Space vs Time, Cache vs Main Memory

Marc Moreno Maza

University of Western Ontario, London, Ontario (Canada)

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Plan

1. Space vs Time

2. Cache vs Main Memory

A computation with input and output values can be modelled in various ways: directed acyclic graph (DAG), straight-line program (SLP).

By computation, we mean the execution of a program, not a program itself, similarly to the instruction stream DAG of a Cilk++ program.

Thus, we assume that all operations (additions, multiplications) to be performed are precisely known.
Our purpose is then on how computer resources are used to realize this computation. To do so, we make use of pebbling games on DAGs.

From now on we consider a connected directed acyclic graph $G = (V, E)$:
- Each vertex represents an operation and its result.
- An edge from a vertex $v_1$ to a vertex $v_2$ indicates that the result of $v_1$ is needed for performing the operation of $v_2$.
- A vertex $v$ of $G$ is an input (resp. output) if it has no predecessors (resp. no successors).
- The sets of inputs and outputs are respectively denoted by $I(G)$ and $O(G)$. Note that these sets are disjoint.

The red pebble game is played on a directed and connected acyclic graph $G = (V, E)$ using four rules:

(R_1) **Input rule**: A pebble can be placed on an input vertex at any time.

(R_2) **Output rule**: Each output vertex must be pebbled at least once.

(R_3) **Compute rule**: A pebble can be placed on or moved to any non-input vertex if all of its immediate predecessors carry pebbles.

(R_4) **Delete rule**: pebble can be removed at any time.

A pebbling strategy determines sequence of rules invoked on vertices of a graph.

A strategy uses space $S$ if it uses at most $S$ pebbles. It uses time $T$ if the input rule and compute rule are invoked $T$ times in total.

The minimum space $S_{\text{min}}$ to pebble the graph $G$ is the smallest space of any strategy that pebbles $G$.

We shall see that the FFT graph exhibits a tradeoff between space and time: the time required when the minimum space is used is strictly more than that required when more space is available.

What is $S_{\text{min}}$? What is $T$ when $S = S_{\text{min}}$? What is $T$ when $S = S_{\text{min}} + 1$?
We have $S_{\text{min}} = 2$. Moreover $S = 2 \implies T \geq 5$ while $S = 3 \implies T \geq 4$.

What is $S_{\text{min}}$?

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FFT graph for 8 input nodes

We have $S_{\min} = 4$. More generally, for the FFT graph on $n = 2^k$ inputs we have $S_{\min} = k + 1$.

Pebbling a complete binary tree (1/4)

What is $S_{\min}$?

Pebbling a complete binary tree (2/4)

Proof (1/2).

1. Each path has $k + 1$ vertices.
2. Initially each path from an input to the output is free of pebbles.
3. Finally, a pebble is on the output and thus all paths contain a pebble.
4. Therefore, there is a last time at which a path is open.
5. When placing a pebble on last input, all paths from other inputs to vertices on the path carry 1 pebble; moreover there are $k$ such paths.
6. Therefore, we have $S_{\min} \geq k + 1$.

Theorem. The complete balanced binary on $n = 2^k$ inputs has $S_{\min} = k + 1 = \log_2(n) + 1$. It can be pebbled in time $T = 2n - 1$ steps, but no fewer.

Pebbling a complete binary tree (3/4)
Proof (2/2).

1. We prove by induction that \( S_{\text{min}} = k + 1 \) holds and that \( T = 2n - 1 \) holds for \( S = S_{\text{min}} \).
2. The property is true for \( n = 1 \), that is, for \( k = 0 \).
3. Assume \( n \geq 1 \). We do the left subtree in time \( 2(n/2) - 1 \) using \( k \) pebbles (by induction) and leave one pebble at its root.
4. We do the right subtree in time \( 2(n/2) - 1 \) using \( k \) pebbles too.
5. Therefore, we have: \( T = 2((n/2) - 2) + 1 \).

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Proof (1/2).

