Hierarchical Comprehensive Triangular Decomposition

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Triangular Decomposition

Consider the linear equations

$$8x_1 + 7x_2 - 1 = 0$$
$$2x_1 + 3x_2 - 2 = 0$$

Triangular decomposition

$$8\mathbf{x_1} + 7x_2 - 1 = 0$$
$$5\mathbf{x_2} - 7 = 0$$

Triangular Decomposition

Consider the non-linear equations

$$3x_1x_2 + 2x_2^2 - 1 = 0$$
$$x_1^2 + 3x_2^2 - 1 = 0$$

Triangular decomposition

$$3\mathbf{x_1}x_2 + 2x_2^2 - 1 = 0$$
$$31\mathbf{x_2^4} - 13x_2^2 + 1 = 0$$

Triangular Decomposition

Let's change the coefficients of the non-linear equations

$$\begin{aligned} 5x_1x_2 + 4x_2^2 - 1 &= 0 \\ x_1^2 + 5x_2^2 - 1 &= 0 \end{aligned}$$

The triangular decomposition looks similar

$$\begin{aligned} 5\mathbf{x_1}x_2 + 4x_2^2 - 1 &= 0 \\ 141\mathbf{x_2^4} - 33x_2 + 1 &= 0 \end{aligned}$$

Triangular Decomposition

Let's choose special coefficients

$$-4x_1x_2 + 8x_2^2 - 1 = 0$$
$$x_1^2 - 4x_2^2 - 1 = 0$$

The triangular decomposition looks different

$$16x_1x_2 + 3 = 0$$
$$32x_2^2 - 1 = 0$$

Comprehensive Triangular Decomposition (CTD)

Consider the parametric polynomial equations

$$\mathbf{u_2} x_1 x_2 + \mathbf{u_1} x_2^2 - 1 = 0$$
$$x_1^2 + \mathbf{u_2} x_2^2 - 1 = 0$$

Comprehensive Triangular Decomposition (CTD)

[Chen, C., Golubitsky, O., Lemaire, F., Moreno Maza, M.and Pan, W, 2007]

- **Step 1.** Compute the "full" solution for $x_1 \succ x_2 \succ u_1 \succ u_2$

$$\begin{cases} u_2\mathbf{x_1}x_2 + u_1x_2^2 - 1 \\ (u_1^2 + u_2^3)\mathbf{x_2^4} - (u2^2 + 2u_1)x_2^2 + 1 \end{cases} \quad \begin{cases} u_2\mathbf{x_1}x_2 + u_1x_2^2 - 1 \\ (2u_1 + u_2^2)\mathbf{x_2^2} - 1 \\ \mathbf{u_1^2} + u_2^3 \end{cases} \quad \begin{cases} \mathbf{x_1^2} - 1 \\ u_1\mathbf{x_2^2} - 1 \\ \mathbf{u_2} \end{cases}$$

- **Step 2.** Partition the paramter space

$$\begin{cases} u_2(u_1^2 + u_2^3) \neq 0 \end{cases} \quad \begin{cases} u_1^2 + u_2^3 = 0 \\ u_2(2u_1 + u_2^2) \neq 0 \end{cases} \quad \begin{cases} u_2 = 0 \\ u_1 \neq 0 \end{cases}$$

- **Step 3.** Find the related triangular decomposition for each sub space

Comprehensive Triangular Decompostion (CTD)

[Chen, C., Golubitsky, O., Lemaire, F., Moreno Maza, M.and Pan, W, 2007]

- **Branch 1.**

$$\begin{cases} u_2\mathbf{x_1}x_2 + u_1x_2^2 - 1 \\ (u_1^2 + u_2^3)\mathbf{x_2^4} - (u2^2 + 2u_1)x_2^2 + 1 \end{cases}, \quad \text{if } u_2(u_1^2 + u_2^3) \neq 0$$

- **Branch 2.**

$$\begin{cases} u_2\mathbf{x_1}x_2 + u_1x_2^2 - 1 \\ (2u_1 + u_2^2)\mathbf{x_2^2} - 1 \end{cases}, \quad \text{if } u_1^2 + u_2^3 = 0 \wedge u_2(2u_1 + u_2^2) \neq 0$$

- **Branch 3.**

$$\begin{cases} \mathbf{x_1^2} - 1 \\ u_1\mathbf{x_2^2} - 1 \end{cases}, \quad \text{if } u_2 = 0 \wedge u_1 \neq 0$$

Hierarchical Strategy

- **Step 1.** Assume u_1 and u_2 are generic, compute CTD

$$\textbf{Branch1}. \quad \begin{cases} u_2\mathbf{x_1}x_2 + u_1x_2^2 - 1 \\ (u_1^2 + u_2^3)\mathbf{x_2^4} - (u2^2 + 2u_1)x_2^2 + 1 \end{cases}, \qquad \text{if } u_2(u_1^2+u_2^3) \neq 0$$

- **Step 2.** Add $u_2(u_1^2 + u_2^3) = 0$ into the original system and regard x_1, x_2, u_1 as the new variable set. Assume u_2 is generic, compute CTD

$$\textbf{Branch2}. \quad \begin{cases} u_2(u_2^2 + 2u_1)\mathbf{x_1}x_2 - u_2^2 - u_1 \\ (u_2^2 + 2u_1)\mathbf{x_2^2} - 1 \end{cases}, \qquad \text{if } u_1^2+u_2^3 = 0 \wedge u_2(u_2+4) \neq 0$$

