Computation of Canonical Forms for Ternary Cubics

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Equivalence under a linear change of variables

$$\boxed{F \sim \bar{F} \Longleftrightarrow \exists g \in GL(m, \mathbb{C}) : \ \bar{F}(\mathbf{x}) = F(g \cdot \mathbf{x})}$$

Example:

- $5x^2 - 2xy + 2y^2$ is equivalent to $x^2 + y^2$
  under the change of variables
  $x \to x + y; \quad y \to y - 2x.$

Problems:

- Find classes of equivalent polynomials.
- Find invariants which characterize each class.
- Find a "simple" canonical form in each class.
- Match a given F with its canonical form.

Symmetry of Polynomials

$$\boxed{g \in GL(m) \text{ is a symmetry of } F \iff F(g\mathbf{x}) = F(\mathbf{x})}$$

- $x^2 + y^2$ is symmetric under any orthogonal map:

$$(x, y) \to \begin{cases} (\cos(\alpha)x + \sin(\alpha)y, \, -\sin(\alpha)x + \cos(\alpha)y) \\ (-x, \, -y) \end{cases}$$

- $x^8 + 14x^4y^4 + y^8$ has a sym. group with 192 elements gen. by:

$$(x, y) \to \begin{cases} (\frac{\sqrt{2}}{2}(1+i)x, \, \frac{\sqrt{2}}{2}(1+i)y) \\ (\frac{\sqrt{2}}{2}i(x+y), \, \frac{\sqrt{2}}{2}(x-y)) \\ (ix, \, y) \end{cases}$$

Problem: Given F find its group of symmetries G_F.

$$\boxed{F \sim \bar{F} \Longrightarrow G_{\bar{F}} = gG_Fg^{-1}}$$

Why classification of polynomials is difficult?

$$\boxed{GL(m,\mathbb{C}) \curvearrowright \mathbb{C}^m \Longrightarrow GL(m,\mathbb{C}) \curvearrowright P^d_m = \mathbb{C}[x^1,\ldots,x^m]^d}$$

P^d_m – a linear space parameterized by $\{c_\alpha\}$ coefficients of polynomials.

$$\dim P^d_m = C^d_{m+d-1}$$

Non-regular action!

- equivalence classes (orbits) have different dimensions.
- equivalence classes are not closed subsets of P^d_m.

$$\Downarrow$$

Continuous invariants $I(c_\alpha)$ do not distinguish classes.

Example.

$$\boxed{(1) \quad x^3 + a\,xz^2 + z^3 - y^2 z \quad \not\to \quad (4) \quad x^3 - y^2 z}$$

$$\text{for } \varepsilon \neq 0 : \quad x \to x, \quad y \to \frac{1}{\varepsilon} y, \quad z \to \varepsilon^2 z :$$

$$(x^3 + a\,xz^2 + z^3 - y^2 z) \longrightarrow (x^3 + a\,\varepsilon^4 xz^2 + \varepsilon^6\, z^3 - y^2 z),$$

$$\lim_{\varepsilon \to 0} (x^3 + a\,\varepsilon^4 xz^2 + \varepsilon^6\, z^3 - y^2 z) = x^3 - y^2 z.$$

$$\boxed{\bar{\mathcal{O}}_{(1)} \supset \mathcal{O}_{(4)}.}$$

Complete classifications of polynomials in m variables of degree d

(known to us).

- $d = 2$ (quadratics m-ary forms): $x_1^2 + \cdots + x_k^2$.
- $m = 2$ (binary forms): $d = 1, 2, 3, 4$.
- $m = 3$ (ternary forms): $d = 1, 2, 3$.

Some references or partial results for cases when $m = 2$, $d = 5, 6, 7, 8$; when $m = 3$, $d = 4$; when $m = 4$, $d = 3$.

Approaches

- **Classical (XIX century)** by Aronhold, Gordan, Caley, ...
  Computation of covariants (rational invariants $I(\mathbf{x}, c_\alpha)$).

- Hilbert
  The rings of covariants and invariants are finitely generated.
  Nullcones.

- **Algebraic Geometry** by Mamford, Kraft, Vinberg, Popov, ...
  Description of the algebraic variety that represents the space of orbits.

- **Algebraic computational algorithms** by Sturmfels, Derkson, ...

Differential Geometry (Moving Frame) Approach.

by P. Olver.

Main Idea

- Consider the graphs of polynomials $u = F(x_1, \ldots, x_m)$ in C^{m+1} dimensional space. Apply Cartan's equivalence method for submanifolds.

$$\Downarrow$$

Algorithms

- To decide whether two polynomials are equivalent.
- If yes find a corresponding linear transformation.
- To find the symmetry group of a given polynomial.

Implementation

- Computing differential invariants.
  differentiation, algebraic operations, multivariate polynomial elimination by hand (inductive approach of moving frame [Kogan, 2000])

- Computing the signature variety, parameterized by differential invariants.
  ranking conversions of regular chains using the PALGIE *algorithm [Boulier, Lemaire, Moreno Maza, 2001]*

Inhomogeneous version

$$\boxed{u = f(p,q) = F(p,q,1) \Longleftrightarrow F(x,y,z) = z^3\, f(\tfrac{x}{z}, \tfrac{y}{z})}$$

Γ_F : $u = F(x,y,z)$ homogeneous poly. in 3 variables of degree 3

$g \downarrow$

$\bar{\Gamma}_F$: $u = F\left(\alpha\, x + \beta\, y + \lambda\, z\,, \gamma\, x + \delta\, y + \mu\, z\,, a\, x + b\, y + \eta\, z\right),$