1. The last open path argument can be used to show that \( S_{\text{min}} \geq m \) holds.
2. To pebble \( P(m) \) with \( m \) pebbles, place pebbles on all inputs.
3. Move the leftmost pebble up one level.
Proof (2/2).
1. Now all vertices one level up can be pebbled using \( m - 1 \) pebbles.
2. Repeat this procedure at all subsequent levels.
3. Each vertex is pebbled once.

Observe that \( S_{\text{min}} \) is about \( \sqrt{n} \), where \( n \) is the number of vertices of \( P(m) \), which is much larger than for binary trees.

For \( k \geq 3 \), the graph \( H_k \) consists of a \( k \)-input tree \( T(k) \), a spine \( S(k) \) of \( k \) vertices connected to the \( k \) outputs of \( H_{k-1} \), an open vertex, and a complete bipartite graph \( BP(k) \), with \( k \) inputs and \( k + 1 \) outputs.

What is \( S_{\text{min}}(H_1) \)? \( S_{\text{min}}(H_2) \)? \( S_{\text{min}}(H_k) \), for \( k \geq 3 \)?
Lemma. $S_{\text{min}}(H_k) = k$ for all $k \geq 1$.

Proof. 
1. Since the tree $T(k)$ needs $k$ pebbles, we have $S_{\text{min}}(H_k) \geq k$.
2. We show $k$ pebbles suffice assuming outputs of $H_{k-1}$ can be pebbled in succession with $k-1$ pebbles.
3. Advance one pebble to tree output, then use $k-1$ pebbles on $H_{k-1}$ to pebble its $k-1$ outputs and advance the $k$ pebbles along the spine.
4. Advance one pebble to open vertex.
5. Put $k$ pebbles on the $BP(k)$ inputs and then pebble one output of $BP(k)$.
6. Repeat the whole process for each additional output of $BP(k)$.

Lemma. $H_k$ requires at least $(k + 1)!$ steps to be pebbled with $S = S_{\text{min}}(H_k)$ pebbles.

Proof. 
1. When $S = S_{\text{min}}(H_k)$, the subgraph $H_{k-1}$ must be repebbled $k + 1$ times.
2. Indeed, pebbling one output of $BP(k)$, removes all pebbles from $H_{k-1}$.

Lemma. $H_k$ can be pebbled in $N(k)$ steps with $k - 1$ pebbles.

Proof. 
1. Base case: $N(2) = 12 = 8 + 10 - 6$.
2. By induction:
   \[
   N(k) = N(k-1) + (k+1) + (k+1) + (k+1) + (k+1) + 4k + 3 \\
   = 2(k-1)^2 + 5(k-1) + 6 + 4k + 3 \\
   = 2k^2 + 5k - 6
   \]
The red-blue pebble game is played on a directed and connected acyclic graph $G = (V, E)$.

- At any point of the game, some vertices have red pebbles, others have blue, others have pebbles of both types, others have no pebbles.
- A configuration is a pair of subsets $(R, B)$ of the vertex set $V$ such that any vertex $v \in R$ (resp. $v \in B$) has a blue pebble (resp. red pebble).
- The initial configuration is the one given by $(\emptyset, I(G))$.
- The final configuration is the one given by $(\emptyset, O(G))$.

The rules of the red-blue pebble game are

- $(R_1)$ Input rule: A red pebble may be placed on any vertex that has a blue pebble.
- $(R_2)$ Output rule: A blue pebble may be placed on any vertex that has a red pebble.
- $(R_3)$ Compute rule: If all immediate predecessors of a vertex $v$ have red pebbles then a red pebble may be placed on $v$.
- $(R_4)$ Delete rule: A pebble red or blue may be removed at any time from any vertex.

Key concepts:

- A transition is an ordered pair of configurations, the second of which follows from the first according to one of the rules $(R_1)$ to $(R_4)$.
- A calculation is a sequence of configurations, each successive pair of which form a transition.
- A complete calculation is one that begins with the initial configuration and ends with the final configuration.
A graph on which the red-blue pebble game is played can model a computation performed on a two-level memory structure, say, a fast memory (or cache) and a slow memory.

