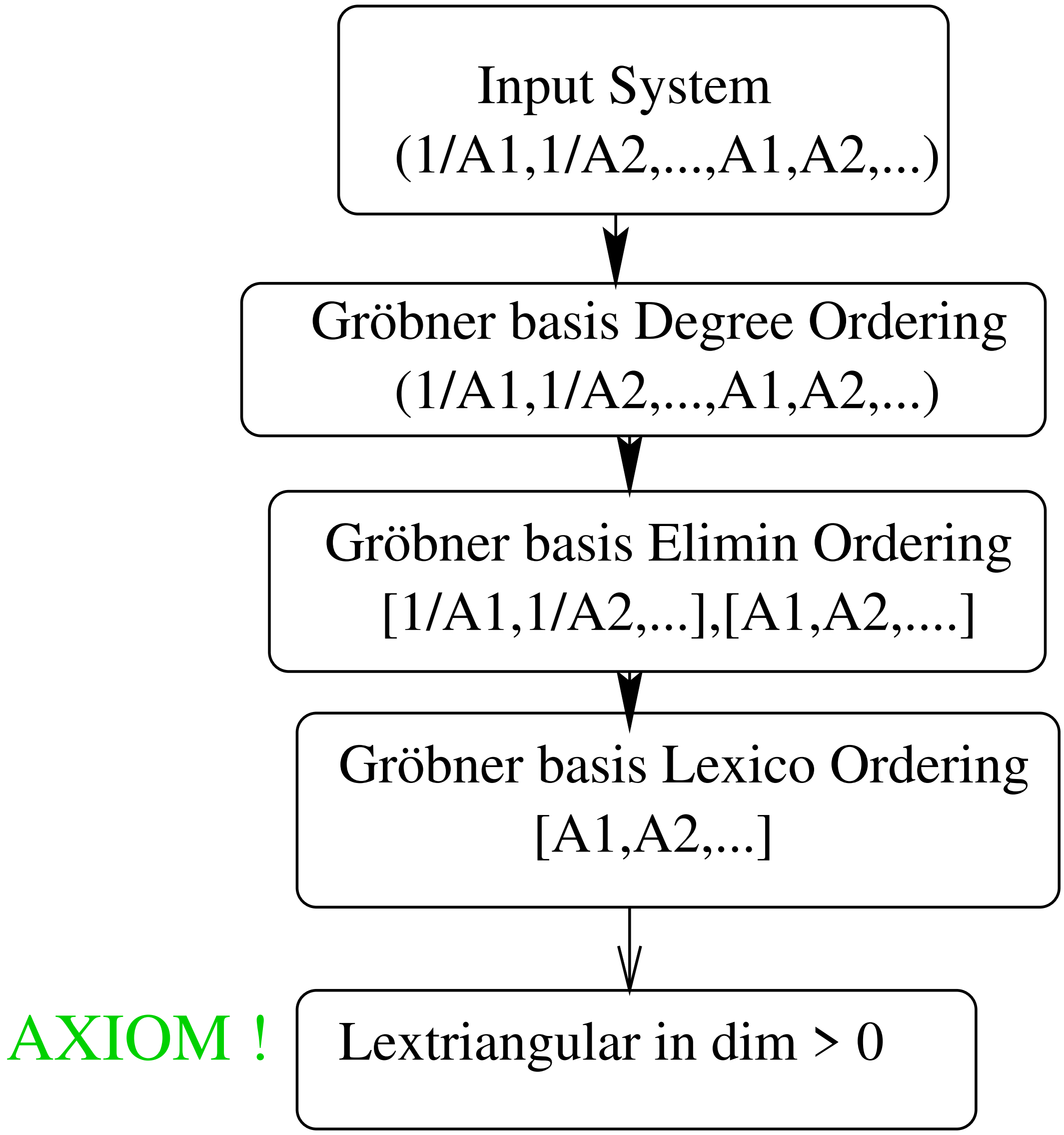
$A_2, \ldots, A_K$ must satisfy the following equations :

$$
\begin{array}{ll}
N=2 & \sum_{k=1}^{K} A_k = 0 \\
N=3 & \sum_{k=2}^{K} \frac{1}{A_k} \left(\sum_{j=1}^{k-1} A_j - \sum_{j=k+1}^{K} A_j \right) = -\frac{1}{4} \\
N=4 & \sum_{k=2}^{K-1} \frac{1}{A_k} \left(\sum_{j=k+1}^{K} A_j \right) \left(\sum_{j=1}^{k-1} A_j \right) = -\frac{1}{8} \\
N=5 & \sum_{i=4}^{K} \sum_{j=2}^{i-1} \frac{1}{A_i A_j} \left(\sum_{k=1}^{j-1} A_k - \sum_{i+1}^{K} A_k \right) \left(\sum_{k=j+1}^{i-1} A_k \right) = \frac{4K-7}{32}
\end{array}
$$

The problem is now to solve the system $S'(K, N)$ with $A_2, \ldots, A_k$ as unknowns, A_i complex number of modulus 1.

If $f(A_1, A_2, \ldots)$ is an equation we known that $\overline{f(A_1, A_2, \ldots)}$ is an equation, that is to say $f(\frac{1}{A_1}, \frac{1}{A_2}, \ldots)$ also (cf cyclic n)

The solving process



Lazard's decompositions

• Every polynomial is viewed as univariate in its main variable $\rightsquigarrow$ triangular sets.

If $T = \{t_1, \dots, t_s\} \subseteq \mathbf{k}[x_1, \dots, x_n]$ is a triangular set define

$$(i)\quad \mathbf{W}(T) := \mathbf{V}(t_1, \dots, t_s) \setminus \mathbf{V}(h)$$

$$(ii)\quad \mathsf{Sat}(T) :=< T >: h^\infty$$

where $h = \mathsf{init}(t_1) \cdots \mathsf{init}(t_s)$.

• If T is a **Lazard set**, then $\mathbf{k}[x_1, \dots, x_n]/\mathsf{Sat}(T)$ is a product of field $\rightsquigarrow$ gcd computations.

For $F \subseteq \mathbf{k}[x_1, \dots, x_n]$ there exist an algorithm to compute $T_1, \dots, T_l$ such that

$$(i)\quad \mathbf{V}(F) = \cup_1^l \, \mathbf{W}(T_i)$$

(ii) this decomposition is **irredundant**

K	Nmax N	dimension
3	2	0
4	2	2
5	3	2
6	3	4
7	4	2
8	5	2

We find also the set of possible value for the free degrees such that all roots are of modulus 1.

The optimization of free degrees is then carried out with a conjugate gradient algorithm (numer.).

The resulting filters are the most regular orthogonal centro-symmetric wavelets known at this time, with a wavelet whose Sobolev exponent is 2.11, which means that it is at least C^1, and possibly C^2. One might think that the flatness and the regularity can achieve any value by considering a large enough K, thus increasing the filter support.

Conclusions

- **wavelets** : the first example of bidimensional orthogonal linear-phase wavelets which are continuously differentiable.
- **filter banks** : the resulting filters have good frequency characteristics, and it has been shown that the remaining free degrees can be chosen so as to optimize various criteria.
- **image compression** : it is interesting to have new filter banks holding simultaneously orthogonality, centrosymmetry and regularity, and in addition with good frequency characteristics. They hold as good results as the best filters known up till now in terms of compression efficiency.
- **computer algebra** : it is one of the major applications of computer algebra in digital signal processing. In addition, the problem has been solved using the most recent techniques for the Gröbner basis and triangular systems computations in positive dimension.