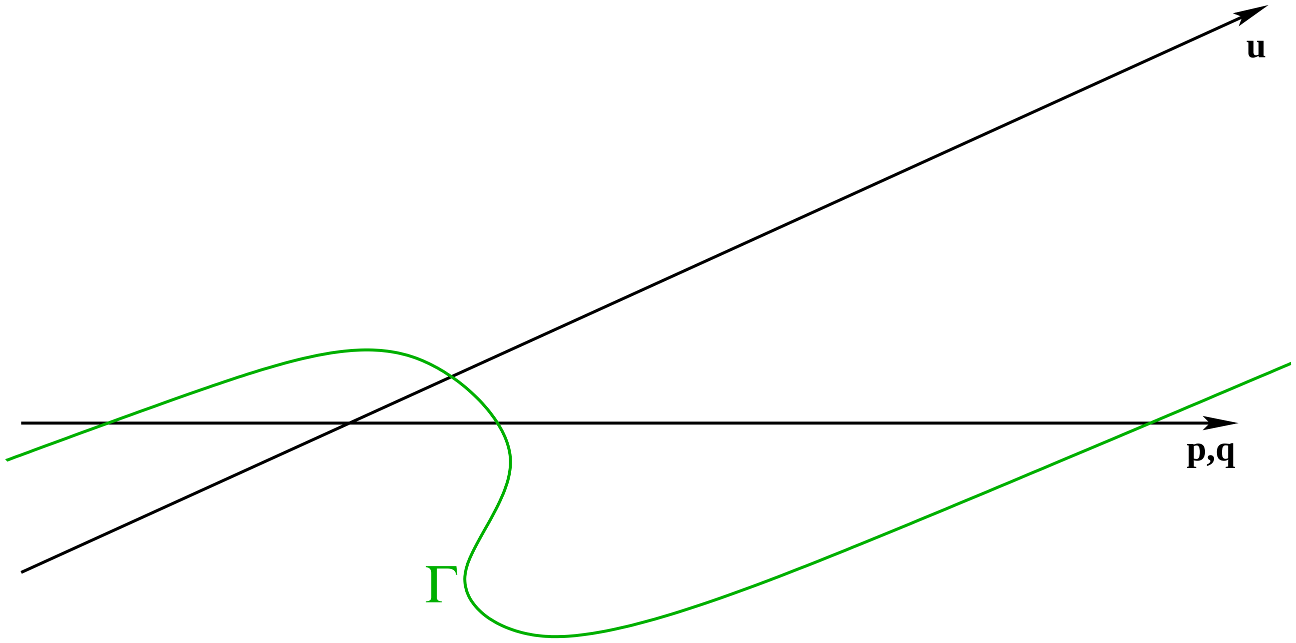
$$\Updownarrow$$

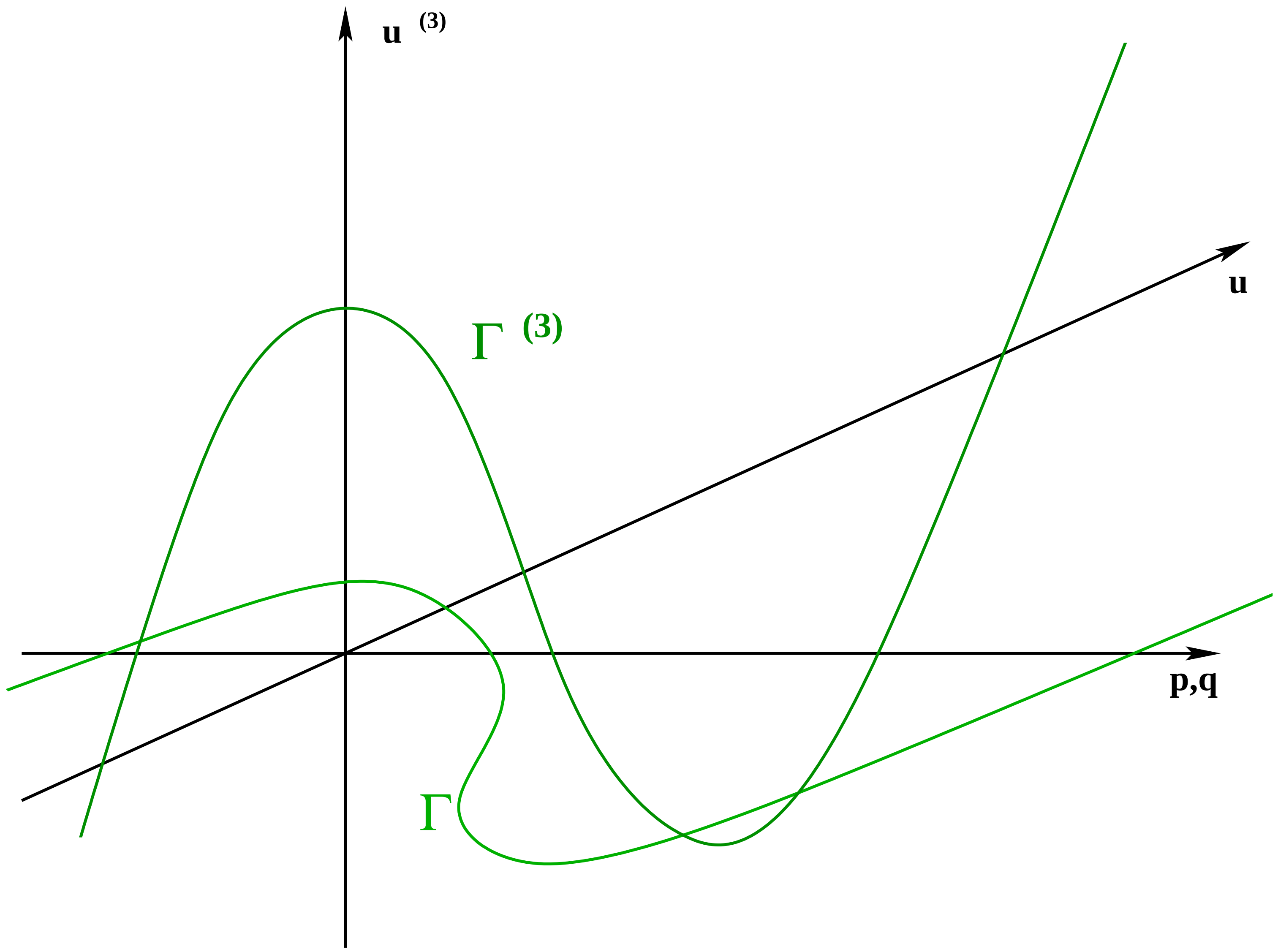
Γ_f : $u = f(p,q)$ inhomogeneous poly. in 2 variables of degree ≤ 3

