(Automatic) Parallelization

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Outline

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First examples

- Can the loop on the right be run in parallel?

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for (i = 1; i < N; i++) {
  a[i] = b[i];
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for (i = 1; i < N; i++) {
  a[i] = b[i];
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- What needs to be true for a loop to be parallelizable?
- Iterations cannot interfere with each other

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for (i = 1; i < N; i++) {
  a[i] = b[i];
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}
```

Detailed answer

- A *flow dependence* occurs when one iteration writes a location that a later iteration reads.

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for (i = 1; i < N; i++) {
  a[i] = b[i];
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```

i = 1	i = 2	i = 3	i = 4	i = 5
W(a[1])	W(a[2])	W(a[3])	W(a[4])	W(a[5])
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R(a[0])	R(a[1])	R(a[2])	R(a[3])	R(a[4])

Detailed answer

- A *flow dependence* occurs when one iteration writes a location that a later iteration reads.
- In the first example of the previous slide, there is a flow dependence from the first statement at iteration `i` to the second statement at iteration `i+1`.

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Other kinds of dependence

- Anti dependence: when an iteration reads a location that a later iteration writes

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for (i = 1; i < N; i++) {
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- Same reason as for flow dependence.

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for (i = 1; i < N; i++) {
  a[i - 1] = b[i];
  c[i] = a[i];
}
```

```
for (i = 1; i < N; i++) {
  a[i] = b[i];
  a[i + 1] = c[i];
}
```

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for (i = 1; i < N; i++) {
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}
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- Output dependence: when an iteration writes a location that a later iteration writes

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for (i = 1; i < N; i++) {
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- Same motivation as for flow dependence.

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for (i = 1; i < N; i++) {
  a[i] = b[i];
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Outline

Data dependence analysis

Data dependence

- A data dependency happens when, in a program, (i) a statement S_2 accesses a memory location that is accessed by a preceding statement S_1, (ii) one of these two accesses is a WRITE, and (iii) there is an execution path from S_1 to S_2.

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3. Output dependence: $O(S_1) \cap O(S_2) \neq \emptyset$, that is, S_1 and S_2 write to the same memory location.

Data dependence analysis

- The *dependence source* is the earlier statement (the statement at the tail of the dependence arrow)

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for (i = 1; i < N; i++) {
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Note that dependences can only go forward in time: always from an earlier iteration to a later iteration.

Data dependence analysis

- The *dependence source* is the earlier statement (the statement at the tail of the dependence arrow)
- The *dependence sink* is the later statement (the statement at the head of the dependence arrow)

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Data dependence analysis

Loop dependence theorem

There exists a dependence from statement S_1 to statement S_2 in a common nest of loops if and only if there exist two index vectors $\mathbf{i}$ and $\mathbf{j}$ for the nest, such that

(1) $\mathbf{i} < \mathbf{j}$ or $\mathbf{i} = \mathbf{j}$ and there is a path from S_1 to S_2 in the body of the loop,
(2) statement S_1 accesses memory location M on index $\mathbf{i}$ and statement S_2 accesses location M on index $\mathbf{j}$, and
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When no such dependence exists within that loop nest, then all iterations can be executed in parallel.

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- How do we represent dependences in loops?

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- How do we represent dependences in loops?
- How do we determine if there are dependences?

Outline

Iteration space graphs

- Loop dependence analysis focuses on finding the dependencies within iterations of a loop or a loop-nest.

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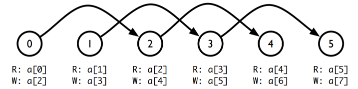
Iteration space graph

The iteration space graphs represent each execution point (a particular statement at a particular iteration) of a loop as a node in a graph. Then, one draws an arrow from one point P to a point Q whenever there is a flow dependence from P to Q.

