Space vs Time, Cache vs Main Memory

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CS 4435 - CS 9624

Plan

1. Space vs Time
2. Cache vs Main Memory

Plan

Pebbling games and computing (1/2)

- A computation with **input** and **output** values can be modelled in various ways: **directed acyclic graph (DAG)**, **straight-line program (SLP)**.
- By computation, we mean the execution of a program, not a program itself, similarly to the **instruction stream DAG** of a `Cilk++` program.
- Thus, we assume that all operations (additions, multiplications) to be performed are precisely known.

Pebbling games and computing (2/2)

- Our purpose is then on how computer resources are used to realize this computatiuon. To do so, we make use of **pebbling games** on DAGs.
- From now on we consider a connected directed acyclic graph $G = (V, E)$:
  - Each vertex represents an **operation** and its **result**.
  - An edge from a vertex v_1 to a vertex v_2 indicates that the result of v_1 is needed for performing the operation of v_2.
  - A vertex v of G is an **input** (resp. **output**) if it has no predecessors (resp, no successors).
  - The sets of inputs and outputs are respectively denoted by $I(G)$ and $O(G)$. Note that these sets are disjoint.

The red pebble game (1/2)

The **red** pebble game is played on a directed and connected acyclic graph $G = (V, E)$ using four rules:

(R_1) **Input rule:** A pebble can be placed on an input vertex at any time.

(R_2) **Output rule:** Each output vertex must be pebbled at least once.

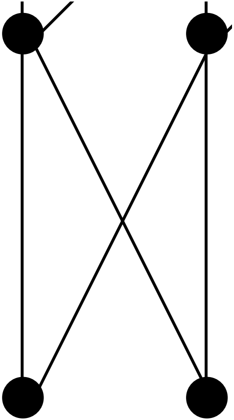
(R_3) **Compute rule:** A pebble can be placed on or moved to any non-input vertex if all of its immediate predecessors carry pebbles.

(R_4) **Delete rule:** pebble can be removed at any time.

The red pebble game (2/2)

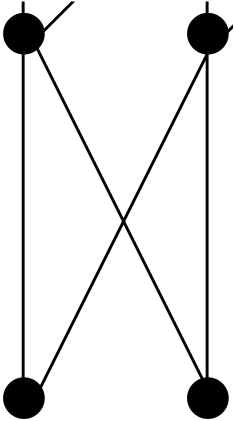
- A **pebbling strategy** determines sequence of rules invoked on vertices of a graph.
- A strategy uses **space** S if it uses at most S pebbles. It uses **time** T if the **input rule** and **compute rule** are invoked T times in total.
- The minimum space $S_{\min}$ to pebble the graph G is the smallest space of any strategy that pebbles G.
- We shall see that the FFT graph exhibits a tradeoff between space and time: the time required when the minimum space is used is strictly more than that required when more space is available.

FFT graph for 2 input nodes



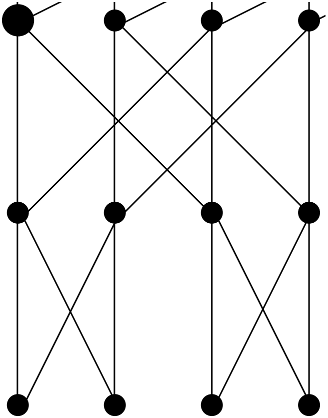
What is $S_{\min}$? What is T when $S = S_{\min}$? What is T when $S = S_{\min} + 1$?

FFT graph for 2 input nodes



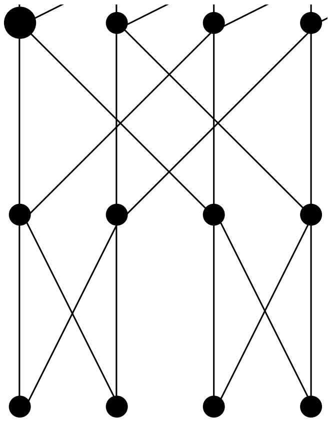
We have $S_{\min} = 2$. Moreover $S = 2 \implies T \geq 5$ while $S = 3 \implies T \geq 4$.