$g \downarrow$

$\bar{\Gamma}_f$: $u = (a\,p + b\,q + \eta)^3\ f\left(\dfrac{\alpha\,p + \beta\,q + \lambda}{a\,p + b\,q + \eta}, \dfrac{\gamma\,p + \delta\,q + \mu}{a\,p + b\,q + \eta}\right).$

Γ is the graph of $u = f(p,q)$, $\dim \Gamma = 2$.

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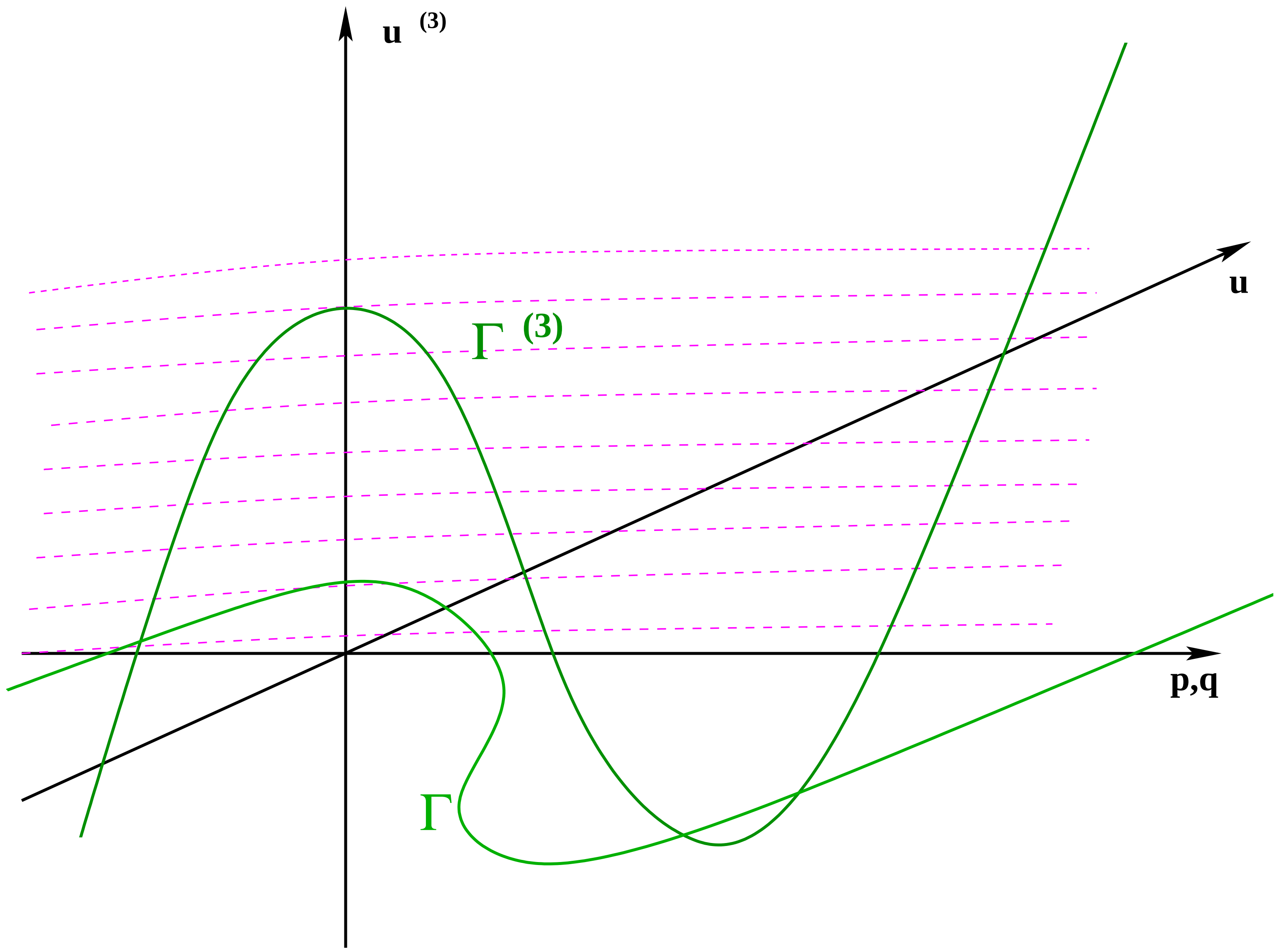
Γ (3)
u
p,q
Γ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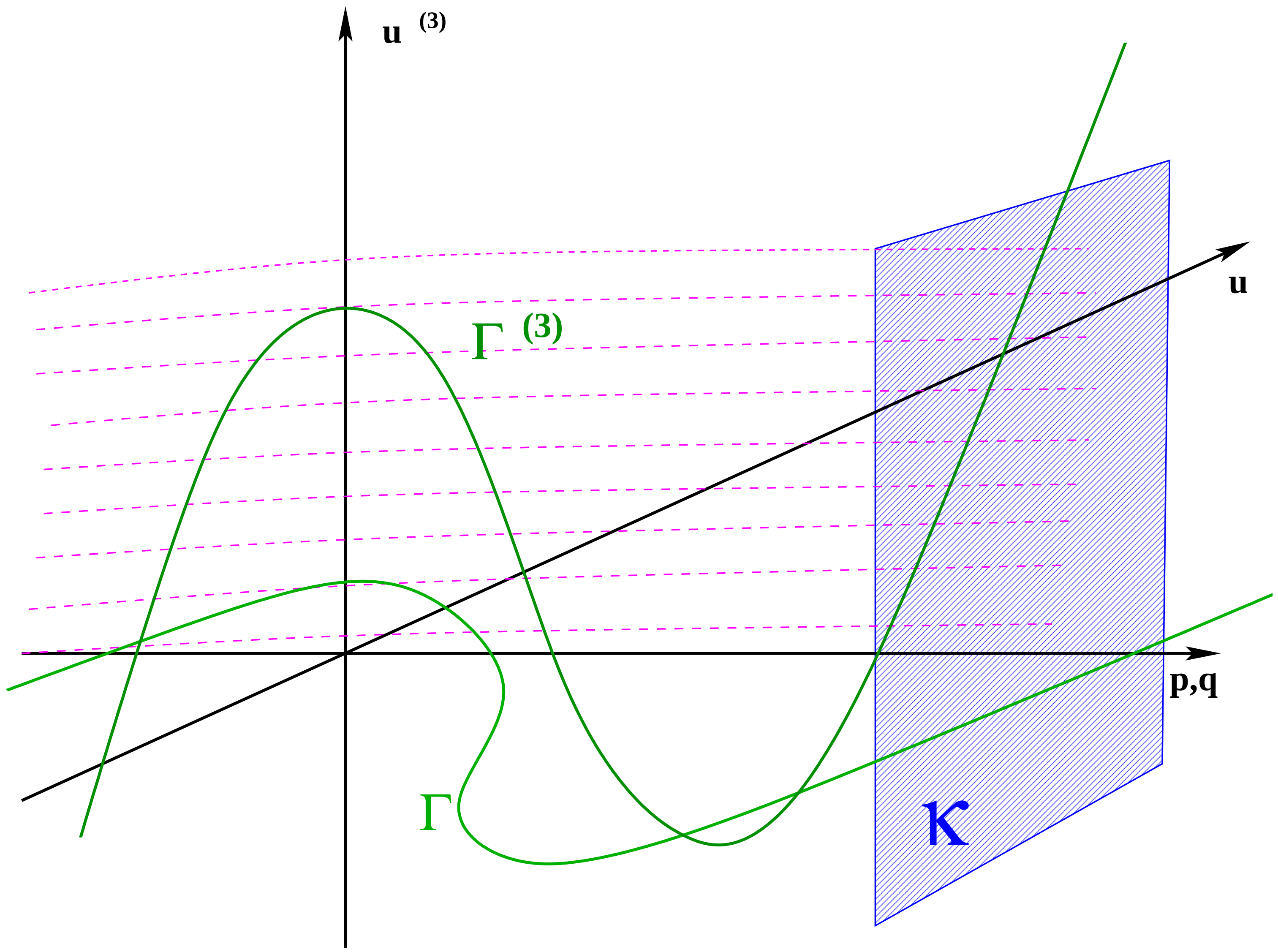
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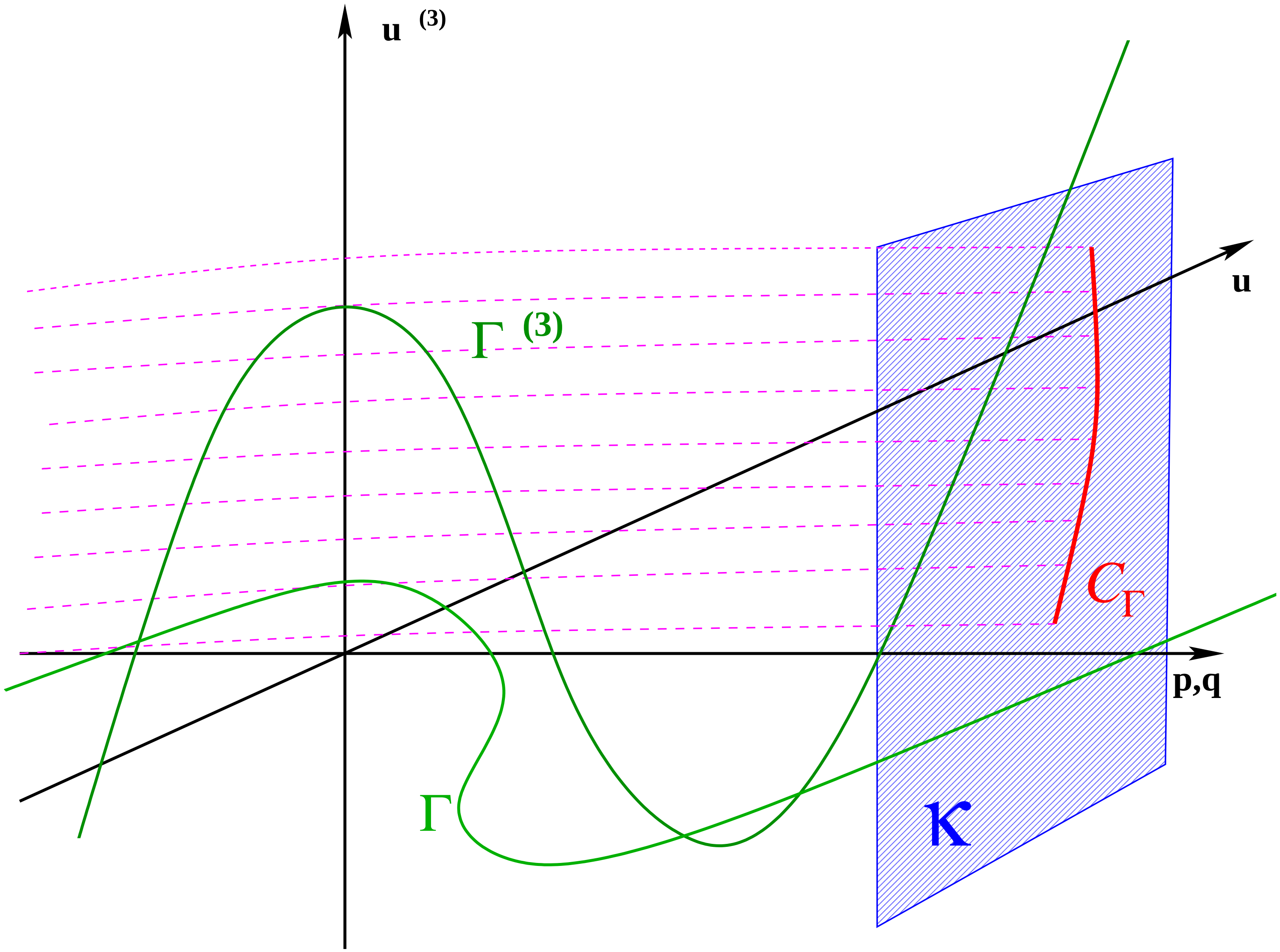