Iteration space graphs

Iteration space graphs

```
for (int i ...)
    a[i + 2] = a[i];
```

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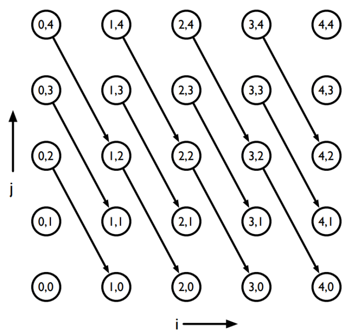

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- The number of arrows in an Iteration space graph can grow exponentially with the number of iterations.

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- To this end, one can use different kinds of arrows for clarity.

Limitations

- The number of arrows in an Iteration space graph can grow exponentially with the number of iterations.
- Can we represent dependences in a more compact way?

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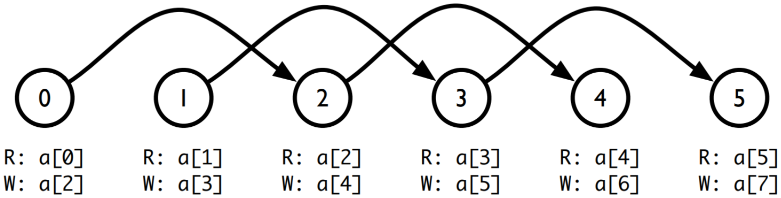
Distance and direction vectors

Distance vector: informal definition

- The distance vector represents each dependence arrow in an iteration space graph as a vector



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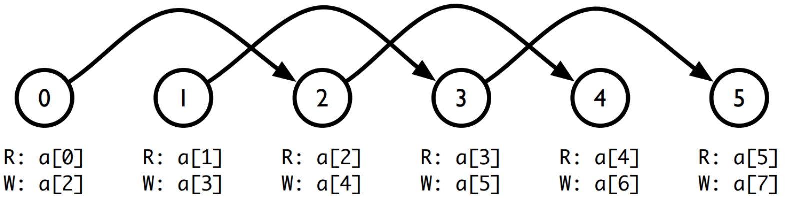
Distance vector: informal definition

- The distance vector represents each dependence arrow in an iteration space graph as a vector
- That is, it captures the "shape" of the dependence, but loses where the dependence originates



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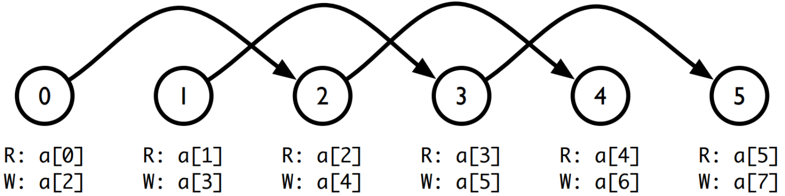

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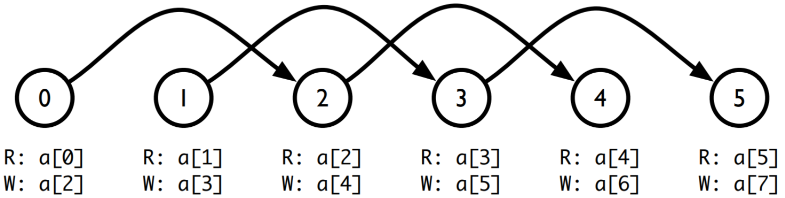

- For the above 1D iteration space graph, the distance vector is (2)
- Indeed, each dependence is 2 iterations forward

Distance and direction vectors

Distance vector

For an n-loop nest, let $\vec{I} = (i_1, \ldots, i_n)$ and $\vec{I'} = (i'_1, \ldots, i'_n)$ be two iterations. The distance vector $d(\vec{I}, \vec{I'})$ from $\vec{I}$ to $\vec{I'}$ is a vector of length n and with $i'_k - i_k$ as k-th coordinate.

