Differential geometry, words combinatorics and polylogarithms

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Classical polylogarithms are holomorphic functions defined on the unit disk $|z| < 1$

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

for every integer $s > 0$. We retrieve the Riemann function as $\zeta(s) := \mathrm{Li}_s(1)$. Also, we obtain classical polylogarithms as iterated integrals of the two differential forms $\omega_0 = dz/z$ et $\omega_1 = dz/(1-z)$:

$$\mathrm{Li}_s(z) = \int_0^z \omega_0^{s-1}\omega_1 \text{ for } z \in \mathbb{C} - \{0,1\} \text{ and } s \geq 1. \tag{1}$$

For instance,

$$\begin{aligned}
\mathrm{Li}_2(z) &= \int_0^z \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{1-z_2} = \int_0^z \omega_0\omega_1 \\
\mathrm{Li}_3(z) &= \int_0^z \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{z_2} \int_0^{z_2} \frac{dz_3}{1-z_3} = \int_0^z \omega_0\omega_0\omega_1
\end{aligned}$$

Therefore, classical polylogarithms are encoded by binary words of the form $w = x_0^* x_1$.

Generalized polylogarithms $\mathrm{Li}_w(z)$ are encoded by arbitrary words $w \in X^*$ over the alphabet $X = \{x_0, x_1\}$. We will show how word combinatorics enables the computation of the asymptotic development and monodromy of these functions together with the relations between the *multiple zeta values* (MZV) obtained after defining $\zeta(w) := \mathrm{Li}_w(1)$.

The link with differential geometry passes through the differential equation satisfied by the generating power series of the polylogarithms $\mathrm{L}(z) := \sum_{w \in X^*} \mathrm{Li}_w(z)\ w$. This equation

$$d\mathrm{L}(z) = (\omega_0 \otimes x_0 + \omega_1 \otimes x_1)\mathrm{L}(z)$$

shows that $d\mathrm{L}(z) \cdot \mathrm{L}^{-1}(z) = \omega_0 \otimes x_0 + \omega_1 \otimes x_1$ is a Maurer-Cartan form with values in the free Lie algebra generated by x_0 et x_1.