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Equivalence and symmetry theorems

Theorem 1. (Equivalence)

$$\bar{F} \sim F \Longleftrightarrow \mathcal{C}_F = \mathcal{C}_{\bar{F}}.$$

Computational problem *decide if two parameterization define the same set (elimination).*

Theorem 2 (Symmetry) Γ_F is the graph of F.

$$\dim G_F = \dim \Gamma_F - \dim \mathcal{C}_F$$

For a generic F: $\dim \mathcal{C}_F = \dim \Gamma_F$ (maximal) $\Rightarrow G_F$ is finite and can be computed explicitly.

$\mathcal{C}_\Gamma$ is parameterizes by diff. invarinats i_1, i_2, i_3.
$A = i_1^3/i_2^2$ is constant on each of the equivalence class!

Example.

The signature $\mathcal{C}_f$ for $f = p^2 + q^2 + 1$ $(F = z\,(x^2 + y^2 + z^2))$:

$$\begin{aligned} i_1|_f &= 90\,\frac{(p^2+q^2+1)^2}{(p^2-3+q^2)^2}, \quad i_2|_f = 270\frac{(p^2+q^2+1)^3}{(p^2-3+q^2)^3}, \\ i_3|_f &= 180\,\frac{(p^2+q^2+1)\,((p^2+q^2+3)^2-12)}{(p^2-3+q^2)^3} \end{aligned}$$

Elimination of p and q $\Rightarrow$ equations for 1-dim'l signature variety V_f:

$$\boxed{i_1\,(i_3 - i_2) + 30\,i_2 = 0, \quad 10\,i_2^2 - i_1^3 = 0.}$$

Classes of ternary cubics:

- Irreducible:
  - Regular(elliptic curves): (1)- *1-paramteric family*; (2); (3).
  - Singular: (4); (5).
- Reducible into
  - a linear and a quadratic factor: (6); (7).
  - three linear factors: (8); *binary form is disguise* (9), (10), (11).

Irreducible cubics.

Regular (Elliptic Curves):

(1) $\mathbf{F \sim x^3 + axz^2 + z^3 - y^2z}, \quad \mathbf{f \sim p^3 + a\,p + 1 - q^2},$