FFT graph for 4 input nodes



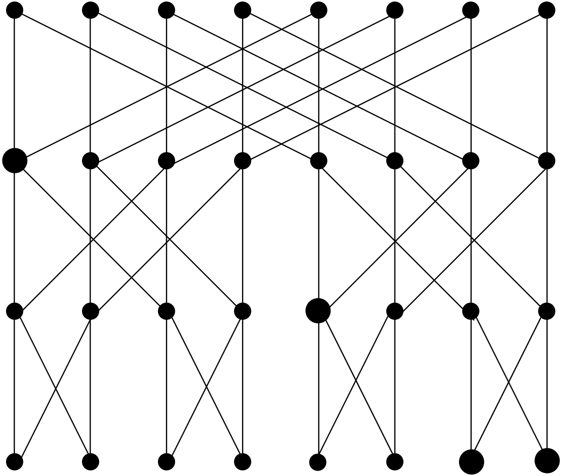
What is $S_{\min}$?

FFT graph for 4 input nodes



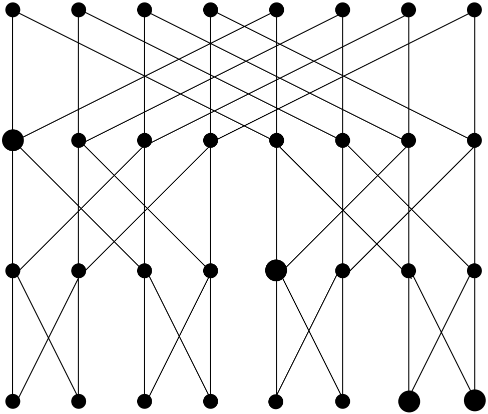
We have $S_{\min} = 3$.

FFT graph for 8 input nodes



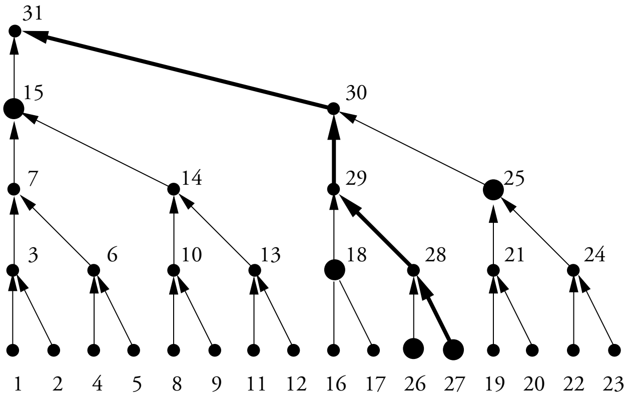
What is $S_{\min}$?

FFT graph for 8 input nodes



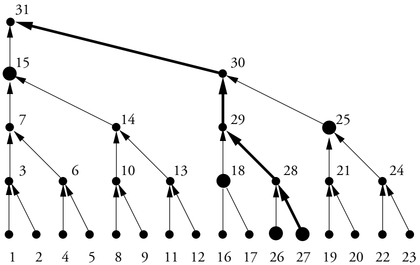
We have $S_{\min} = 4$. More generally, for the FFT graph on $n = 2^k$ inputs we have $S_{\min} = k + 1$.

Pebbling a complete binary tree (1/4)



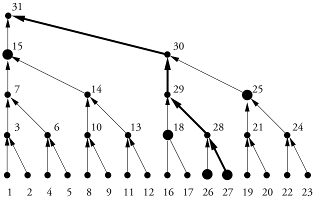
What is $S_{\min}$?

Pebbling a complete binary tree (2/4)



Theorem. The complete balanced binary on $n = 2^k$ inputs has $S_{\min} = k + 1 = \log_2(n) + 1$. It can be pebbled in time $T = 2n - 1$ steps, but no fewer.

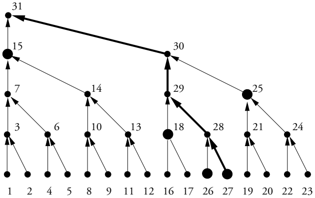
Pebbling a complete binary tree (3/4)



Proof (1/2).

1. Each path has $k + 1$ vertices.
2. Initially each path from an input to the output is free of pebbles.
3. Finally, a pebble is on the output and thus all paths contain a pebble.
4. **Therefore**, there is a last time at which a path is open.
5. When placing a pebble on last input, all paths from other inputs to vertices on the path carry 1 pebble; moreover there are k such paths.
6. **Therefore**, we have $S_{\min} \geq k + 1$.

