Big Prime Field FFTs on the GPU

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Outline

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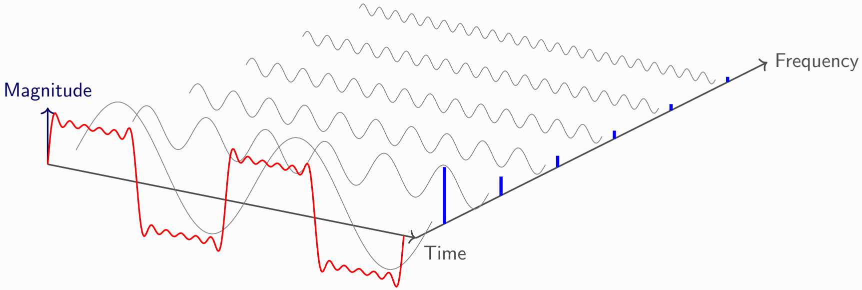
What does Fourier transform do?



Examples of a function sampled over a finite time interval:

- pressure of a sound wave,
- a radio signal,
- daily temperature readings.

Fourier transform over $\mathbb{C}$

- An extension of the Fourier series when the period approaches infinity.
- It can be studied for complex values of ξ:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi t} f(t)\, dt$$

How does FT help solving a problem?

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\text{(problem)} \arrow[r, "\text{Fourier transform}"] \arrow[d, dotted, "\text{solve}"', "\text{(hard)}"] & \text{(transformed problem)} \arrow[d, "\text{solve}"', "\text{(easy)}"] \\
\text{(solution)} & \text{(transformed solution)} \arrow[l, "\text{inverse Fourier transform}"']
\end{tikzcd}
$$

Discrete Fourier Transform (DFT) over $\mathbb{C}$

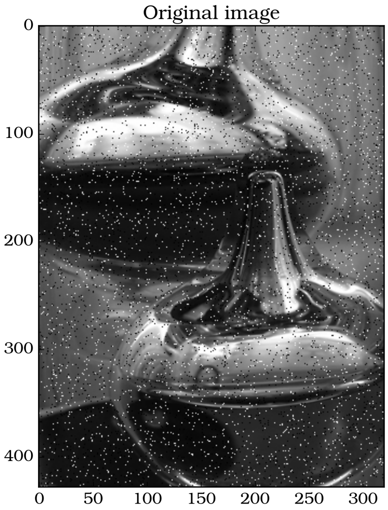
DFT transforms a sequence of N complex numbers $\{\mathbf{x_n}\} := x_0, x_1, \ldots, x_{N-1}$ into another sequence of complex numbers, $\{\mathbf{X_k}\} := X_0, X_1, \ldots, X_{N-1},$:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-\frac{2\pi i}{N}kn} = \sum_{n=0}^{N-1} x_n \cdot [\cos(\frac{2\pi kn}{N}) - i \cdot \sin(\frac{2\pi kn}{N})]$$

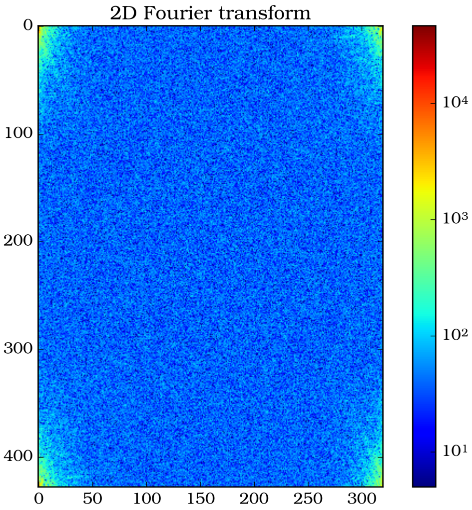
Applications of DFT over $\mathbb{C}$

- Digital signal processing,
- Solving partial differential equations,
- Fast polynomial multiplication,
- Multiplying large integers.

Example: Denoising image (1/4)

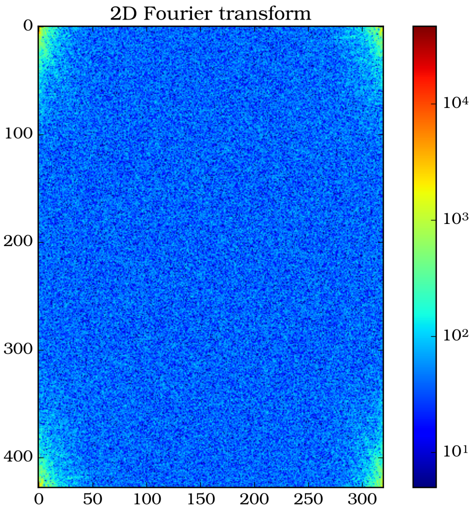


$\Rightarrow$

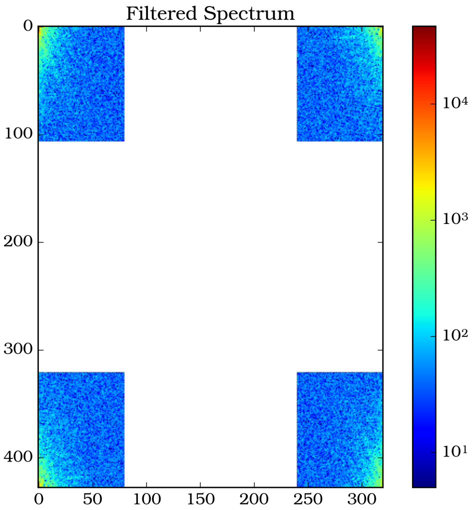


https://www.scipy-lectures.org/intro/scipy/auto_examples/solutions/plot_fft_image_denoise.html

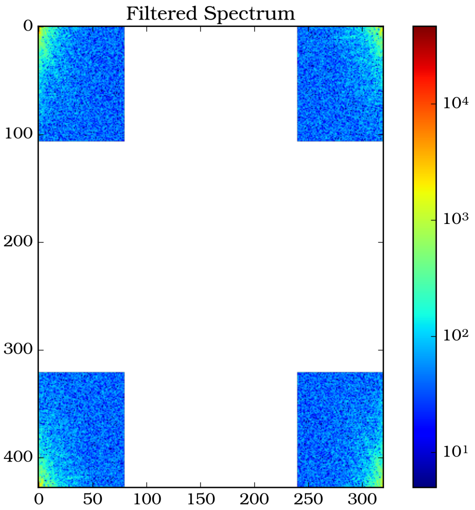
Example: Denoising image (2/4)



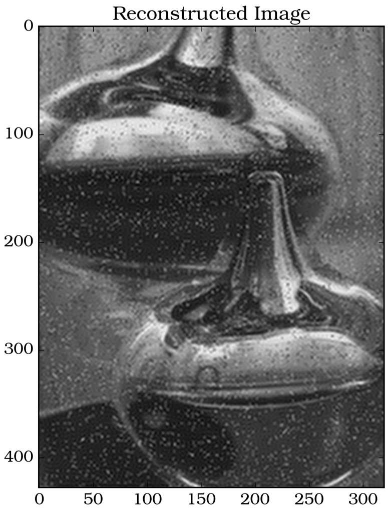
$\Rightarrow$



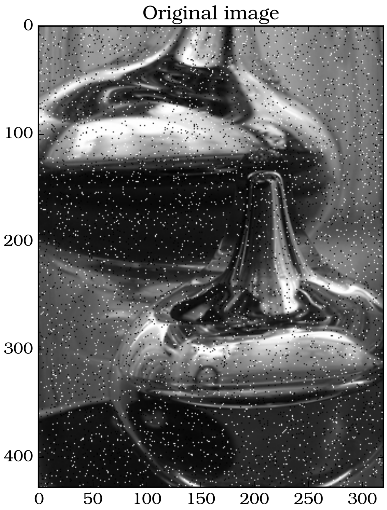
Example: Denoising image (3/4)



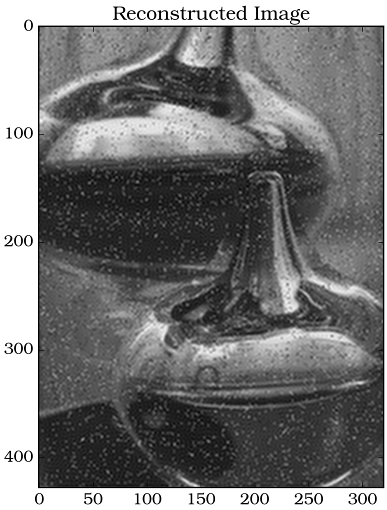
$\Rightarrow$



Example: Denoising image (4/4)



$\Rightarrow$



Complex DFT in practice

- DFT deals with a finite amount of data, so it can be implemented in computers
- In practice, it is realized both as a software, or through dedicated hardware.