non-equivalent for different values of a^3*;*

$a \neq 0$ *(else* $F \sim (3)$*)*, $a^3 \neq -27/4$ *(else* $F \sim (5)$*)*,

$|G_F| = 18 \times 3$

$\boxed{675\, i_1^3 + (10\, a)^3\, i_2^2 = 0.}$

(2) $\mathbf{F \sim x^3 + xz^2 - y^2z}, \quad \mathbf{f \sim p^3 + p - q^2},$

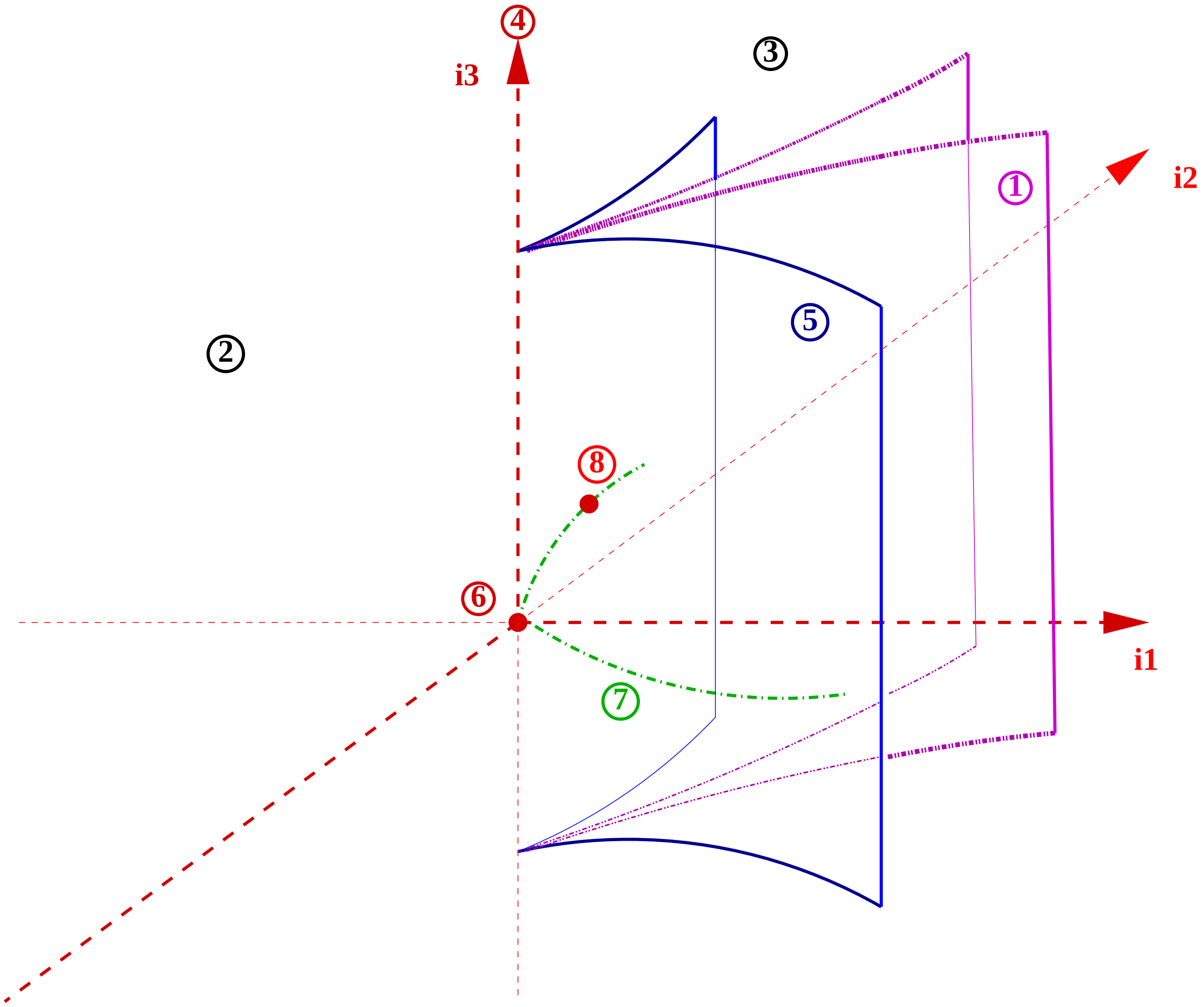
$|G_F| = 36 \times 3,$

$\boxed{i_2 = 0.}$

(3) $\mathbf{F \sim x^3 + z^3 - y^2z}, \quad \mathbf{f \sim p^3 + 1 - q^2},$

$|G_F| = 54 \times 3,$

$\boxed{i_1 = 0.}$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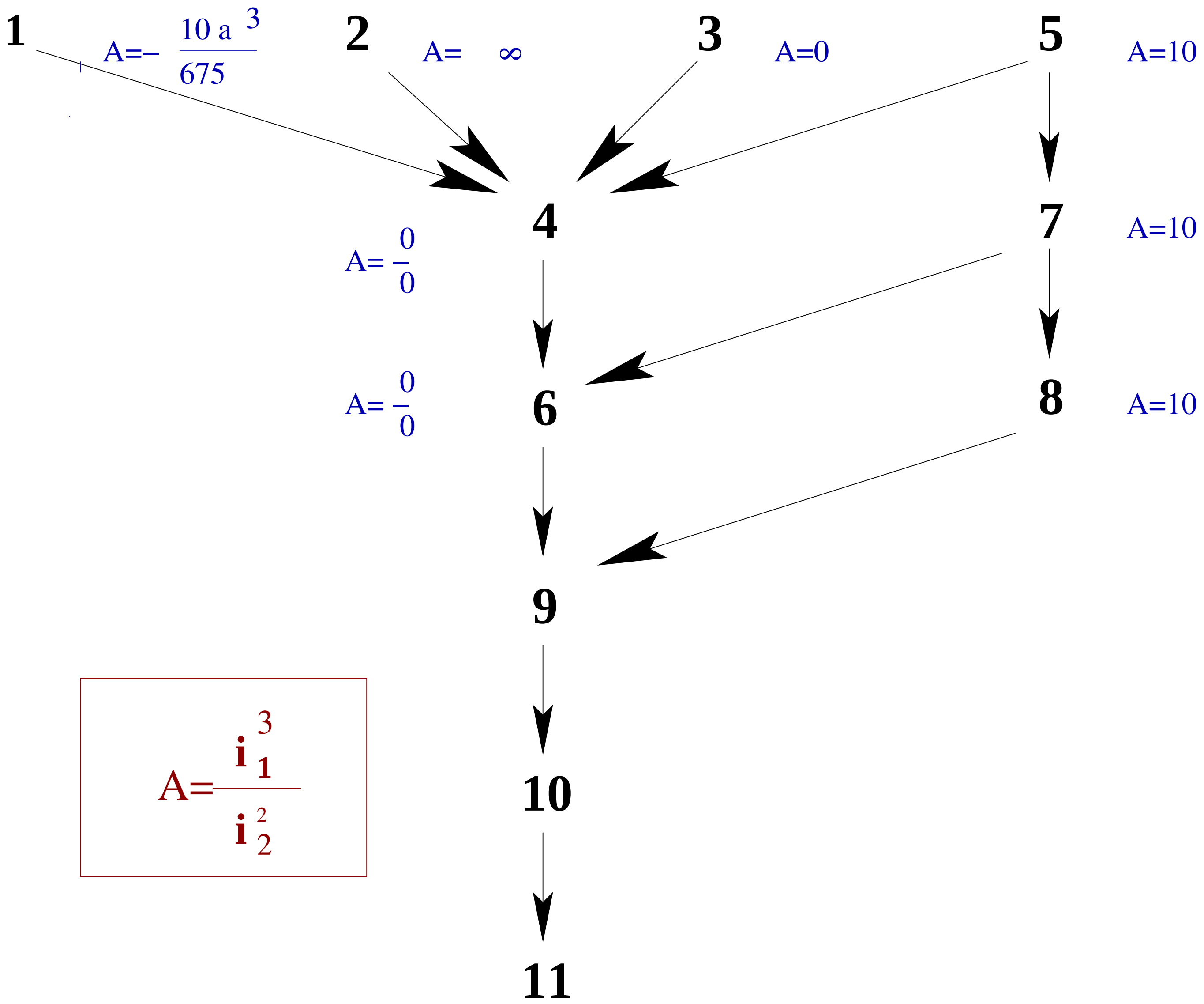
1
A=− 10 a^3 / 675
2
A= ∞
3
A=0
5
A=10
4
A= 0/0
7
A=10
6
A= 0/0
8
A=10
9
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11
A= i_1^3 / i_2^2