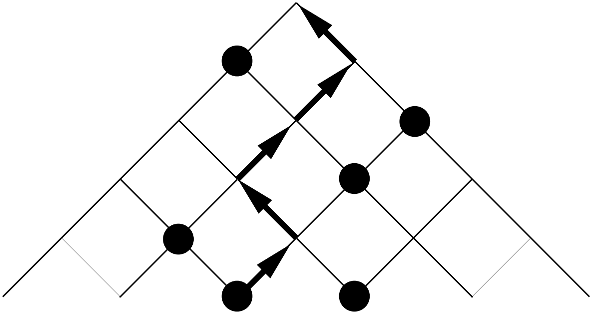
Pebbling a complete binary tree (4/4)



Proof (2/2).

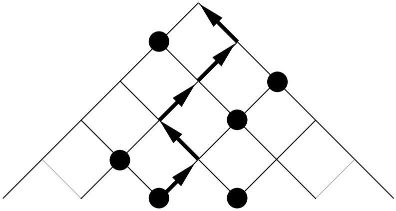
1. We prove by induction that $S_{\min} = k + 1$ holds and that $T = 2n - 1$ holds for $S = S_{\min}$.
2. The property is true for $n = 1$, that is, for $k = 0$.
3. Assume $n \geq 1$. We do the left subtree in time $2(n/2) - 1$ using k pebbles (by induction) and leave one pebble at its root.
4. We do the rigth subtree in time $2(n/2) - 1$ using k pebbles too.
5. **Therefore**, we have: $T = 2((n/2) - 2 + 1$.

Pebbling the pyramid graph (1/4)



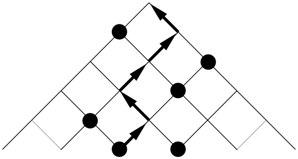
What is $S_{\min}$?

Pebbling the pyramid graph (2/4)



Theorem. For the pyramid graph $P(m)$ on m inputs, we have $S_{\min} = m$. With $S = S_{\min}$ pebbles, the graph $P(m)$ can be pebbled in time $T = n$, where $n = m(m+1)/2$ is the number of vertices of $P(m)$.

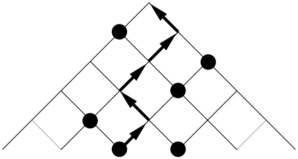
Pebbling the pyramid graph (3/4)



Proof (1/2).

1. The last open path argument can be used to show that $S_{\min} \geq m$ holds.
2. To pebble $P(m)$ with m pebbles, place pebbles on all inputs.
3. Move the leftmost pebble up one level.

Pebbling the pyramid graph (4/4)

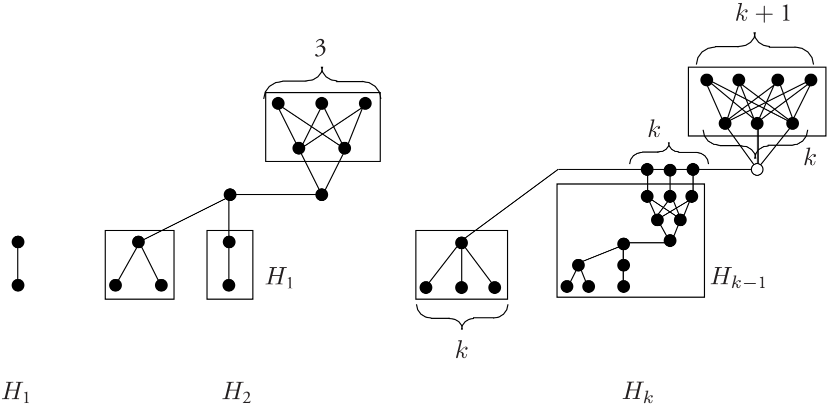


Proof (2/2).

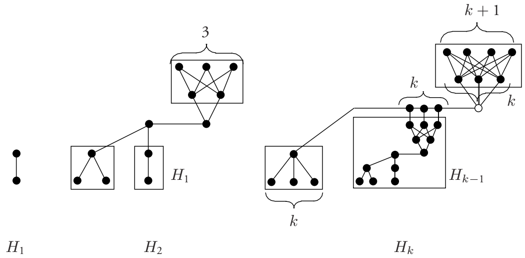
1. Now all vertices one level up can be pebbled using $m - 1$ pebbles.
2. Repeat this procedure at all subsequent levels.
3. Each vertex is pebbled once.