- **Step 3.** Add $u_2(u_2 + 4) = 0, u_2(u_1^2 + u_2^3) = 0$ into the original system and regard x_1, x_2, u_1, u_2 as the new variable set. Compute CTD

$$\textbf{Branch3}. \quad \begin{cases} 16\mathbf{x_1}x_2 + 3 \\ 32\mathbf{x_2^2} - 1 \end{cases}, \qquad \text{if } u_2 = -4 \wedge u_1 = 8$$

$$\textbf{Branch4}. \quad \begin{cases} \mathbf{x_1^2} - 1 \\ u_1\mathbf{x_2^2} - 1 \end{cases}, \qquad \text{if } u_2 = 0 \wedge u_1 \neq 0$$

Experiment
Comparing HCTD and CTD

	benchmark	d	n	time		ratio
				HCTD	CTD	
1.	*MontesS10*	3	4	0.421	0.359	1.173
2.	*Maclane*	3	7	5.242	4.009	1.308
3.	*MontesS11*	3	3	0.858	0.655	1.310
4.	*MontesS9*	3	3	0.693	0.468	1.474
5.	*S3*	4	3	2.618	1.436	1.823
6.	*zhou5*	4	5	5.616	2.902	1.935
7.	*F6*	4	1	0.296	0.14	2.114
8.	*Hereman-8-8*	3	5	96.439	10.468	9.213
9.	*F4*	4	2	11.637	0.375	31.032
10.	*MontesS1*	2	2	0.016	0.	
11.	*S1*	3	2	`timeout`	4.04	
12.	*Neural*	1	3	`timeout`	0.188	
13.	*Gerdt*	3	4	`timeout`	0.842	

- Intel(R) Core(TM) i5 processor (3.20GHz CPU), 2.5 GB RAM
- Windows 7 (32 bit), Maple 17

Experiment
Comparing HCTD and CTD

	benchmark	d	n	time		ratio
				HCTD	CTD	
14.	*Bronstein*	2	2	0.015	0.219	0.068
15.	*F3*	4	1	0.063	0.905	0.070
16.	*MontesS12*	2	6	0.593	7.925	0.075
17.	*MontesS14*	1	4	0.452	4.353	0.104
18.	*F2*	2	2	0.032	0.234	0.137
19.	*zhou6*	3	3	0.031	0.218	0.142
20.	*SBCD13*	1	3	0.015	0.094	0.160
21.	*Hereman-2*	1	7	0.093	0.468	0.199
22.	*MontesS15*	4	8	0.187	0.889	0.210
23.	*AlkashiSinus*	3	6	0.094	0.437	0.215
24.	*MontesS13*	3	2	0.078	0.265	0.294
25.	*MontesS7*	1	3	0.046	0.156	0.295
26.	*zhou1*	3	4	0.047	0.156	0.301
27.	*MontesS6*	2	2	0.015	0.047	0.319
28.	*zhou2*	6	7	0.671	2.09	0.321
29.	*MontesS5*	4	4	0.078	0.187	0.417
30.	*F5*	3	2	0.046	0.11	0.418
31.	*F8*	4	4	0.437	1.014	0.431
32.	*Lanconelli*	7	4	0.28	0.546	0.513
33.	*SBCD23*	1	3	0.202	0.344	0.587
34.	*MontesS16*	3	12	1.198	1.825	0.656
35.	*MontesS2*	1	3	0.	0.	1
36.	*MontesS4*	2	2	0.	0.	1
37.	*MontesS3*	1	2	0.	0.031	
38.	*MontesS8*	2	2	0.	0.094	
39.	*F7*	3	2	0.	0.016	
40.	*S2*	4	1	44.544	`timeout`	

Some difficult benchmarks

	benchmark	d	n	time	
				HCTD	CTD
41.	*Lazard-ascm2001*	3	4	`timeout`	`timeout`
42.	*Leykin-1*	4	4	`timeout`	`timeout`
43.	*Cheaters-homotopy-easy*	4	3	`timeout`	`timeout`
44.	*Cheaters-homotopy-hard*	5	2	`timeout`	`timeout`
45.	*Lazard-ascm2001*	3	4	`timeout`	`timeout`
46.	*MontesS18*	2	3	`timeout`	`timeout`
47.	*Pavelle*	4	4	`timeout`	`timeout`
48.	*p3p*	5	2	`timeout`	`timeout`
49.	*z3*	6	11	`timeout`	`timeout`

Timings for getting partial answers

	benchmark	d	n	time				
				Step 1	*Step*2	*Step*3	*Step*4	*Step*5
41.	*Lazard-ascm2001*	3	4	0.936	**timeout**			
42.	*Leykin-1*	4	4	0.203	20.436	**timeout**		
43.	*Cheaters-homotopy-easy*	4	3	3.681	**timeout**			
44.	*Cheaters-homotopy-hard*	5	2	39.640	**timeout**			
45.	*Lazard-ascm2001*	3	4	0.858	**timeout**			
46.	*MontesS18*	2	3	0.327	**timeout**			
47.	*Pavelle*	4	4	0.234	**timeout**			
48.	*p3p*	5	2	0.	0.	0.015	6.549	**timeout**
49.	*z3*	6	11	0.094	**error**			

Summary

- Triangular Decomposition
- Comprehensive Triangular Decomposition
- Hierarchical Comprehensive Triangular Decomposition

Thank You!