A quick review

Fourier transform over $\mathbb{C}$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi t} f(t)\, dt$$

$$\Downarrow$$

Discret Fourier transform (DFT) over $\mathbb{C}$:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot [\cos(\frac{2\pi kn}{N}) - i \cdot \sin(\frac{2\pi kn}{N})]$$

$$\Downarrow$$

DFT over finite field

DFT over finite field

- We now turn our focus to finite fields.
- Then, the next question is what are finite fields?

Field

A set on which addition, subtraction, multiplication, and division are defined.

Examples of field

- $\mathbb{C}$ is a field
- $\mathbb{Z}$ is not a field as inverse is not defined for every element!

Finite field

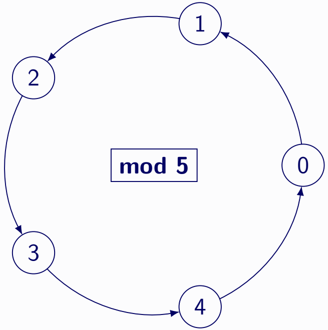
A field with a finite number of elements.

Prime field

- $\mathbb{Z}_p/\mathbb{Z} = \{0,\ 1,\ \ldots,\ p-1\}$
- The sum, the difference, and the product are computed, then, reduced modulo p.
- The modular inverse is computed using "extended Euclidean algorithm".

Example: $\mathbb{Z}_5/\mathbb{Z} = \{0,\ 1,\ 2,\ 3,\ 4\}$

- $3 + 4 \equiv 2 \pmod 5$
- $3 - 4 \equiv -1 \pmod 5 \equiv 4 \pmod 5$
- $3 * 2 \equiv 1 \pmod 5$
- $3^{-1} \equiv 2 \pmod 5$



Driving applications of prime fields

Coding theory, cryptography, and solving systems of polynomial equations.

Can we compute DFT over a prime field?

It can be proven that most attributes of DFT over $\mathbb{C}$ also hold over prime fields.

Definition

For prime field $\mathbb{Z}_p$, with $\omega \in \mathcal{R}$ a N-th root of unity, the following linear map is the DFT_N at ω:

$$\vec{a} = (a_0, \ldots, a_{N-1})^T \xrightarrow{\Omega} \vec{b} = (b_0, \ldots, b_{N-1})^T$$

with matrix Ω defined as follows:

$$\Omega = (\omega^{jk})_{0 \le j,k \le N-1} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(N-1)} & \omega^{2(N-1)} & \ldots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

Example: RSA crpyptosystem (1/3)

Alice	EAVESDROPPER	Bob
Generates public key (n, e) Generate private key d	Publishes public key (n,e) $\longrightarrow$	Receives public key (n, e)
$\downarrow$		$\downarrow$
Keeps her secret private!		Encrypts his message M using (n, e)
$\downarrow$	EAVESDROPPER	$\downarrow$
Retrieves M by using d to decrypt C	$\longleftarrow$ Sends C to Alice	Encrypted cipher C

Example: RSA crpyptosystem (2/3)

- Find three very large positive integers e, d and n such that for $0 \leq m < n$ we have:

$$\underbrace{(m^e)^d \equiv m \pmod{n}}_{\text{modular exponentiation}}$$

- The value of n is product of two randomly generated primes p and q.
- Encryption: $\underbrace{C \equiv m^e \pmod{n}}_{\text{modular exponentiation}}$
- Decryption: $\underbrace{C^d \equiv (m^e)^d \equiv m \pmod{n}}_{\text{modular exponentiation}}$

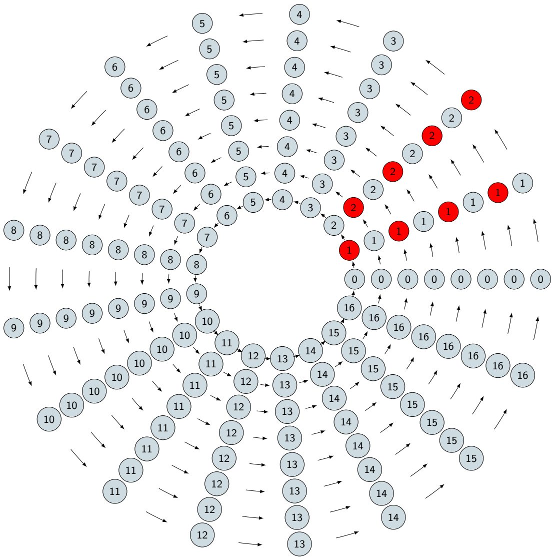
Example: RSA crpyptosystem (3/3)

Alice	EAVESDROPPER	Bob
Generate public key (n, e): $n = 7 * 11 = 77$ and $e = 13$ Generate private key $d = 7$	Publishes public key (n,e) $\longrightarrow$	Receives public key (n, e)
$\downarrow$		$\downarrow$
Keeps her secret private!		Encrypt $M = 15$: $C \equiv m^e \equiv 15^{13} \equiv 64 \pmod{77}$
$\downarrow$	EAVESDROPPER	$\downarrow$
Retrieve M: $C^d \equiv (m^e)^d \equiv 64^7 \equiv 15 \pmod{n}$	$\longleftarrow$ Sends C to Alice	Encrypted cipher C

Comuting DFT_8 for

$\vec{v} = [1, 2, 1, 2, 1, 2, 1, 2]$

- $n = 8$ and $n^{-1} = 15$,
- $\omega = 2$ (as $\omega^8 \equiv 1 \pmod{17}$), and
- $\omega^{-1} = 9$ (as $\omega\omega^{-1} \equiv 1 \pmod{17}$).

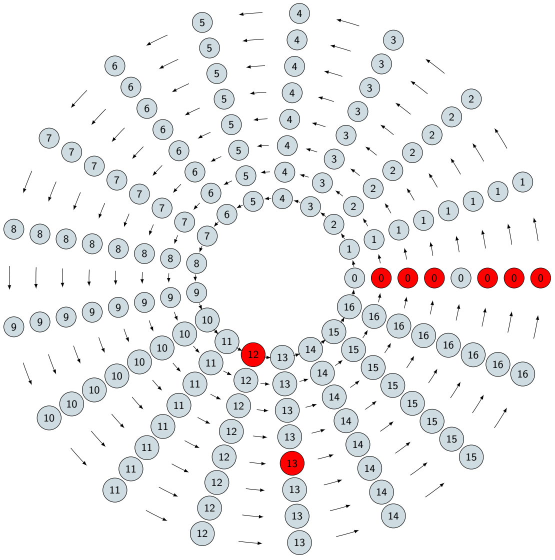


Example: computing DFT over $\mathbb{Z}_{17}$ (II)

$$\Omega = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 15 & 13 & 9 \\ 1 & 4 & 16 & 13 & 1 & 4 & 16 & 13 \\ 1 & 8 & 13 & 2 & 16 & 9 & 4 & 15 \\ 1 & 16 & 1 & 16 & 1 & 16 & 1 & 16 \\ 1 & 15 & 4 & 9 & 16 & 2 & 13 & 8 \\ 1 & 13 & 16 & 4 & 1 & 13 & 16 & 4 \\ 1 & 9 & 13 & 15 & 16 & 8 & 4 & 2 \end{bmatrix}$$

$$\begin{aligned} DFT_8(\vec{v}) &= \Omega\vec{v} \\ &= \Omega([1,2,1,2,1,2,1,2]) \\ &= [12,0,0,0,13,0,0,0] \end{aligned}$$