Observe that $S_{\min}$ is about $\sqrt{n}$, where n is the number of vertices of $P(m)$, which is much larger than for binary trees.

Extreme Tradeoffs (1/7)

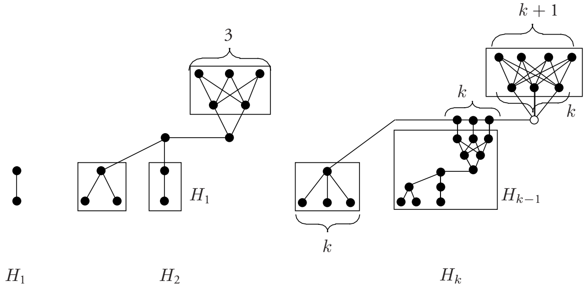


Extreme Tradeoffs (2/7)



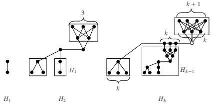
For $k \geq 3$, the grap H_k consists of a k- input tree $T(k)$, a **spine** $S(k)$ of k vertices connected to the k outputs of H_{k-1}, an **open** vertex, and a complete bipartite graph $BP(k)$, with k inputs and $k+1$ outputs.

Extreme Tradeoffs (3/7)



What is $S_{\min}(H_1)$? $S_{\min}(H_2)$? $S_{\min}(H_k)$, for $k \geq 3$?

Extreme Tradeoffs (4/7)

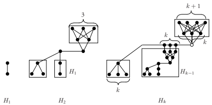


Lemma. $S_{\min}(H_k) = k$ for all $k \geq 1$.

Proof .

1. Since the tree $T(k)$ needs k pebbles, we have $S_{\min}(H_k) \geq k$.
2. We show k pebbles suffice assuming outputs of H_{k-1} can be pebbled in succession with $k-1$ pebbles.
3. Advance one pebble to tree output, then use $k-1$ pebbles on H_{k-1} to pebble its $k-1$ outputs and advance the k pebbles along the spine.
4. Advance one pebble to open vertex.
5. Put k pebbles on the $BP(k)$ inputs and then pebble one output of $BP(k)$.
6. Repeat the whole process for each additional output of $BP(k)$.

Extreme Tradeoffs (5/7)



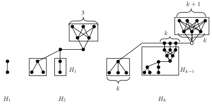
Lemma. H_k has $N(k) = 2k^2 + 5k - 6$ vertices, for all $k \geq 2$.

Proof .

1. Base case: $N(2) = 12 = 8 + 10 - 6$.
2. By induction:

$$
\begin{aligned}
N(k) &= N(k-1) + (k+1) + k + 1 + (k + k + 1) \\
&= N(k-1) + 4k + 3 \\
&= 2(k-1)^2 + 5(k-1) - 6 + 4k + 3 \\
&= 2k^2 + 5k - 6
\end{aligned}
$$

Extreme Tradeoffs (6/7)

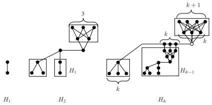


Lemma. H_k requires at least $(k+1)!$ steps to be pebbled with $S = S_{\min}(H_k)$ pebbles.

Proof .

1. When $S = S_{\min}(H_k)$, the subgraph H_{k-1} must be repebbled $k+1$ times.
2. Indeed, pebbling one output of $BP(k)$, removes all pebbles from H_{k-1}.

Extreme Tradeoffs (7/7)



Lemma. H_k can be pebbled in $N(k)$ steps with $k+1$ pebbles.

Proof .

1. Inductive Hypothesis: When $k+1$ pebbles are used, assume all outputs of H_{k-1} can be pebbled in succession using $k+1$ pebbles without repebbling any vertices.
2. We advance k pebbles to the inputs of the $BP(k)$ without repebbling any vertices.
3. The remaining pebble is used to pebble outputs of the $BP(k)$ in succession.

Plan

1. Space vs Time
2. Cache vs Main Memory

The Red-Blue Pebble Game (1/3)

The **red**-**blue** pebble game is played on a directed and connected acyclic graph $G = (V, E)$.