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Direction vector

With the same notations as above, the direction vector $D(\vec{I}, \vec{I'})$ is defined as a vector of length n such that

$$D(\vec{I}, \vec{I'})_k = \begin{cases} \text{“} < \text{”} \text{ if } d(\vec{I}, \vec{I'})_k > 0 \\ \text{“} = \text{”} \text{ if } d(\vec{I}, \vec{I'})_k = 0 \\ \text{“} > \text{”} \text{ if } d(\vec{I}, \vec{I'})_k < 0 \end{cases}$$

Distance and direction vectors

Iteration space graphs

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for (int i ...)
   for (int j ...)
      a[i + 1][j - 2] = a[i][j] +
        a[i - 1][j - 2];
```

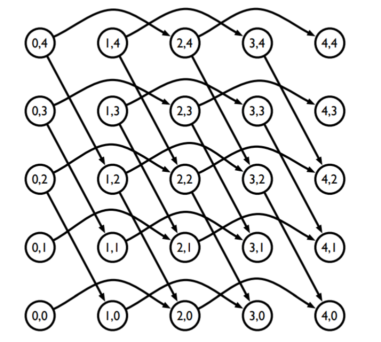

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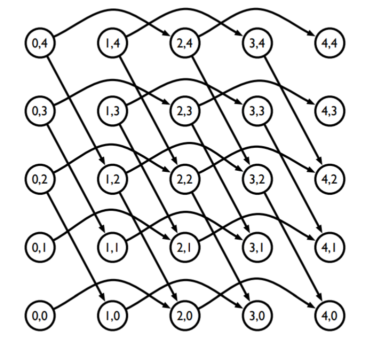

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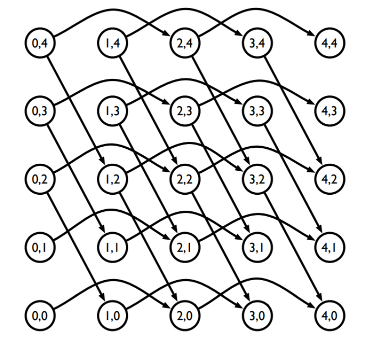

Distance and direction vectors

The distance vectors are: $(1, -2), (2, 0)$

The direction vectors are: $(<, >), (<, =)$

Distance and direction vectors

Problems with distance vectors

Cannot always summarize as easily:

```
for (i = 0; i < N; i++)
a[2*i] = a[i];
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Nevertheless:

- Direction vectors lose a lot of information, but do capture some useful information: Which dimension and direction the dependence is in.

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```

Nevertheless:

- Direction vectors lose a lot of information, but do capture some useful information: Which dimension and direction the dependence is in.
- Often, the only information we need to determine (in order to decide whether an optimization is legal) is captured by direction vectors.

More examples (1/3)

```
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        a[i][j + 1] = a[i][j] + 1;
```

More examples (1/3)

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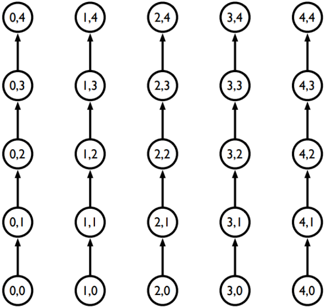

Can parallelize i loop but not j loop

More examples (2/3)

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for (int i = 0; i < n; i++)
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            a[i+1][j] = a[i-1][j] + 1
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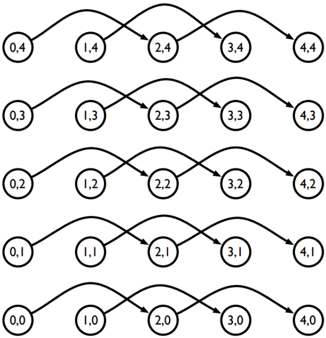

Can parallelize j loop but not i loop

More examples (3/3)

```
for (int i = 0; i < n; i++)
   for (int j = 0; j < n; j++)
       for (int k = 0; j < n; j++)
           a[i+1][j][k] = a[i][j][k+1] + B
```

More examples (3/3)

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for (int i = 0; i < n; i++)
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```

- Two iterations (i_0, j_0, k_0) and $(i_0 + \Delta i, j_0 + \Delta j, k_0 + \Delta k)$ access the same memory location, whenever we have:

$$i_0 + 1 = i_0 + \Delta i;\ j_0 = j_0 + \Delta j;\ k_0 = k_0 + \Delta k + 1.$$

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- That is:

$$\Delta i = 1;\ \Delta j = 0;\ \Delta k = -1.$$

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- That is:

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- The corresponding direction vector is: $(<, =, >)$.
- The j-loop can be vectorized.