Example: DFT as an evaluation-interpolation scheme

$$f(x) = 1x^0 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7$$

$$\left.\begin{aligned}
y_0 &= f(\omega^0) = 2\\
y_1 &= f(\omega^1) = 8\\
y_2 &= f(\omega^2) = 14\\
y_3 &= f(\omega^3) = 6\\
y_4 &= f(\omega^4) = 13\\
y_5 &= f(\omega^5) = 3\\
y_6 &= f(\omega^6) = 12\\
y_7 &= f(\omega^7) = 1
\end{aligned}\right\} \mathit{INTERPOLATE}((y_0, y_1, \cdots, y_7)) = \mathrm{DFT}^{-1}((y_0, y_1, \cdots, y_7)) = [1, 2, 3, 4, 5, 6, 7, 8]$$

DFT-based polynomial multiplication over $\mathbb{Z}_p$

$$
\begin{array}{ccccc}
f(x) = \sum_{i=0}^{n-1} a_i x_i & \longrightarrow & \vec{a} & \xrightarrow{\mathrm{DFT}_n(\omega)} & \mathrm{DFT}(\vec{a}) \\
g(x) = \sum_{i=0}^{n-1} b_i x_i & \longrightarrow & \vec{b} & \xrightarrow{\mathrm{DFT}_n(\omega)} & \mathrm{DFT}(\vec{b}) \\
\downarrow & & & & \downarrow \\
f(x)g(x) \equiv \sum_{i=0}^{n-1} c_i x_i \pmod{x^n - 1} & \rightarrow & \vec{c} & \xleftarrow{\mathrm{DFT}_n^{-1}(\omega^{-1})} & \mathrm{DFT}(\vec{a}) * \mathrm{DFT}(\vec{b})
\end{array}
$$

Example: DFT-based polynomial multiplication over $\mathbb{Z}_{17}$

$$
\begin{aligned}
f(x) &= 1x^0 + 2x^1 + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 \\
g(x) &= 8x^0 + 7x^1 + 6x^2 + 5x^3 + 4x^4 + 3x^5 + 2x^6 + 1x^7 \\
f(x)g(x) &\equiv 6x^0 + 3x^1 + 8x^2 + 4x^3 + 8x^4 + 3x^5 + 6x^6 + 0x^7 \pmod{p}
\end{aligned}
$$

Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the vector of cofficients for $f(x)$, $g(x)$, and $f(x)g(x)$, respectively.

$$
\begin{array}{lllcrcl}
\vec{a} & = & [1,2,3,4,5,6,7,8] & \xrightarrow{\mathrm{DFT}_8(\omega)} & \mathrm{DFT}_8(\vec{a}) & = & [2,8,14,6,13,3,12,1] \\
\vec{b} & = & [8,7,6,5,4,3,2,1] & \xrightarrow{\mathrm{DFT}_8(\omega)} & \mathrm{DFT}_8(\vec{b}) & = & [2,9,3,11,4,14,5,16] \\
\vec{c} & = & [6,3,8,4,8,3,6,0] & \xleftarrow{\mathrm{DFT}_8^{-1}(\omega^{-1})} & (\mathrm{DFT}_8(\vec{a}) * \mathrm{DFT}_8(\vec{b})) & = & [4,4,8,15,1,8,9,16]
\end{array}
$$

What do we achieve by computing DFT via FFT?

Fast Fourier transform (FFT): a fast divide-and-conquer algorithm

- Naively computing DFT over a vector of size n takes $O(n^2)$.
- Efficient DFT computation is done via fast Fourier transform (FFT) which takes $O(n \log n)$.
- First mentioned by Gauss (1805), popularized by IBM fellows Cooley and Tukey (1965).
- Using FFT, DFT-based polynomial multiplication leads to faster division, gcd, and factorization!

Problem

Problem

- Some computations over prime fields need high accuracy.
- This can only be achieved if computing is done directly over prime fields of a large characteristic.

Our work

- CUDA implementation of arithmetic over $\mathbb{Z}/p\mathbb{Z}$ for p of size of at least 8 or 16 machine words.
- CUDA implementation of FFT over $\mathbb{Z}/p\mathbb{Z}$.
- Theoretical and practical comparison of our approach vs. an approach based on small prime fields.

Outline

1. Fourier transforms
2. Fürer's trick: beyond Cooley-Tukey factorization
3. Implementation challenges
4. Experimental Comparison

Outline

Cooley-Tukey factorization formula

- Expressed in tensor notation, for $J, K > 1$ and $n = JK$, we have:

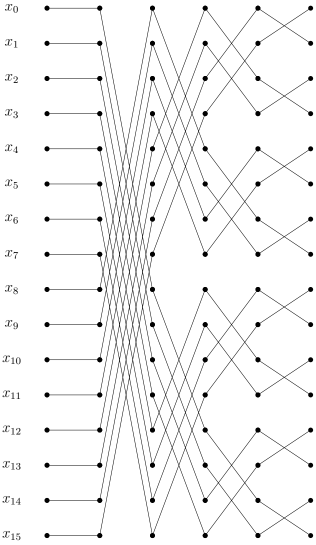
$$\mathrm{DFT}_n = \mathrm{DFT}_{JK} \quad = \quad (\mathrm{DFT}_J \otimes I_K) D_{J,K} (I_J \otimes \mathrm{DFT}_K) L_J^{JK}$$

  where

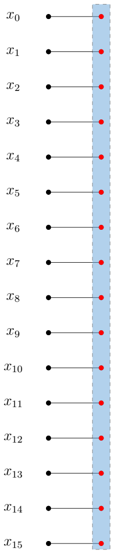
$$\begin{aligned} D_{J,K} &= \bigoplus_{j=0}^{J-1} \mathrm{diag}(1, \omega^j, \ldots, \omega^{j(K-1)}), \\ L_J^{JK} &= \mathbf{x}[iJ + j] \mapsto \mathbf{x}[jJ + i], (0 \leq j < J, 0 \leq i < K.) \end{aligned}$$

- Various fast Fourier transform algorithms can be derived from this formula.
- We can greatly benefit from sparsity of factorized matrices.

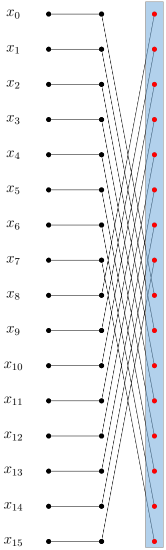
Example: FFT algorithm and data locality



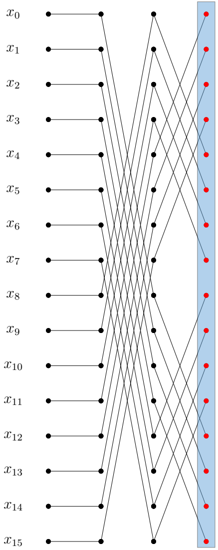
Example: FFT algorithm and data locality: radix-2 (1/5)



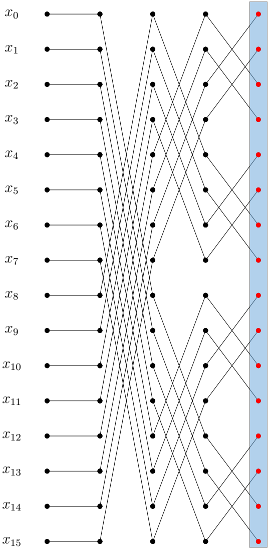
Example: FFT algorithm and data locality: radix-2 (2/5)



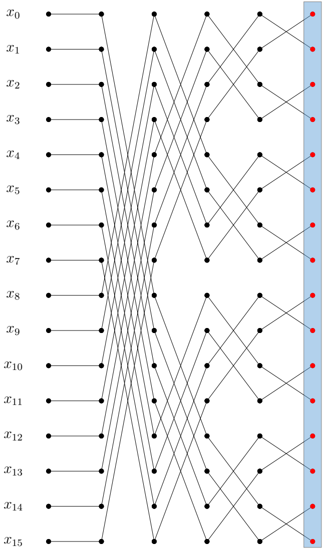
Example: FFT algorithm and data locality: radix-2 (3/5)