- Each vertex represents an operation and its result.
- An edge from a vertex $v_1$ to a vertex $v_2$ indicates that the result of $v_1$ is needed for performing the operation of $v_2$.
- An operation can be performed only if all operands reside in cache (or fast memory).
- The maximum allowable number of red (or blue) pebbles on the graph at any point in the game corresponds to the number of words available for use in the fast (or slow) memory, respectively.

In what follows, the fast memory can only hold $S$ words, where $S$ is a constant, while the slow memory is arbitrarily large.

- For any given connected DAG, we are interested in the I/O time, denoted by $Q$, which is the minimum number of transitions according to Rules $(R_1)$ or $(R_2)$ required by any complete calculation.
- In the original work of (J.W. Hong, H.T. Kung, 1981) a "static problem" is associated with the red-blue pebble game, the S-Partitioning Problem. Then lower bounds for the S-Partitioning Problem lead to lower bounds for the red-blue pebble game.

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To establish bounds like those (but weaker) of (J.W. Hong, H.T. Kung, 1981) we will follow a simpler approach due to J.E. Savage (see his book Models of Computations) reducing to the red pebble game.
Theorem. Assume $S \geq 3$. For the $n$-point FFT graph we have $Q \log(S) \in \Omega(n \log(n))$. Moreover, there is a pebbling strategy for which $Q \log(S) \in \Theta(n \log(n))$ holds.

Decomposing the 16-point FFT graph with $n = 16$ and $S = 4$.

Theorem. For any DAG $G$ encoding an algorithm multiplying two square matrices of order $n$ (based on an $\Theta(n^3)$-work algorithm) and for every pebbling strategy $P$ for $G$ in the red-blue pebble game that uses $S \geq 3$ red pebbles, the I/O-time satisfies the following lower bound:

$$Q_P \in \Omega(n^3/\sqrt{S}).$$

Furthermore, for $S \geq 3$, there exists a pebbling strategy for which we have:

$$Q_P \in \theta(n^3/\sqrt{S}).$$

Let $G = (V, E)$ be a directed and connected acyclic graph. Let $X, Y \subseteq V$ be two proper subsets of $V$, hence $X \neq \emptyset$ and $Y \neq \emptyset$ hold.

- A subset $D \subseteq V$ is a dominator set for $X$ if for every path from a vertex of $I(G)$ to a vertex of $X$ has at least one vertex in $D$.

- The minimum set of $X$ is the set of vertices $v \in X$ such that none of the successors of $v$ belongs to $X$.

- We say that $Y$ depends on $X$ whenever there exists $(v, w) \in X \times Y$ such that $(v, w) \in E$ holds.
Let $G = (V, E)$ be a directed and connected acyclic graph and $S$ be a positive integer. A partition $\{V_1, \ldots, V_h\}$ of $V$ is called an $S$-partition of $G$ if the following properties hold for each $i = 1 \cdots h$:

- $V_i$ admits a dominator set $D_i$ with $|D_i| \leq S$.
- The minimum set $M_i$ of $V_i$ satisfies $|M_i| \leq S$.
- There is no cyclic dependence among $V_1, \ldots, V_h$.

Consider a red-blue pebble game on $G$ using at most $S$ red pebbles. Denote by $C$ a complete calculation. There exists an integer $h \geq 2$ and a sequence of $h$ consecutive subcalculations $C_1, C_2, \ldots, C_h$ such that the following holds:

- for each $i = 1 \cdots (h - 1)$, the subcalculation $C_i$ has exactly $S$ transitions using Rules ($R_1$) or ($R_2$),
- $C_h$ has at most $S$ transitions using Rules ($R_1$) or ($R_2$).

Theorem. Let $G = (V, E)$ be a directed and connected acyclic graph. Any complete calculation red-blue pebble game on $G$ using at most $S$ red pebbles is associated with a $2S$-partition of $G$ such that

$$S \cdot h \geq q \geq S \cdot (h - 1).$$

where $q$ is the I/O time required by the calculation and $h$ is the number of parts in the $2S$-partition.