Outline

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Delinearization

Linearized multi-dimensional array

```
for (int i = 0; i < n; i ++)
    for (int j = i + 1; j < n; j ++)
        A[i * n + j] =
                A[n * n - n + j - 1];
```

$$\begin{cases} 0 \le i_1 < n \\ i_1 + 1 \le j_1 < n \\ 0 \le i_2 < n \\ i_2 + 1 \le j_2 < n \\ i_1 * n + j_1 = n^2 - n + j_2 - 1 \end{cases} \tag{1}$$

Delinearized multi-dimensional array

```
for (int i = 0; i < n; i ++)
    for (int j = i + 1; j < n; j ++)
        A[i][j] = A[n - 1][j - 1];
```

$$\begin{cases} 0 \le i_1 < n \\ i_1 + 1 \le j_1 < n \\ 0 \le i_2 < n \\ i_2 + 1 \le j_2 < n \\ i_1 = n - 1 \\ j_1 = j_2 - 1 \end{cases} \tag{2}$$

Type of array references

- We assume that arrays are multi-dimensional, thus delinearization has been applied.

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for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
       for (int k = 0; j < n; j++)
            a [5][i+1][j] = a[N][i][k] + B
```

In the above loop body, the pair `[5]`, `[N]` is ZIV, the pair `[i+1]`,`[i]` is SIV and the pair `[j]`,`[k]` is MIV.

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- Consider two statements S_1 and S_2 accessing an array A, one of them for writing.

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Strong SIV

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Strong SIV Test

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- A dependence exists between two references only if $|d| \leq U - L$, where U and L are the loop upper and lower bounds for the index i.
- Otherwise, no dependence exists.

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parallel do $I = 1, N$
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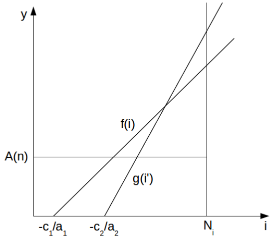

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- With the example on the next slide, we have $i = n - i' + 1$ with $1 \leq i, i' \leq n$.
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$$i + i' = n + 1$$
- To break this dependence (assuming n is even for simplicity) we split the loop into two: $1 \leq i \leq n/2$ and $n/2 + 1 \leq i \leq n$.

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enddo parallel do $I = N/2 + 1, N$
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enddo

Symbolic SIV test

do $I = L_1, U_1$
$S_1 : \quad A(a_1 * I + c_1) = \dots$
enddo
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- A dependence exists if the following dependence equation is satisfied

$$a_1\, i - a_2\, j = c_2 - c_1,$$

for some index value of i, s.t. $L_1 \leq i \leq U_1$, and j, s.t. $L_2 \leq j \leq U_2$.

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- Hence, for $a_1 > 0$, there is a dependence only if

$$\begin{array}{ll} a_1\, L_1 - a_2\, U_2 \leq c_2 - c_1 \leq a_1\, U_1 - a_2\, L_2, & \text{if} \quad a_2 > 0; \\ a_1\, L_1 - a_2\, L_2 \leq c_2 - c_1 \leq a_1\, U_1 - a_2\, U_2, & \text{if} \quad a_2 < 0. \end{array}$$

Symbolic SIV test: idea of the proof

- The solutions of the **linear Diophantine** equation

$$a_1\, x - b_1\, y = b_0 - a_0$$

provide the values i, j of the index variable of the previous symbolic SIV test.

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- The solutions of the Diophantine equation are given by

$$\begin{array}{rcl} x_k & = & e_a \left(\frac{b_0 - a_0}{g}\right) + k\, \frac{b_1}{g} \\ y_k & = & e_b \left(\frac{b_0 - a_0}{g}\right) + k\, \frac{a_1}{g} \end{array}$$

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- For dependence to exist, these solutions must occur in the region defined by the loop bounds.

Outline

Key notions (1/2)

- The *polyhedral model* is mathematical framework for analyzing, scheduling and optimizing for-loop nests.

do $i = 0, N-1$
  do $j = 0, N-1$
S_1 : $A[i,j] += u[i] * v[i]$
  enddo
enddo
do $k = 0, N-1$
  do $\ell = 0, N-1$
S_2 : $x[k] += A[\ell, k] * y[\ell]$
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Iteration domain

$$0 \leq i < N$$
$$0 \leq j < N$$
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$$0 \leq \ell < N$$

Dependence equation

$$i = \ell$$
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- It views the iterations of a for-loop nest as the integer points of a polyhedral set.
- It makes a number of natural assumptions, in particular: every *array reference* is an affine expression in the loop counters, loop bounds, array dimension sizes and possibly other constants.

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Dependence polyhedron for $S_1 \to S_2$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{array}\right) \begin{bmatrix} i \\ j \\ k \\ \ell \\ N \\ 1 \end{bmatrix} \begin{array}{c} \\ \\ \ge 0 \\ \\ \hline = 0 \\ \\ \end{array}$$

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- The *iteration domain* is defined by the value ranges of the loop counters.
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Key notions (2/2)

- The *iteration domain* is defined by the value ranges of the loop counters.
- The *dependence equations* are deduced from the array references of the pair of statements under study.
- The *dependence polyhedron* collects the equality and inequality constraints from the iteration domain and the dependence equations. Obviously, this polyhedron has integer points, this there is an output dependence.

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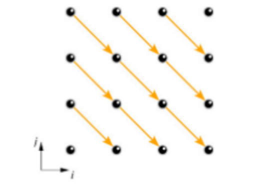
Automatic parallelization: plain multiplication (1/2)

Serial dense univariate polynomial multiplication