- At any point of the game, some vertices have **red** pebbles, others have **blue**, others have pebbles of both types, others have no pebbles.
- A **configuration** is a pair of subsets (R, B) of the vertex set V such that any vertex $v \in R$ (resp. $v \in B$) has a **blue** pebble (resp. **red** pebble).
- The **initial configuration** is the one given by $(\emptyset, I(G))$.
- The **final configuration** is the one given by $(\emptyset, O(G))$.

The Red-Blue Pebble Game (2/3)

The rules of the **red**-**blue** pebble game are

(R_1) **Input rule:** A **red** pebble may be placed on any vertex that has a **blue** pebble.

(R_2) **Output rule:** A **blue** pebble may be placed on any vertex that has a **red** pebble.

(R_3) **Compute rule:** If all immediate predecessors of a vertex v have **red** pebbles then a **red** pebble may be placed on v.

(R_4) **Delete rule:** A pebble **red** or **blue** may be removed at any time from any vertex.

The Red-Blue Pebble Game (3/3)

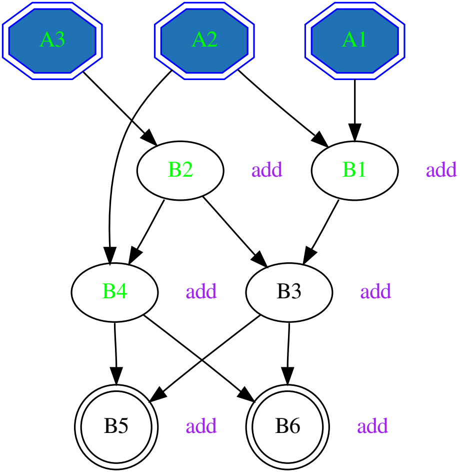
Key concepts:

- A **transition** is an ordered pair of configurations, the second of which follows from the first according to one of the rules (R_1) to (R_4).
- A **caculation** is a sequence of configurations, each successive pair of which form a transition.
- A **complete caculation** is one that begins with the initial configuration and ends with the final configuration.

Application to cache complexity (1/7)

- A graph on which the **red**-**blue** pebble game is played can model a computation performed on a two-level memory structure, say, a fast memory (or cache) and a slow memory.
- Each vertex represents an **operation** and its **result**.
- An edge from a vertex v_1 to a vertex v_2 indicates that the result of v_1 is needed for performing the operation of v_2.
- An operation can be performed only if **all operands reside in cache** (or fast memory).
- The maximum allowable number of **red** (or **blue**) pebbles on the graph at any point in the game corresponds to the number of words available for use in the fast (or slow) memory, respectively.

Application to cache complexity (2/7)



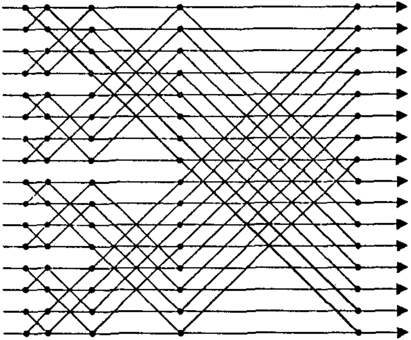
Application to cache complexity (3/7)

- Placing a **red** pebble using Rule (R_3) corresponds to performing an operation and storing the result in cache
- Placing a **blue** pebble using Rule (R_2) corresponds to storing a copy of a result (currently in the fast memory) into the slow memory.
- Placing a **red** pebble using Rule (R_1) corresponds to retrieving a copy of a result (currently in the slow memory) into the fast memory.
- Removing a red or **red** or **blue** pebble using Rule (R_4) means freeing a memory location in the fast or slow memory respectively.