Application to ternary cubics

$$F(x, y, z),\ \deg F = 3$$

- Fast algorithm to determine the class of F.
- An algorithm to compute a change of variables from F to its canonical form.
- Classification of the symmetry groups.
- A geometric description of the equivalence classes, which depicts information about the size of the symmetry group and inclusions of the closures of the classes.

Conclusions

- Differential invariants for polynomials = covariants in the classical sense.
- The set of differential invariants that parameterize signature depends only on the **group action** and **the number of variables**, but **not on the degree**.
- For m=2,3 the complete set of invariants is computed.

Further projects

- new classifications (e. g. for binary forms),
- other group actions,
- other fields ($\mathbb{R}$, finite fields).

More results with moving frames

- Binary forms: $F(x, y)$, $\deg F = n$ (F is homogeneous).
  - complete set of differential invariants (P. Olver).
  - algorithm (coded in MAPLE) to compute G_F (Olver, Kogan).
- Ternary forms $F(x, y, z)$, $\deg F = n$.
  - complete set of differential invariants (Kogan).
  - necessary and sufficient for F to be equivalent to $x^n + y^n + z^n$ (Kogan, thanks to Schost, Lecerf).

Irreducible cubics.

Regular (Elliptic Curves):

(1) $\mathbf{F \sim x^3 + axz^2 + z^3 - y^2z}, \quad \mathbf{f \sim p^3 + a\,p + 1 - q^2}$,
non-equivalent for different values of a^3*;*
$a \neq 0$ *(else* $F \sim (3)$*)*, $a^3 \neq -27/4$ *(else* $F \sim (5)$*)*,
$|G_F| = 18 \times 3$
$\boxed{675\, i_1^3 + (10\, a)^3\, i_2^2 = 0.}$

(2) $\mathbf{F \sim x^3 + xz^2 - y^2z}, \quad \mathbf{f \sim p^3 + p - q^2}$,
$|G_F| = 36 \times 3$,
$\boxed{i_2 = 0.}$

(3) $\mathbf{F \sim x^3 + z^3 - y^2z}, \quad \mathbf{f \sim p^3 + 1 - q^2}$,
$|G_F| = 54 \times 3$,
$\boxed{i_1 = 0.}$

Singular:

(4) $\mathbf{F} \sim \mathbf{x^3 - y^2 z}, \quad \mathbf{f} \sim \mathbf{p^3 - q^2},$

$G_F \sim x \to x,\ y \to \alpha y,\ z \to \alpha^{-2} z,$ (1-dim'l)

$\boxed{i_1 = 0, \quad i_2 = 0.}$

(5) $\mathbf{F} \sim \mathbf{x^2(x+z) - y^2 z}, \quad \mathbf{f} \sim \mathbf{p^2(p+1) - q^2}$

$|G_F| = 6 \times 3$

$\boxed{i_1^3 - 10\, i_2^2 = 0.}$

Reducible cubics:

A linear and an irreducible quadratic factor:

(6) $\mathbf{F \sim z(x^2 + yz)}, \quad \mathbf{f \sim (p^2 + q)}$

$G_F \sim$ non-commutative 2-dim'l (affine) group:

$x \to x + \alpha z$, $y \to -2\alpha x + y - \alpha^2 z$, $z \to z$,

$x \to \beta x$, $y \to \beta^4 y$, $z \to \beta^{-2} z$,

$\boxed{i_1 = 0,\ i_2 = 0,\ i_3 = 0.}$

(7) $\mathbf{F \sim z(x^2 + y^2 + z^2)}, \quad \mathbf{f \sim p^2 + q^2 + 1}$

$G_F \sim$ rotation in the xy plane (1-dim'l)

$\boxed{i_1(i_3 - i_2) + 30\, i_2 = 0,\ 10\, i_2^2 - i_1^3 = 0.}$

Three linear factors:

(8) non-coplaner $\Longleftrightarrow \mathbf{F} \sim \mathbf{xyz}, \quad \mathbf{f} \sim \mathbf{p\,q}$;

$G_F \sim \mathbb{R}^2 : \{x \to \alpha x,\, y \to \beta y,\, z \to \frac{1}{\alpha\beta} z\}$.

$\boxed{i_1 = 90,\, i_2 = 270,\, i_3 = 180.}$

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(9) different, coplaner $\Leftrightarrow \mathbf{F} \sim \mathbf{x\,y\,(x+y)}, \mathbf{f} \sim \mathbf{p\,q(p+q)}$

$G_F \sim$ 3-dim'l $\{z \mapsto \alpha x + \beta y + \gamma z\} \times G_{xy(x+y)}$,

$(G_{xy(x+y)} \sim S_3 \times Z_3 \subset GL(2,\mathbb{C}) \curvearrowright (x,y)$ preserves $xy(x+y)$.