```
for(i=0; i<=n; i++){
  c[i] = 0; c[i+n] = 0;
  for(j=0; j<=n; j++)
    c[i+j] += a[i] * b[j];
}
```



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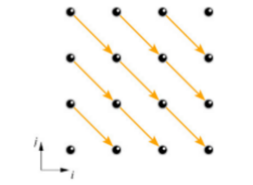

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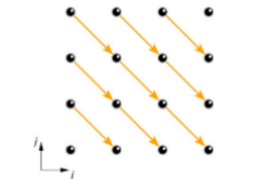

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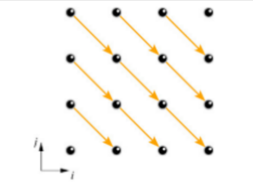

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- Hence, this suggests a change of coordinates with one new coordinate being $p = i + j$, where p stands for *processor*. What can be the second coordinate?

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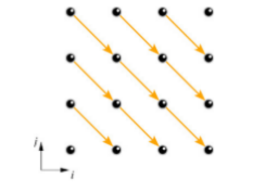

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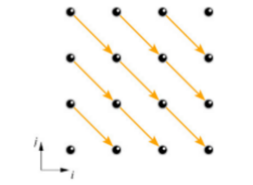
- Call it t for *time*. The map $M : (i, j) \longmapsto (p, t)$ must be linear, given by a unimodular matrix (thus with determinant equal to 1 or -1).

Automatic parallelization: plain multiplication (1/2)

Serial dense univariate polynomial multiplication