Application to cache complexity (4/7)

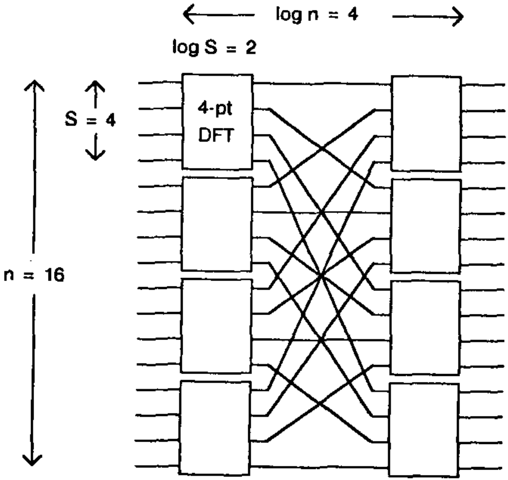
- In what follows, the fast memory can only hold S words, where S is a constant, while the slow memory is arbitrarily large.
- For any given connected DAG, we are interested in the **I/O time**, denoted by Q, which is the minimum number of transitions according to Rules (R_1) or (R_2) required by any complete calculation.
- In the original work of (J.W. Hong, H.T. Kung, 1981) a "static problem" is associated with the **red-blue** pebble game, the *S-Partitioning Problem*. Then lower bounds for the S-Partitioning Problem lead to lower bounds for the **red-blue** pebble game.
- To establish bounds like those (but weaker) of (J.W. Hong, H.T. Kung, 1981) we will follow a simpler approach due to J.E. Savage (see his book *Models of Computations*) reducing to the **red** pebble game.

Application to cache complexity (5/7)



Theorem. Assume $S \geq 3$. For the n-point FFT graph we have $Q \log(S) \in \Omega(n \log(n))$. Moreover, there is a pebbling strategy for which $Q \log(S) \in \Theta(n \log(n))$ holds.

Application to cache complexity (6/7)



Decomposing the 16-point FFT graph with $n = 16$ and $S = 4$.

Application to cache complexity (7/7)

Theorem. For any DAG G encoding an algorithm multiplying two square matrices of order n (based on an $\Theta(n^3)$-work algorithm) and for every pebbling strategy $\mathcal{P}$ for G in the **red**-**blue** pebble game that uses $S \geq 3$ **red** pebbles, the I/O-time satisfies the following lower bound:

$$Q_{\mathcal{P}} \in \Omega(n^3/\sqrt{S}).$$

Furthermore, for $S \geq 3$, there exists a pebbling strategy for which we have:

$$Q_{\mathcal{P}} \in \theta(n^3/\sqrt{S}).$$

The S-Partitioning Problem (1/6)

Let $G = (V, E)$ be a directed and connected acyclic graph. Let $X, Y \subset V$ be two proper subsets of V, hence $X \neq \emptyset$ and $Y \neq \emptyset$ hold.

- A subset $D \subset V$ is a **dominator set** for X if for every path from a vectex of $I(G)$ to a vertex of X has at least one vertex in D.
- The **minimum set** of X is the set of vertices $v \in X$ such that none of the successors of v belongs to X.
- We say that Y **depends on** X whenever there exists $(v, w) \in X \times Y$ such that $(v, w) \in E$ holds.

The S-Partitioning Problem (2/6)

Let $G = (V, E)$ be a directed and connected acyclic graph and S be a positive integer. A partition $\{V_1, \ldots, V_h\}$ of V is called an S-partition of G if the following properties hold for each $i = 1 \cdots h$:

1. V_i admits a **dominator set** D_i with $\mid D_i \mid \leq S$,
2. the **minimum set** M_i of V_i satisfies $\mid M_i \mid \leq S$,
3. There is **no cyclic dependence** among $V_1, \ldots, V_h$.

The S-Partitioning Problem (3/6)

Consider a **red**-**blue** pebble game on G using at most S **red** pebbles.

Denote by C a complete calculation. There exists an integer $h \geq 2$ and a sequence of h consecutive subcalculations $C_1, C_2, \ldots, C_h$ such that the following holds:

- for each $i = 1 \cdots (h - 1)$, the subcalculation C_i has exactly S transitions using Rules (R_1) or (R_2),
- C_h has at most S transitions using Rules (R_1) or (R_2),

The S-Partitioning Problem (4/6)

For each $i = 1 \cdots (h-1)$, define V_i to be the largest subset of V with the following properties:

- During the subcalculation C_i each vertex of V_i receives a **red** pebble thanks to Rules (R_1) or (R_3).
- At the end of the subcalculation C_i each vertex of $v \in V_i$
  - either has **red** pebbles or **blue** pebbles that are placed on v during C_i,
  - or v has a successor in V_i
- v does not belong to any V_j for $j = (i+1) \cdots h$.