Sketch of proof.

- $\{V_1, V_2, \ldots, V_h\}$ (as defined before) is a $2S$-partition of $V$,
- For each $i = 1 \cdots (h - 1)$, exactly $S$ transitions with Rules ($R_1$) or ($R_2$) correspond to $V_i$,
- No more than $S$ transitions with Rules ($R_1$) or ($R_2$) correspond to $V_h$,
- The inequalities follow.
Corollary. Let $G = (V, E)$ be a directed and connected acyclic graph. Let $P(2S)$ be the minimum number of parts in a $2S$-partition of $V$. Then we have:

$$Q \geq S (P(2S) - 1).$$

Using this Corollary, lower bounds for $P(2S)$ translate into lower bounds for $Q$.

The $S$-span of the DAG $G = (V, E)$, denoted by $\rho(S, G)$, is

- the maximum number of vertices of $G$ that can be pebbled with $S$ red pebbles in the red pebble game,
- maximized over all initial placements of $S$ red pebbles,
- which means that the initialization rule is disallowed.

**Theorem.** We have: $\left\lceil \frac{Q}{S} \right\rceil \rho(2S, G) \geq |V| - |I(G)|$.

**Sketch of proof (1/3).**

1. Let $\mathcal{P}$ be a pebbling strategy with $S$ pebbles.
2. Divide $\mathcal{P}$ into consecutive sequential sub-pebblings (or calculations) $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_h$, where each sub-pebbling has $S$ I/O operations (rules $(R_1)$ and $(R_2)$) except possibly the last one.
3. Thus we have $h = \left\lceil \frac{Q}{S} \right\rceil$.

**Sketch of proof (2/3).**

1. The upper bound on $R$ is developed by adding $S$ new red pebbles and showing that we may use these new pebbles to move all I/O operations in each sub-pebbling $\mathcal{P}_j$ to either the beginning or the end of the sub-pebbling without changing the number of computation steps or I/O operations.
2. Consider a vertex $v$ carrying a red pebble at some time during $\mathcal{P}_j$ and which is pebbled for the first time with a blue pebble during $\mathcal{P}_j$.
3. Instead of pebbling $v$ with a blue pebble, we use a new red pebble to keep red until its last output operation which is preserved and moved to the end of $\mathcal{P}_j$. 

**Theorem.** We have: $\left\lceil \frac{Q}{S} \right\rceil \rho(2S, G) \geq |V| - |I(G)|$. 

**Sketch of proof (2/3).**

1. Let $\mathcal{P}$ be a pebbling strategy with $S$ pebbles.
2. Divide $\mathcal{P}$ into consecutive sequential sub-pebblings (or calculations) $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_h$, where each sub-pebbling has $S$ I/O operations (rules $(R_1)$ and $(R_2)$) except possibly the last one.
3. Thus we have $h = \left\lceil \frac{Q}{S} \right\rceil$.
4. We shall exhibit an upper bound $R$ to the number of vertices of $G$ pebbled with red pebbles in any sub-pebbling $\mathcal{P}_j$.
5. This will satisfy $h R \geq |V|$. 

(Moreno Maza)
Reduction to the **red** pebble game (4/4)

**Theorem.** We have: \[ \lceil Q/S \rceil \rho(2S, G) \geq |V| - |I(G)|. \]

**Sketch of proof (3/3).**

1. Consider a vertex \( v \) carrying a **blue** pebble at the start of \( P_j \) and that is given a **red** during \( P_j \); consider the first pebbling of this kind for \( v \).
2. Then, we use a **new red** pebble instead.
3. This allows us to move this input operation at the beginning of \( P_j \), without violating the precedence conditions of \( G \).
4. It follows that that the number of vertices that are pebbled with **red** pebbles during any **computations** of \( P_j \) is at most \( \rho(2S, G) \).
5. Therefore, by definition of a pebbling strategy, we have \[ h\rho(2S, G) \geq |V| - |I(G)|. \]