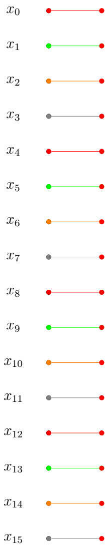
Example: FFT algorithm and data locality: radix-2 (4/5)



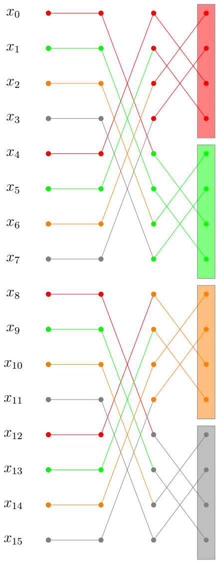
Example: FFT algorithm and data locality: radix-2 (5/5)



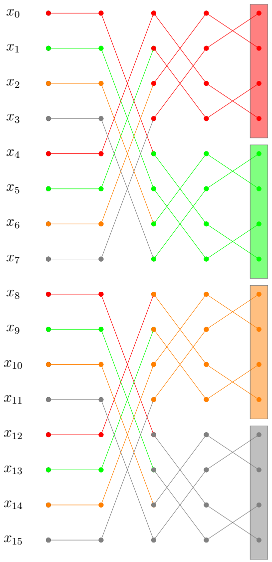
Example: FFT algorithm and data locality: blocking (1/5)



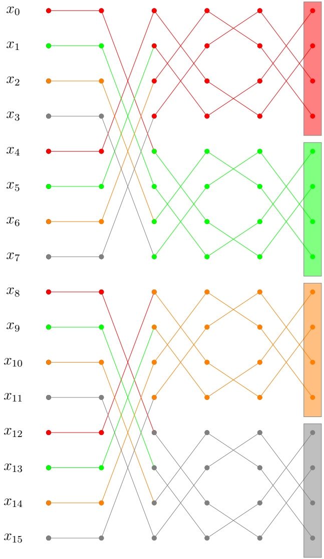
Example: FFT algorithm and data locality: blocking (2/5)



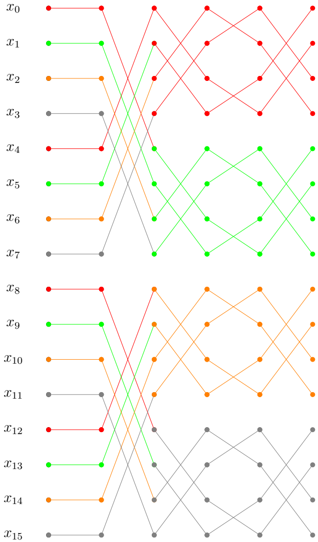
Example: FFT algorithm and data locality: blocking (3/5)



Example: FFT algorithm and data locality: blocking (4/5)



Example: FFT algorithm and data locality: blocking (5/5)



Complexity analysis (1/2)

Analysis of FFT over a prime field

$$\mathrm{DFT}_{JK} = \underbrace{(\mathrm{DFT}_J \otimes I_K)}_{K \text{ DFT's of size } J} \quad \underbrace{D_{J,K}}_{N \text{ mults by a power of } \omega} \quad \underbrace{(I_J \otimes \mathrm{DFT}_K)}_{J \text{ DFT's of size } K} \quad L_J^{JK}$$

- For $f \in \mathbb{Z}/p\mathbb{Z}[x]$ of degree at most $N-1$, computing $\mathrm{DFT}_N(f)$ at ω by a FFT amounts to
  - $N\log(N)$ additions in $\mathbb{Z}/p\mathbb{Z}$,
  - $N/2\log(N)$ multiplications by a power of ω in $\mathbb{Z}/p\mathbb{Z}$.
- If p spans k machine words, the cost of each addition remains linear ($\mathcal{O}(k)$ word ops.), but multiplication by a power of ω becomes a bottleneck as k grows ($O(\mathsf{M}(k))$ word ops).
- Can we reduce cost of some multiplications to a cost of an addition?

Complexity analysis (2/2)

Fürer's trick

- Let $N = K^e$ for some "small" K, $J = \frac{N}{K} = K^{e-1}$ and $\eta = \omega^{N/K}$.
- Assume that multiplying any element of $\mathbb{Z}/p\mathbb{Z}$ by a power of η is as cheap as an addition.
- Radix r fits in one machine word (32 or 64 bits wide depending on the device).
- This latter result holds whenever p is a prime of the form $p = r^k + 1$ (*a generalized Fermat number*).

Applying Furer's trick to CT factorization

- Using $p = r^k + 1$, with $K = 2k$, $\mathrm{DFT}_N(\omega)$ amounts to:

$$O(N\,\log_2(N)\,k \;+\; N\,\log_k(N)\,\mathsf{M}(k)) \text{ word ops.}$$

- Without our assumption, the same DFT would run in $O(N\,\log_2(N)\,\mathsf{M}(k))$ word ops.
- Using $p = r^k + 1$ results in a speedup factor of $\log(K)$.

Outline

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2. Fürer's trick: beyond Cooley-Tukey factorization
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Outline

Compute Unified Device Architecture (CUDA)

- A platform developed by NVIDIA corporation.
- Provides an API for writing scalable parallel programs on GPUs

The CUDA execution and programming model

- Streaming multiprocessors (SMs): building blocks of GPUs
- Kernel: The function that is executed on the GPU.
- Device: the GPU that executes kernels.
- Warp: The GPU scheduler deploys every 32 threads together for execution.

On-chip Memory:

- Registers
- L1 cache
- Shared memory

Off-chip memory:

- Global memory
- L2 cache
- Local memory

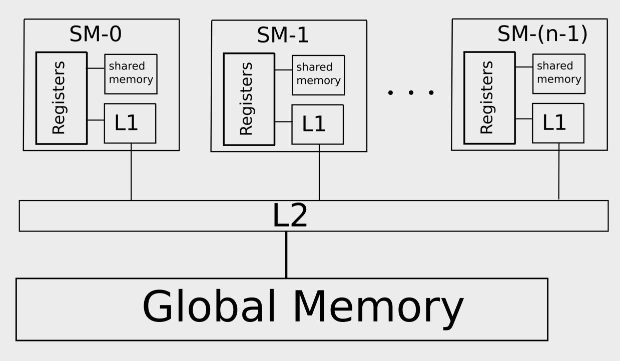


Figure: The CUDA memory model for CC 2.x and higher.

Choice of FFT algorithm:

- Using GPU as a block processor (i.e., lots of SM's working together).
- Block parallelism can be realized by $I_J \otimes \mathrm{DFT}_K$.
- We use the six-step recursive FFT algorithm

$$\mathrm{DFT}_N = L_K^N (I_J \otimes \mathrm{DFT}_K) L_J^N D_{K,J} (I_K \otimes \mathrm{DFT}_J) L_K^N.$$

- We expand $I_K \otimes \mathrm{DFT}_J$ to turn all DFT computations to base-case DFT_K.

Parallelization of arithmetic operations.

- Initial idea: using multiple threads for computing one operation!
- High overhead will not improve the performance before a threshold! (think of $p > 2^{2048}$)
- Solution: we need to compute operations in batches; one big number handled by one thread.

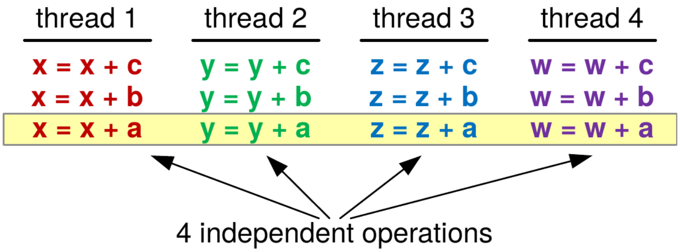


Figure: Better performance at lower occupancy, Vasily Volkov

Implementation challenges (2/6, continued)

Maximizing global memory efficiency.

