Differential geometry, words combinatorics and polylogarithms

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Classical polylogarithms are holomorphic functions defined on the unit disk $|z| < 1$

$$\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$$

for every integer $s > 0$. We retrieve the Riemann function as $\zeta(s) := \text{Li}_s(1)$. Also, we obtain classical polylogarithms as iterated integrals of the two differential forms $\omega_0 = dz/z$ et $\omega_1 = dz/(1 - z)$:

$$\text{Li}_s(z) = \int_0^z \omega_0^{s-1} \omega_1$$ for $z \in \mathbb{C} - \{0, 1\}$ and $s \geq 1$. \hfill (1)

For instance,

$$\text{Li}_2(z) = \int_0^z \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{1 - z_2} = \int_0^z \omega_0 \omega_1$$

$$\text{Li}_3(z) = \int_0^z \frac{dz_1}{z_1} \int_0^{z_1} \frac{dz_2}{z_2} \int_0^{z_2} \frac{dz_3}{1 - z_3} = \int_0^z \omega_0^2 \omega_1$$

Therefore, classical polylogarithms are encoded by binary words of the form $w = x_0^s x_1$.

Generalized polylogarithms $\text{Li}_w(z)$ are encoded by arbitrary words $w \in X^*$ over the alphabet $X = \{x_0, x_1\}$. We will show how word combinatorics enables the computation of the asymptotic development and monodromy of these functions together with the relations between the multiple zeta values (MZV) obtained after defining $\zeta(w) := \text{Li}_w(1)$.

The link with differential geometry passes through the differential equation satisfied by the generating power series of the polylogarithms $L(z) := \sum_{w \in X^*} \text{Li}_w(z) \ w$. This equation

$$dL(z) = (\omega_0 \otimes x_0 + \omega_1 \otimes x_1)L(z)$$

shows that $dL(z) \cdot L^{-1}(z) = \omega_0 \otimes x_0 + \omega_1 \otimes x_1$ is a Maurer-Cartan form with values in the free Lie algebra generated by $x_0$ et $x_1$. 

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