Lemma. The set $\{V_1, V_2, \ldots, V_h\}$ is a $2S$-partition of V.

The S-Partitioning Problem (5/6)

Theorem. Let $G = (V, E)$ be a directed and connected acyclic graph. Any complete calculation **red**-**blue** pebble game on G using at most S **red** pebbles is associated with a $2S$- partiton of G such that

$$S\,h \geq q \geq S\,(h-1).$$

where q is the I/O time required by the calculation and h is the number of parts in the $2S$- partiton.

Sketch of proof.

1. $\{V_1, V_2, \ldots, V_h\}$ (as defined before) is a $2S$-partition of V,
2. For each $i = 1 \cdots (h-1)$, exactly S transitions with Rules (R_1) or (R_2) correspond to V_i,
3. No more than S transitions with Rules (R_1) or (R_2) correspond to V_h.
4. The inequalities follow.

The S-Partitioning Problem (6/6)

Corollary. Let $G = (V, E)$ be a directed and connected acyclic graph. Let $P(2S)$ be the minimum number of parts in a $2S$-partition of V. Then we have:

$$Q \geq S\,(P(2S) - 1).$$

Using this Corollary, lower bounds for $P(2S)$ translate into lower bounds for Q.

Reduction to the **red** pebble game (1/4)

The *S*-**span** of the DAG $G = (V, E)$, denoted by $\rho(S, G)$, is

- the maximum number of vertices of G that can be pebbled with S red pebbles in the **red** pebble game,
- maximized over all initial placements of S red pebbles,
- which means that the initialization rule is disallowed.

Theorem. We have: $\lceil Q/S \rceil \; \rho(2S, G) \;\; \geq \;\; | \, V \, | - | \, I(G) \, |$.

Reduction to the red pebble game (2/4)

Theorem. We have: $\lceil Q/S \rceil \, \rho(2S, G) \;\; \geq \;\; \mid V \mid - \mid I(G) \mid$.

Sketch of proof (1/3).

1. Let $\mathcal{P}$ be a pebbling strategy with S pebbles.
2. Divide $\mathcal{P}$ into consecutive sequential sub-pebblings (or calculations) $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_h$, where each sub-pebbling has S I/O operations (rules (R_1) and (R_2)) except possibly the last one.
3. Thus we have $h = \lceil Q/S \rceil$.
4. We shall exhibit an upper bound R to the number of vertices of G pebbled with **red** pebbles in any sub-pebbling $\mathcal{P}_j$
5. This will satisfy $h\,R \geq \mid V \mid$.

Reduction to the **red** pebble game (3/4)

Theorem. We have: $\lceil Q/S \rceil \, \rho(2S, G) \geq |V| - |I(G)|$.

Sketch of proof (2/3).

1. The upper bound on R is developed by adding S new red pebbles and showing that we may use these new pebbles to move all I/O operations in each sub-pebbling $\mathcal{P}_j$ to either the beginning or the end of the sub-pebbling without changing the number of computation steps or I/O operations.
2. Consider a vertex v carrying a **red** pebble at some time during $\mathcal{P}_j$ and which is pebbled for the first time with a **blue** pebble during $\mathcal{P}_j$.
3. Instead of pebbling v with a **blue** pebble, we use a **new red** pebble to keep **red** until its last output operation which is preserved and moved to the end of $\mathcal{P}_j$,.

Reduction to the **red** pebble game (4/4)

Theorem. We have: $\lceil Q/S \rceil \; \rho(2S, G) \;\; \geq \;\; |\, V \,| - |\, I(G) \,|$.

Sketch of proof (3/3).

1. Consider a vertex v carrying a **blue** pebble at the start of $\mathcal{P}_j$ and that is given a **red** during $\mathcal{P}_j$; consider the first pebbling of this kind for v.
2. Then, we use a **new red** pebble instead.
3. This allows us to move this input operation at the beginning of $\mathcal{P}_j$, without violating the precedence conditions of G.
4. It follows that that the number of vertices that are pebbled with **red** pebbles during any **computations** of $\mathcal{P}_j$ is at most $\rho(2S, G)$.
5. Therefore, by definition of a pebbling strategy, we have $h\rho(2S, G) \;\geq |\, V \,| - |\, I(G) \,|$.