- Consider $\vec{X}$ as a vector of N elements of $\mathbb{Z}/p\mathbb{Z}$
- Assume that consecutive digits of each element are stored in adjacent memory addresses.
- View it as the row-major layout of a $N \times k$ matrix.
- Problem: performance is hurt due to increased memory overhead.
- Solution: perform a stride permutation (transposition) L_k^{kN} on all input vectors.
- Result: increasing memory (load/store) efficiency.

$$\begin{bmatrix} \vec{X}_0 \\ \vec{X}_1 \\ \vdots \\ \vec{X}_{N-1} \end{bmatrix} = \begin{bmatrix} \vec{X}_{(0,0)} & \vec{X}_{(0,1)} & \vec{X}_{(0,2)} & \cdots & \vec{X}_{(0,k-1)} \\ \vec{X}_{(1,0)} & \vec{X}_{(1,1)} & \vec{X}_{(1,2)} & \cdots & \vec{X}_{(1,k-1)} \\ & & & & \\ \vec{X}_{(N-1,0)} & \vec{X}_{(N-1,1)} & \vec{X}_{(N-1,2)} & \cdots & \vec{X}_{(N-1,k-1)} \end{bmatrix}_{(N \times k)} \xrightarrow{L_k^{kN}} \begin{bmatrix} \vec{X}_{(0,0)} & \vec{X}_{(1,0)} & & \vec{X}_{(N-1,0)} \\ \vec{X}_{(0,1)} & \vec{X}_{(1,1)} & & \vec{X}_{(N-1,1)} \\ \vec{X}_{(0,2)} & \vec{X}_{(1,2)} & & \vec{X}_{(N-1,2)} \\ \vdots & \vdots & & \vdots \\ \vec{X}_{(0,k-1)} & \vec{X}_{(1,k-1)} & & \vec{X}_{(N-1,k-1)} \end{bmatrix}_{(k \times N)}$$

Implementation challenges (3/6)

Memory-bound kernels.

- Problem: Performance is limited by frequent accesses to memory.
- Solutions:
  - minimizing memory latency (through buffering),
  - maximizing occupancy (#active warps on each SM) to hide latency, and
  - maximizing IPC (instructions per clock cycle): exploiting ILP.
  - avoiding use of shared memory (to keep occupancy high),
  - turn off L1 cache for operations that do not reuse data: all add, sub
  - keep all data on global memory

Implementation challenges (4/6)

Register spilling

- Problem: Using many registers per thread can lower the occupancy,
- Solution: register-intensive kernels are broken into multiple smaller ones.

Implementation challenges (5/6)

Maximizing occupancy

- Problem: For the same application, different GPUs need different kernel parameters for achieving peak performance.
- Solution: Design kernels that are oblivious to the size of a thread block.
- We choose size of thread blocks for maximizing bandwidth-related performance metrics (read and write throughput).

Implementation challenges (6/6)

Effect of GPU instructions on performance

- Initial idea: implementation based on 64-bit instructions (64-bit radix should be better!)
- Problem: Compiler converts all instructions to a sequence of 32-bit equivalents, this conversion can have negative impact on the overall performance (specially, 64-bit multiplication!).
- Solution: Using 32-bit arithmetic provides more opportunities for optimization such as ILP

Argument for NOT using 64-bit arithmetic

- Let's see how GPU behaves in each case.
- There is a cost for converting between 64-bit types to and from all other 32-bit types.
- All 64-bit instructions have a significantly lower IPC count compared to their 32-bit counterparts. Table 1 is taken from Pages 85-86 of CUDA C PROGRAMMING GUIDE.

Instruction throughput comparison

Table: Number of Results per Clock Cycle per Multiprocessor

Instruction /CC	3.0, 3.2	3.5, 3.7	5.0, 5.2	5.3	6.0	6.1	6.2	7.0
32-bit integer add, extended-precision add, subtract, extended-precision subtract	160	160	128	128	64	128	128	64
32-bit integer multiply, multiply-add, extended-precision multiply-add	32	32	Multiple inst.	Multiple inst.	Multiple inst.	Multiple inst.	Multiple inst.	64
32-bit bitwise AND, OR, XOR	160	160	128	128	64	128	128	64
Type conversions from and to 64-bit types	8	32	4	4	16	4	4	16
32-bit floating point add, multiply, multiply-add	192	192	128	128	64	128	128	64
64-bit floating point add, multiply, multiply-add	8	64	4	4	32	4	4	32
32-bit integer shift	32	64	64	64	32	64	64	64
compare, minimum, maximum	160	160	64	64	32	64	64	64

A lesson hard-learned: better to avoid 64-bit instructions in CUDA

"Forward looking GPU integer performance"

One of the answers verified by NVIDIA on this subject:

```
" Yes, 64-bit arithmetic is accomplished via instruction sequences generated
by the compiler (on all current CUDA GPUs). There is no native 64 bit integer
add or multiply instruction ... "
Posted 06/29/2016 04:52 AM, by txbob
```

https://devtalk.nvidia.com/default/topic/948014/forward-looking-gpu-integer-performance/?offset=14

Profiling addition for $N = 2^{16}$ (random input) for $P_3 = (2^{63} + 2^{34})^8$ ($K = 16$)

Metric Description	Avg. non-transposed	Avg. transposed
Achieved Occupancy	92%	87%
Executed IPC	0.13	0.72
Global Memory Load Efficiency	25%	99%
Global Memory Store Efficiency	25%	100%
Instruction Replay Overhead	3.22	0.36
Global Memory Replay Overhead	1.78	0.10
Device Memory Read Throughput	26 GB/s	30 GB/s
Device Memory Write Throughput	13 GB/s	15 GB/s
Total Kernel Time	2.395ms	0.504ms

Table: Profiling results for transposed and non-transposed addition in $\mathbb{Z}/p\mathbb{Z}$

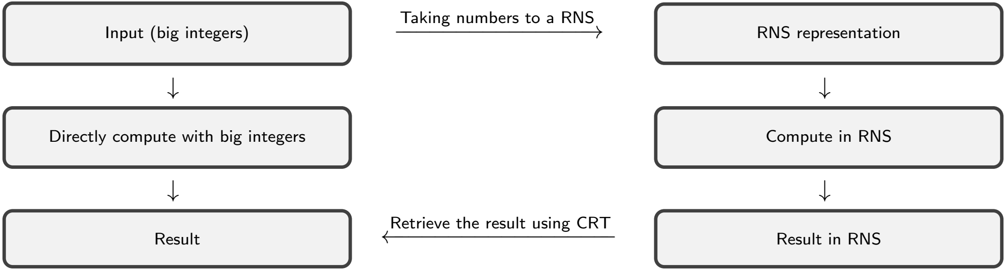
Profiling multiplication by a power of radix ($x\ r^s$) for $N = 2^{16}$ for $P_3 = (2^{63} + 2^{34})^8$ ($K = 16$)

Metric Description	Avg. non-transposed	Avg. transposed
Achieved Occupancy	91%	57%
Executed IPC	0.18	4.61
Global Memory Load Efficiency	25%	86%
Global Memory Store Efficiency	25%	87%
Instruction Replay Overhead	3.19	0.04
Global Memory Replay Overhead	1.22	0.01
Device Memory Read Throughput	14 GB/s	20 GB/s
Device Memory Write Throughput	18 GB/s	20 GB/s
Total Kernel Time	3.331ms	0.380ms

Table: Profiling results for transposed and non-transposed $x\ r^s$ in $\mathbb{Z}/p\mathbb{Z}$

Outline

Outline


Input (big integers)
Taking numbers to a RNS
RNS representation
Directly compute with big integers
Compute in RNS
Result
Retrieve the result using CRT
Result in RNS

Big prime vs. small prime: RNS and CRT (2/2)

Example of RNS-CRT scheme for 4-bit numbers

- If M is the maximum absolute value that will be used in computation, a RNS/CRT scheme needs primes $M < p_1 \times \cdots \times p_e$.
- For $0 \leq x, y < B$, the $M = max(x * y) = (2^B - 1)^2$.
- With $B = 4$, $M = (2^4 - 1)^2 = 225$, then, we can pick $p1 = 3, p2 = 7, p3 = 11$ (as $225 < 3 \times 7 \times 11$).