```
for(i=0; i<=n; i++){
  c[i] = 0; c[i+n] = 0;
  for(j=0; j<=n; j++)
    c[i+j] += a[i] * b[j];
}
```



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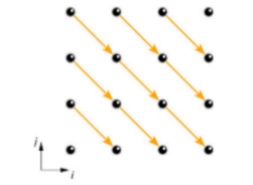

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- One possible choice is $t(i, j) = n - j$.

Automatic parallelization: plain multiplication (2/2)

- What should be the loop bounds in the new system of coordinates?

Asynchronous parallel dense univariate polynomial multiplication

```
parallel_for (p=0; p<=2*n; p++){
  c [ p ] =0;
  for (t=max(0,n-p); t<= min(n,2*n-p)
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```

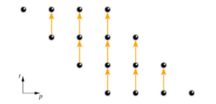

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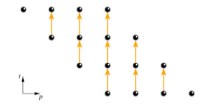

Automatic parallelization: plain multiplication (2/2)

- What should be the loop bounds in the new system of coordinates?
- On the left, see our input iteration domain with the coordinate change
- Some linear transformations produces the new iteration domain with the inverse of our change of coordinates

$$\begin{cases} i & \geq & 0 \\ i & \leq & n \\ j & \geq & 0 \\ j & \leq & n \\ t & = & n-j \\ p & = & i+j, \end{cases}$$

$$\begin{cases} i & = & p+t-n \\ j & = & -t+n \\ t & \geq & -p+n \\ t & \leq & -p+2n \\ n & \geq & t \\ 0 & \leq & t \\ p & \geq & 0 \\ p & \leq & 2n. \end{cases}$$

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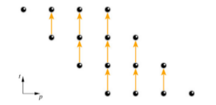

Generating parametric code & use of tiling techniques

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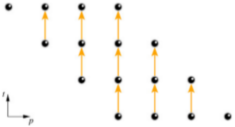

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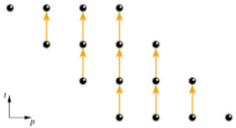

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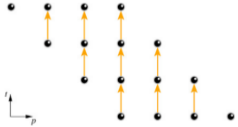

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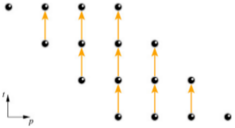

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- We group the virtual processors (or threads) into 1D blocks, each of size B. Each thread is known by its block number b and a local coordinate u in its block.
- Blocks represent good units of work which have good locality property.
- This yields the following constraints: $0 \le u < B$, $p = bB + u$.

Generating parametric code: using tiles

We apply `RegularChains:-QuantifierElimination` on the left system (in order to get rid off i, j) leading to the relations on the right:

$$\left\{\begin{array}{c} o < n \\ 0 \le i \le n \\ 0 \le j \le n \\ t = n - j \\ p = i + j \\ 0 \le b \\ o \le u < B \\ p = bB + u, \end{array}\right. \qquad \left\{\begin{array}{c} B > 0 \\ n > 0 \\ 0 \le b \le 2n/B \\ 0 \le u < B \\ 0 \le u \le 2n - Bb \\ p = bB + u, \end{array}\right. \tag{3}$$

From where we derive the following program:

```
for (p=0; p<=2*n; p++) c [ p ] = 0;
parallel_for (b=0; b<= 2 n / B; b++) {
    parallel_for (u=0; u<=min(B-1, 2*n - B * b); u++) {
       p = b * B + u;
       for (t=max(0,n-p); t<=min(n,2*n-p) ;t++)
           c [ p ] = c [ p ] + a [ t+p-n ] * b [ n-t ];
   }
}
```

References

[1] C. Bastoul. "Code Generation in the Polyhedral Model Is Easier Than You Think". In: *Proceedings of the 13th International Conference on Parallel Architectures and Compilation Techniques*. PACT '04. IEEE Computer Society, 2004, pp. 7–16.

[2] A. Größlinger, M. Griebl, and C. Lengauer. "Quantifier elimination in automatic loop parallelization". In: *J. Symb. Comput.* 41.11 (2006), pp. 1206–1221.