(10) two repeated $\Leftrightarrow \mathbf{F} \sim \mathbf{x^2\,y}, \quad \mathbf{f} \sim \mathbf{p^2 q}$

$G_F \sim$ 4-dim'l: $\{x \to \alpha\, x, \quad y \to \frac{1}{\alpha^2} y, \quad z \to \beta\, x + \gamma\, y + \delta\, z\}$.

(11) three repeated $\Leftrightarrow \mathbf{F} \sim \mathbf{x^3}, \quad \mathbf{f} \sim \mathbf{p^3}$.

$G_F \sim$ 4-dim'l $GL(2,\mathbb{C}) \times Z_3$ $(GL(2,\mathbb{C}) \curvearrowright (y,z)$ and $Z_3 \curvearrowright x)$.

(9), (10) and (11) are binary forms in disguise.

An example

Parameterization of $\mathcal{C}_F$:

$$\left\{\begin{array}{rcl}
0 & = & (3p+4)(-q+p)(q+p)(3p^3+2p^2+3pq^2-2q^2)-6(-3pq^2-q^2+p^2)^2\mathbf{I_1} \\
0 & = & (-q+p)(q+p)(81p^6+972p^5q^2+72p^5+1269p^4q^2+32p^4-144p^3q^2+ \\
 & & +972p^3q^4+1107p^2q^4-64p^2q^2+72pq^4+135q^6+32q^4) \\
 & & -6(-3pq^2-q^2+p^2)^3\mathbf{I_2} \\
0 & = & (16p^2+72p^3+108p^4+54p^5-16q^2+72pq^2+81p^2q^2+ \\
 & & +27p^3q^2+27q^4)(-q+p)^2(q+p)^2-9(-3pq^2-q^2+p^2)^3\mathbf{I_3}
\end{array}\right.$$

Cartesian equation of $\mathcal{C}_F$:

$$7200\mathbf{I_1}^3-1692\mathbf{I_1}^2-504\mathbf{I_1}\mathbf{I_2}-3780\mathbf{I_1}\mathbf{I_3}-12\mathbf{I_2}^2-180\mathbf{I_2}\mathbf{I_3}-675\mathbf{I_3}^2+1440\mathbf{I_1}+40\mathbf{I_2}+300\mathbf{I_3}.$$

Ranking conversions

- For $\mathcal{R} = x > y > z > s > t$ and $\overline{\mathcal{R}} = t > s > z > y > x$ we have:

$$\mathsf{palgie}(\begin{cases} x - t^3 \\ y - s^2 - 1 \\ z - s\,t \end{cases}, \mathcal{R}, \overline{\mathcal{R}}) \quad = \quad \begin{cases} s\,t - z \\ (x\,y + x)s - z^3 \\ z^6 - x^2 y^3 - 3x^2 y^2 - 3x^2 y - x^2 \end{cases}$$

- For $\mathcal{R} = \cdots > v_{xx} > v_{xy} > \cdots > u_{xy} > u_{yy} > v_x > v_y > u_x > u_y > v > u$ we $\overline{\mathcal{R}} = \cdots u_x > u_y > u > \cdots > v_{xx} > v_{xy} > v_{yy} > v_x > v_y > v$ we have:

$$\mathsf{pardi}(\begin{cases} v_{xx} - u_x \\ 4\,u\,v_y - (u_x\,u_y + u_x\,u_y\,u) \\ u_x^2 - 4\,u \\ u_y^2 - 2\,u \end{cases} \mathcal{R}, \overline{\mathcal{R}}) \quad = \quad \begin{cases} u - v_{yy}^2 \\ v_{xx} - 2\,v_{yy} \\ v_y\,v_{xy} - v_{yy}^3 + v_{yy} \\ v_{yy}^4 - 2\,v_{yy}^2 - 2\,v_y^2 + 1 \end{cases}$$

PARDI, PODI, PALGIE

Input: In $\mathbf{k}[X]$

- two rankings $\mathcal{R}, \overline{\mathcal{R}}$ over X,
- a $\mathcal{R}$-triangular C set such that $\mathbf{Sat}(C)$ is prime.

Output: a $\overline{\mathcal{R}}$-triangular set $\overline{C}$ such that de $\mathbf{Sat}(C) = \mathbf{Sat}(\overline{C})$.

Algo: three cases:

PALGIE: *Prime ALGebraic IdEal* implemented in Aldor, C and Maple,

PODI: *Prime Ordinary Differential Ideal*, implemented in C,

PARDI: *Prime pARtial Differential Ideal*, implemented in Maple.

$P := C;\ \overline{C} := \emptyset$
$H := \{\mathsf{init}(p, \mathcal{R})\ \textbf{for}\ p \in C\}$
$\textbf{while}\ (P \neq \emptyset)\ \textbf{repeat}$
$\quad p := \textbf{first}\ P;\ P := \textbf{rest}\ P$
$\quad p := \mathsf{red}(p, \overline{C})$
$\quad (p, P', H') := \mathsf{ensureRank}(p, \overline{\mathcal{R}}, C)$
$\quad (P, H) := (P \cup P', H \cup H')$
$\quad p = 0 \Longrightarrow \textbf{iterate}$
$\quad v := \mathsf{mvar}(p)$
$\quad \textbf{if}\ (\forall q \in \overline{C})\ \mathsf{mvar}(q) \neq v\ \textbf{then}$
$\quad\quad \overline{C} := \overline{C} \cup \{p\}$
$\quad \textbf{else}$
$\quad\quad (g, P', H') := \mathsf{gcd}(p, \overline{C}_v, \overline{C}_v^-, C)$
$\quad\quad (P, H) := (P \cup P', H \cup H')$
$\quad\quad \overline{C} := \overline{C} \setminus \{\overline{C}_v\} \cup \{g\}$
$\quad \overline{C} := \mathsf{saturate}(\overline{C}, H)$
$\textbf{return}\ \overline{C}$