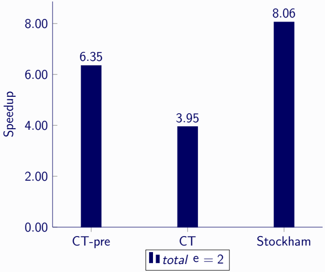
$$
\begin{array}{cccccc}
x & = 14 & \xrightarrow{RNS\ (3,7,11)} & x' & = (2,0,3) \\
y & = 10 & \xrightarrow{RNS\ (3,7,11)} & y' & = (1,3,10) \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
xy & = 140 & \xrightarrow{RNS\ (3,7,11)} & x'y' & = (2,0,8) \\
 & & \xleftarrow{CRT\ (3,7,11)} & &
\end{array}
$$

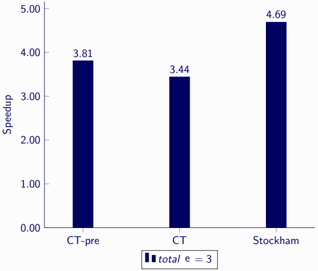
Big prime vs. Small prime

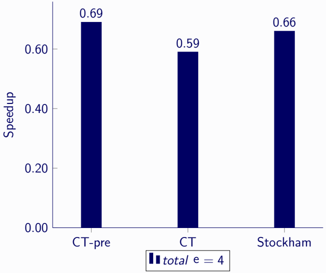
Computing FFT: big prime vs. small prime

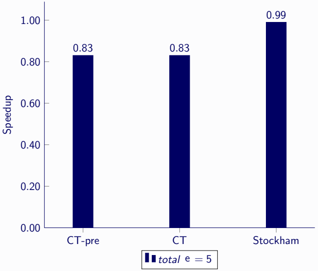
- Small prime approach: pairwise different primes $p_1, \ldots, p_k$
  1. compute image f_i of f in $\mathbb{Z}/p_1\mathbb{Z}[x], \ldots, \mathbb{Z}/p_k\mathbb{Z}[x]$ (*projection*)
  2. compute $\mathrm{DFT}_N(f_i)$ at ω_i in $\mathbb{Z}/p_i\mathbb{Z}[x]$
  3. combine the results using the CRT (*recombination*)
- The small primes are $\dfrac{\text{machine-word size}}{2}$, it is fair to use $2k$ of them!
- Small prime FFTs from the CUMODP library compute DFT_{2^n} for $8 \leq n \leq 26$:
  - the Cooley-Tukey FFT,
  - the Cooley-Tukey FFT with pre-computed powers of ω,
  - the Stockham FFT.
- Tests completed on a NVIDA GTX-1080Ti (3584 CUDA cores @1.5 GHZ, Memory @5 GHZ)

Benchmarking for $P_3 = (2^{63} + 2^{34})^8$ $(K = 16)$

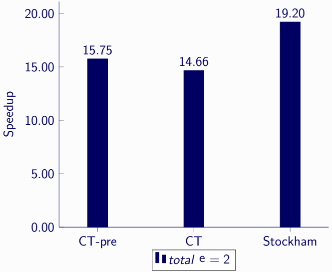


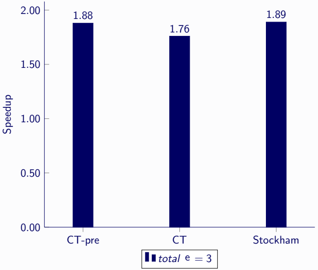


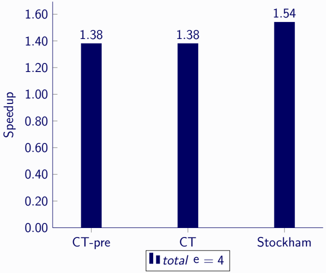




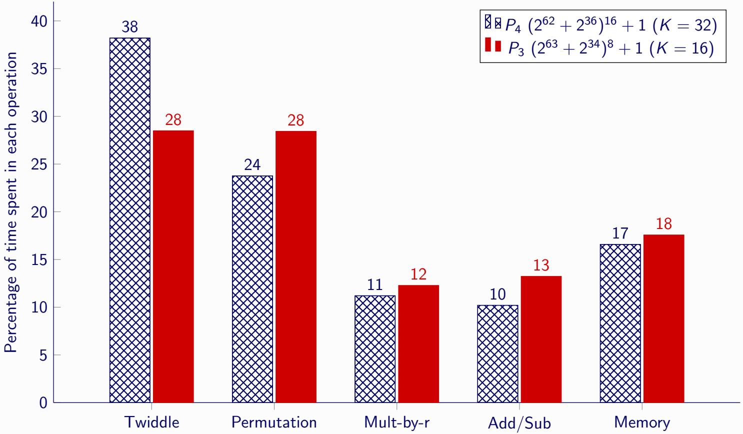
Benchmarking for $P_4 = (2^{62} + 2^{36})^{16}$ $(K = 32)$







Profiling results for DFT_{K^4} with P_3 $(K = 16)$ and P_4 $(K = 32)$



Performance analysis:

Performance analysis:

- Larger p in $\mathbb{Z}/p\mathbb{Z} \Rightarrow$ a higher number of cheap multiplications!
- Beyond a certain point, we have more expensive multiplications!

Addressing performance bottlenecks

- Multiplication algorithm
- Suboptimal use of device due to 64-bit arithmetic (emulated on CUDA GPUs).
- Difficulty in adapting the code for new primes (mostly due to difficulties in multiplication).

Concluding remarks

Work in progress

- Moving to a complete implementation based on 32-bit arithmetic.
- Improving multiplication in $\mathbb{Z}/p\mathbb{Z}$ is work in progress.
- By choosing a larger prime, say $k = 32$, or $k = 64$, we hope to cover other ranges of sizes.
- We have been working on a multi-core implementation and have managed to use FFT for multiplying elements of $\mathbb{Z}/p\mathbb{Z}$ for k large enough ($k \geq 64$).

Conclusions

- Big prime field arithmetic is required by many advanced algorithms.
- Arithmetic modulo a big prime can be efficiently computed on GPUs.
- We have been the first in putting Fürer's ideas into practice and experimentally verify them.
- Computing FFT in $\mathbb{Z}/p\mathbb{Z}$ is competitive with a CRT-based approach for a range of sizes.
- Multiplication in $\mathbb{Z}/p\mathbb{Z}$ (except for the case of a mult by a power of r) remains a bottleneck.

Thank You!

Your Questions?

Appendix

Main References

- Faster integer multiplication by Fürer
- Modern computer algebra 3rd edition by von zur Gathen and Gerhard
- Fast polynomial arithmetic by Moreno Maza and Pan
- How to write fast numerical code by Chellappa, Franchetti, and Puschel
- Fast Fourier transforms by Franchetti and Puschel
- CUDA C programming guide 8.0 by NVIDIA Corporation

Sparse-radix generalized Fermat numbers (SRGFN)

- Generalized Fermat numbers $a^{2^n} + b^{2^n}$ where $a > 1$, $b \geq 0$ and $n \geq 0$.
- For $F_n(a) = a^{2^n} + 1$, a is a 2^{n+1}-th primitive root of unity $\mathbb{Z}/F_n(a)\mathbb{Z}$,
- A SRGFN is any $F_n(r)$ where r is $2^w + 2^u$ or $2^w - 2^u$, for $w > u \geq 0$

Representing elements of $\mathbb{Z}/F_n(r)\mathbb{Z}$

- Let $p = F_n(r)$, and $k = 2^n$ s.t. $p = r^k + 1$ holds
- Represent $x \in \mathbb{Z}/p\mathbb{Z}$ as $\vec{x} = (x_{k-1}, x_{k-2}, \ldots, x_0)$ such that:

$$x \equiv x_{k-1}\, r^{k-1} + x_{k-2}\, r^{k-2} + \cdots + x_0 \mod p.$$

1. either $x_{k-1} = r$ and $x_{k-2} = \cdots = x_1 = 0$, or
2. $0 \leq x_i < r$ for all $i = 0 \cdots (k-1)$

Addition and subtraction in $\mathbb{Z}/p\mathbb{Z}$

Add(sub) with carry for $x, y \in \mathbb{Z}/p\mathbb{Z}$ represented by $\vec{x}, \vec{y}$ with k coefficients.

Multiplication in $\mathbb{Z}/p\mathbb{Z}$

- Associate $x, y \in \mathbb{Z}/p\mathbb{Z}$ with polynomials $f_x, f_y \in \mathbb{Z}[T]$
- For large k, $f_x f_y \mod T^k + 1$ can be computed in $\mathbb{Z}[T]$ by fast algorithms
- For small k, say $k \leq 16$, using plain multiplication is reasonable.

Multiplication by r^i for some $0 < i < 2k$ in $\mathbb{Z}/p\mathbb{Z}$

- Recall that $r^{2k} \equiv 1$, $r^k \equiv -1$, and $r^{k+i} \equiv -r^i$, for $0 < i < k$
- Assume that $0 < i < k$ holds and $0 \leq x < r^k$ holds in $\mathbb{Z}$, then:

$$
\begin{aligned}
x\, r^i &\equiv x_{k-1}\, r^{k-1+i} + \cdots + x_0\, r^i \mod p \\
&\equiv \sum_{j=0}^{j=k-1} x_j r^{j+i} \mod p \equiv \sum_{h=i}^{h=k-1+i} x_{h-i} r^h \mod p \\
&\equiv \sum_{h=i}^{h=k-1} x_{h-i} r^h - \sum_{h=k}^{h=k-1+i} x_{h-i} r^{h-k} \mod p
\end{aligned}
$$

Finding primitive roots of unity in $\mathbb{Z}/p\mathbb{Z}$

Computing ω:

- **Goal**: speed up $x\omega^i$ for $x \in \mathbb{Z}/p\mathbb{Z}$.
- Let $N = 2^\ell$, s.t. $N \mid p - 1$
- Assume that $g^N = 1$ for $g \in \mathbb{Z}/p\mathbb{Z}$
- Write $p = qN + 1$.
- Pick a random $\alpha \in \mathbb{Z}/p\mathbb{Z}$
- Let $\omega = \alpha^q$.
- Fermat's little theorem:
  - if $\omega^{N/2} = 1 \Rightarrow \omega^N = 1$,
  - if $\omega^{N/2} = -1$, pick another α.

Algorithm 1 Primitive N-th root $\omega \in \mathbb{Z}/p\mathbb{Z}$ s.t. $\omega^{N/2k} = r$

procedure $\textsc{PRoot}(N, r, k, g)$
  $\alpha := g^{N/2k}$
  $\beta := \alpha$
  $j := 1$
  while $\beta \neq r$ **do**
    $\beta := \alpha\beta$
    $j := j + 1$
  end while
  $\omega := g^j$
  return (ω)
end procedure

Host entry point function

Template HostGeneralOperation$(\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \vec{\mathtt{U}}, \mathtt{N}, \mathtt{k}, \mathtt{r}, \mathtt{b})$

Input:

- an integer b giving the size of 1D thread block,
- Positive integer k, r, and N
- Vectors $\vec{\mathtt{X}}$ and $\vec{\mathtt{Y}}$, each of size N

Output:

- vector $\vec{\mathtt{U}}$ storing the result ($\vec{\mathtt{U}} :=$ operation$(\vec{\mathtt{X}}, \vec{\mathtt{Y}})$).

$\vec{\mathtt{X}} :=$ HostTranspose$(\vec{\mathtt{X}}, \mathtt{N}, \mathtt{k})$

$\vec{\mathtt{Y}} :=$ HostTranspose$(\vec{\mathtt{Y}}, \mathtt{N}, \mathtt{k})$

KernelGeneralOperation$<<< \frac{\mathtt{N}}{\mathtt{b}}, \mathtt{b} >>>(\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \vec{\mathtt{U}}, \mathtt{N}, \mathtt{k}, \mathtt{r})$

return $\vec{\mathtt{U}}$

Template KernelGeneralOperation($\vec{X}, \vec{Y}, \vec{U}, N, k, r$)

local: stride := N

local: offset:=0

local: vectors $\vec{x}$, $\vec{y}$, $\vec{u}$ each storing k digits, all initialized to zero.

local: tid := blockIdx.x∗blockSize.x+threadIdx.x

for ($0 \leq i < k$) **do**

  offset:=tid +i*stride

  $\vec{x}[i] := \vec{X}[\texttt{offset}]$ ▷ Reading the digit with the index i of element $\vec{X}_{tid}$.

  $\vec{y}[i] := \vec{Y}[\texttt{offset}]$ ▷ Reading the digit with the index i of element $\vec{Y}_{tid}$.

end for

$\vec{u}$:= DeviceGeneralOperation($\vec{x}, \vec{y}, k, r$). ▷ each thread computing one element of the final result.

for ($0 \leq i < k$) **do**

  offset:=tid +i*stride

  $\vec{U}[\texttt{offset}] := \vec{u}[i]$

end for

return ▷ End of Kernel

Host entry point for DFT_{K^e}

Algorithm 4 HostDFTGeneral($\vec{\mathtt{X}}, \vec{\Omega}, \mathtt{N}, \mathtt{K}, \mathtt{k}, \mathtt{s}, \mathtt{r}, \mathtt{b}$)

local: $\mathtt{m} := \mathtt{e}$ where $N = K^e$, $\mathtt{j} := 0$
if $\mathtt{e} \mod 2 = 1$ **then**
    HostGeneralStridePermutation($\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \mathtt{K}^1, \mathtt{N}, \mathtt{k}, \mathtt{s}, \mathtt{b}$)
end if
for $(0 \leq \mathtt{i} < \mathtt{m}$ by $2)$ **do**
    HostDFTK2($\vec{\mathtt{X}}, \vec{\Omega}, \mathtt{N}, \mathtt{K}, \mathtt{k}, \mathtt{s}, \mathtt{r}$)
    KernelTwiddleMultiplication($\vec{\mathtt{X}}, \vec{\Omega}, \mathtt{N}, \mathtt{K}, \mathtt{k}, \mathtt{s} := 2, \mathtt{r}$)
    HostGeneralStridePermutation($\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \mathtt{K}^2, \mathtt{N}, \mathtt{k}, \mathtt{s}, \mathtt{b}$)
    $\vec{\mathtt{X}}[0 : \mathtt{kN} - 1] := \vec{\mathtt{Y}}[0 : \mathtt{kN} - 1]$
    HostDFTK2($\vec{\mathtt{X}}, \vec{\Omega}, \mathtt{N}, \mathtt{K}, \mathtt{k}, \mathtt{s}, \mathtt{r}$)
    HostGeneralStridePermutation($\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \mathtt{K}^2, \mathtt{N}, \mathtt{k}, \mathtt{s}, \mathtt{b}$)
    $\vec{\mathtt{X}}[0 : \mathtt{kN} - 1] := \vec{\mathtt{Y}}[0 : \mathtt{kN} - 1]$
end for

Algorithm 5 HostDFTGeneral$(\vec{\mathtt{X}}, \vec{\Omega}, \mathtt{N}, \mathtt{K}, \mathtt{k}, \mathtt{s}, \mathtt{r}, \mathtt{b})$

if $(\mathtt{e} \mod 2 = 1)$ **then**
    KernelTwiddleMultiplication$(\vec{\mathtt{X}}, \vec{\Omega}, \mathtt{N}, \mathtt{K}, \mathtt{k}, \mathtt{s} := 2, \mathtt{r})$
    HostGeneralStridePermutation$(\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \mathtt{K}^1, \mathtt{N}, \mathtt{k}, \mathtt{s}, \mathtt{b})$
    $\mathtt{X}[0 : \mathtt{kN} - 1] := \vec{\mathtt{Y}}[0 : \mathtt{kN} - 1]$
    KernelBaseDFTKAllSteps$(\vec{\mathtt{X}}, \mathtt{N}, \mathtt{K}, \mathtt{r})$
    HostGeneralStridePermutation$(\vec{\mathtt{X}}, \vec{\mathtt{Y}}, \mathtt{K}^1, \mathtt{N}, \mathtt{k}, \mathtt{s}, \mathtt{b})$
    $\vec{\mathtt{X}}[0 : \mathtt{kN} - 1] := \vec{\mathtt{Y}}[0 : \mathtt{kN} - 1]$
end if
return $\vec{\mathtt{X}}$

Mixed-radix conversion

Mixed-radix representation

- For pairwise different primes $p_1, \ldots, p_s$:

$$b_i \in \mathbb{Z}/p_i\mathbb{Z}, \quad 0 \leq b_i < p_i, \quad (1 \leq i \leq s).$$

- Then $(b_1, b_2, \ldots, b_s)$ is *mixed-radix representation* of $n \in \mathbb{Z}$:

$$\begin{cases} n = b_1 + b_2 p_1 + b_3 p_1 p_2 + \cdots + b_s p_1 \cdots p_{s-1} \\ 0 \leq n < p_1 p_2 \cdots p_s \end{cases}$$

Reconstructing n from $(b_1, b_2, \ldots, b_s)$

- pre-compute $m_1 := p_1,\ m_2 := p_1 p_2,\ \ldots,\ m_s := m$,
- compute $u_i := b_i m_i$ (stored in i machine-words),
- compute the sum $n := u_1 + u_2 + \ldots + u_s$

MRR or CRT?

- The CRT defines a ring isomorphism:

$$\mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s\mathbb{Z} \cong \mathbb{Z}/(p_1 \times \ldots \times p_s)\mathbb{Z}$$

- The MRR defines a bijection:

$$\mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s\mathbb{Z} \mapsto [0, p_1 p_2 \cdots p_s[$$

  which preserves the order (mapping lex-order to $<$)

- Both take $\Theta(k^2)$ machine-word operations.
- The MRR is interesting for modular methods for real numbers
- We use MRR map instead of the CRT.

Computing equivalent results

- Compute a DFT_N in $\mathbb{Z}/p\mathbb{Z}$ with $p = r^k + 1 \to \mathcal{O}(N \log_K(N) k^2)$
- Compute s DFT_N over small prime fields. $\longrightarrow \mathcal{O}(sN \log_2(N))$
- Compute a MRR on results of small prime FFTs $\longrightarrow \mathcal{O}(sNk^2)$

Example: computing DFT-16 based on DFT-2

- Expanding DFT_{16} based on the six-step FFT algorithm:

$$\begin{aligned}\mathrm{DFT}_{16} &= L_2^{16}(I_8 \otimes \mathrm{DFT}_2)L_8^{16}D_{2,8}^{16}(I_2 \otimes \mathrm{DFT}_8)L_2^{16},\\ \mathrm{DFT}_8 &= L_2^{8}(I_4 \otimes \mathrm{DFT}_2)L_4^{8}D_{2,4}^{8}(I_2 \otimes \mathrm{DFT}_4)L_2^{8},\\ \mathrm{DFT}_4 &= L_2^{4}(I_2 \otimes \mathrm{DFT}_2)L_2^{4}D_{2,2}^{4}(I_2 \otimes \mathrm{DFT}_2)L_2^{4},\end{aligned}$$

- Following multiplications by twiddle factors are required:

  1. DFT_{16} with $\omega_0 = \omega^{N/K} = r$,
  2. DFT_8 needs $\omega_1 = \omega^{(N/K)^2} = r^2$,
  3. DFT_4 needs $\omega_2 = \omega^{(N/K)^4} = r^4$.

- We have:

$$\begin{aligned}D_{2,8}^{16} &= (1,1,1,1,1,1,1,1,r^0,r^1,r^2,r^3,r^4,r^5,r^6,r^7),\\ D_{2,4}^{8} &= (1,1,1,1,r^0,r^2,r^4,r^6),\\ D_{2,2}^{4} &= (1,1,r^0,r^4).\end{aligned}$$

Addition and subtraction in $\mathbb{Z}/p\mathbb{Z}$

- $x, y \in \mathbb{Z}/p\mathbb{Z}$ represented by $\vec{x}, \vec{y}$

Algorithm 6 Computing $x + y \in \mathbb{Z}/p\mathbb{Z}$ for $x, y \in \mathbb{Z}/p\mathbb{Z}$

procedure $\textsc{BigPrimeFieldAddition}(\vec{x}, \vec{y}, r, k)$

1: compute $z_i = x_i + y_i$ in $\mathbb{Z}$, for $i = 0, \ldots, k-1$,

2: let $z_k = 0$,

3: for $i = 0, \ldots, k-1$, compute the quotient q_i and the remainder s_i in the Euclidean division of z_i by r, then replace (z_{i+1}, z_i) by $(z_{i+1} + q_i, s_i)$,

4: if $z_k = 0$ then return $(z_{k-1}, \ldots, z_0)$,

5: if $z_k = 1$ and $z_{k-1} = \cdots = z_0 = 0$, then let $z_{k-1} = r$ and return $(z_{k-1}, \ldots, z_0)$,

6: let i_0 be the smallest index, $0 \le i_0 \le k$, such that $z_{i_0} \ne 0$, then let $z_{i_0} = z_{i_0} - 1$, let $z_0 = \cdots = z_{i_0 - 1} = r - 1$ and return $(z_{k-1}, \ldots, z_0)$.

end procedure

- Associate $x, y \in \mathbb{Z}/p\mathbb{Z} \longrightarrow f_x, f_y \in \mathbb{Z}[T]$
- For large k, can compute $\underline{f_x f_y \mod T^k + 1 \text{ in } \mathbb{Z}[T]}$ by fast algorithms
- For small k, say $k \leq 8$, using plain multiplication is reasonable.

Algorithm 7 Computing $xy \in \mathbb{Z}/p\mathbb{Z}$ for $x, y \in \mathbb{Z}/p\mathbb{Z}$

procedure $\textsc{BigPrimeFieldMultiplication}(f_x, f_y, r, k)$

1: We compute the polynomial product $f_u = f_x f_y$ in $\mathbb{Z}[T]$ modulo $T^k + 1$.

2: Writing $f_u = \sum_{i=0}^{k-1} u_i T^i$, we observe that for all $0 \leq i \leq k-1$ we have $0 \leq u_i \leq kr^2$ and compute a representation $\overrightarrow{u_i}$ of u_i in $\mathbb{Z}/p\mathbb{Z}$ explained in Section **??**.

3: We compute $u_i r^i$ in $\mathbb{Z}/p\mathbb{Z}$ using the method of Section **??**.

4: Finally, we compute the sum $\sum_{i=0}^{k-1} u_i r^i$ in $\mathbb{Z}/p\mathbb{Z}$ using Algorithm 6.

end procedure

Example: RNS and CRT

p_i	$N_i = \frac{p_1 p_2 p_3}{p_i}$	$N_i \pmod{p_i}$	$M_i \equiv \frac{1}{N_i} \pmod{p_i}$	$N_i M_i$
3	77	2	2	154
7	33	5	3	99
11	21	10	10	210

$$
\begin{array}{cccccc}
x & = 14 & \xrightarrow{RNS\ (3,7,11)} & x' & = (2,0,3) \\
y & = 10 & \xrightarrow{RNS\ (3,7,11)} & y' & = (1,3,10) \\
\downarrow & \downarrow & & \downarrow & \downarrow \\
xy & = 140 & \xrightarrow{RNS\ (3,7,11)} & x'y' & = (2,0,8) \\
 & & \xleftarrow{CRT\ (3,7,11)} & &
\end{array}
$$