The Fork-Join Model

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CS4402 - CS9635, February 9, 2024

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Plan

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- In this course, we will be using OpenCilk which is freely available in source from the above URL.

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- Cilk's runtime features a provably efficient work-stealing scheduler.
- A number third-party libraries are known to work with OpenCilk out of the box for parallel execution, see OpenCilk-powered libraries.
- OpenCilk includes the Cilkscale performance analyzer.

Nested Parallelism in Cilk

```
int fib(int n)
{
   if (n < 2) return n;
   int x, y;
   x = cilk_spawn fib(n-1);
   y = fib(n-2);
   cilk_sync;
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- The named child function `cilk_spawn fib(n-1)` may execute in parallel with its parent executes `fib(n-2)`.
- `Cilk++` keywords `cilk_spawn` and `cilk_sync` grant permissions for parallel execution. They do not command parallel execution.

Loop Parallelism in Cilk

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

```
// indices run from 0, not 1
cilk_for (int i=1; i<n; ++i) {
    for (int j=0; j<i; ++j) {
        double temp = A[i][j];
        A[i][j] = A[j][i];
        A[j][i] = temp;
    }
}
```

The iterations of a `cilk_for` loop may execute in parallel.

Serial Semantics (1/2)

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  - ↳ The `C` elision of a `Cilk` (resp. `Cilk++`) is a correct implementation of the semantics of the program.

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  - ↳ Moreover, on one processor, a parallel Cilk program scales down to run nearly as fast as its C elision.
- To obtain the serialization of a Cilk program

```
#define cilk_for for
#define cilk_spawn
#define cilk_sync
```

Serial Semantics (2/2)

```
int fib (int n) {
  if (n<2) return (n);
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  }
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```

Cilk++ source

```
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Serialization

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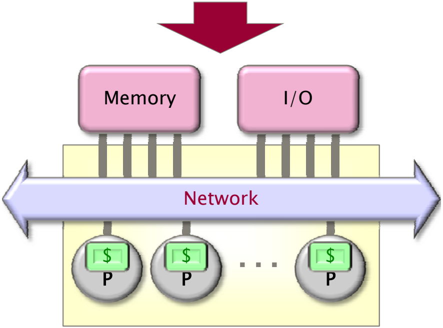

Scheduling (2/2)

- `Cilk` randomized work-stealing scheduler load-balances the computation at run-time. Each processor maintains a ready deque:
  - ↳ A ready deque is a double ended queue, where each entry is a procedure instance that is ready to execute.

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  - ↳ Deletion from the top of the deque corresponds to that procedure instance being stolen.
- A mathematical proof guarantees near-perfect linear speed-up on applications with sufficient parallelism, as long as the architecture has sufficient memory bandwidth.
- A `spawn/return` in `Cilk` is over 100 times faster than a Pthread `create/exit` and less than 3 times slower than an ordinary C function call on a modern Intel processor.

Outline

```
template<typename T> void multiply_iter_par(int ii, int jj, int kk,
    T* C)
{
    cilk_for(int i = 0; i < ii; ++i)
        cilk_for(int j = 0; j < jj; ++j)
            for (int k = 0; k < kk; ++k)
               C[i * jj + j] += A[i * kk + k] + B[k * jj + j];
}
```

Does not scale up well due to a poor locality and uncontrolled granularity.

```
template<typename T> void multiply_rec_seq_helper(int i0, int i1, int j0,
    int j1, int k0, int k1, T* A, ptrdiff_t lda, T* B, ptrdiff_t ldb, T* C,
    ptrdiff_t ldc)
{
    int di = i1 - i0;
    int dj = j1 - j0;
    int dk = k1 - k0;
    if (di >= dj && di >= dk && di >= RECURSION_THRESHOLD) {
        int mi = i0 + di / 2;
        multiply_rec_seq_helper(i0, mi, j0, j1, k0, k1, A, lda, B, ldb, C, ldc);
        multiply_rec_seq_helper(mi, i1, j0, j1, k0, k1, A, lda, B, ldb, C, ldc);
    } else if (dj >= dk && dj >= RECURSION_THRESHOLD) {
        int mj = j0 + dj / 2;
        multiply_rec_seq_helper(i0, i1, j0, mj, k0, k1, A, lda, B, ldb, C, ldc);
        multiply_rec_seq_helper(i0, i1, mj, j1, k0, k1, A, lda, B, ldb, C, ldc);
    } else if (dk >= RECURSION_THRESHOLD) {
        int mk = k0 + dk / 2;
        multiply_rec_seq_helper(i0, i1, j0, j1, k0, mk, A, lda, B, ldb, C, ldc);
        multiply_rec_seq_helper(i0, i1, j0, j1, mk, k1, A, lda, B, ldb, C, ldc);
    } else {
        for (int i = i0; i < i1; ++i)
            for (int k = k0; k < k1; ++k)
                for (int j = j0; j < j1; ++j)
                    C[i * ldc + j] += A[i * lda + k] * B[k * ldb + j];
    }
}
```

```
template<typename T> inline void multiply_rec_seq(int ii, int jj, in
    T* B, T* C)
{
    multiply_rec_seq_helper(0, ii, 0, jj, 0, kk, A, kk, B, jj, C, jj
}
```

Multiplying a 4000x8000 matrix by a 8000x4000 matrix

- on 32 cores = 8 sockets x 4 cores (Quad Core AMD Opteron 8354) per socket.
- The 32 cores share a L3 32-way set-associative cache of 2 Mbytes.

#core	Elision (s)	Parallel (s)	speedup
8	420.906	51.365	8.19
16	432.419	25.845	16.73
24	413.681	17.361	23.83
32	389.300	13.051	29.83

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The fork-join parallelism model

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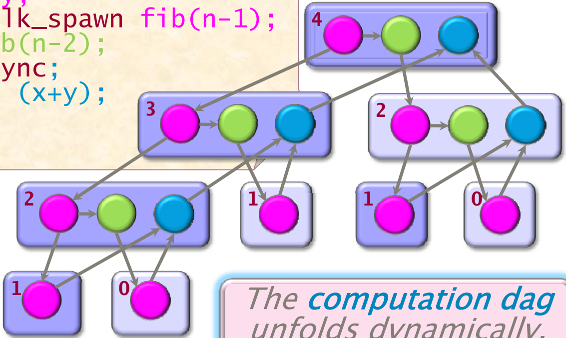
int fib (int n) {
  if (n<2) return (n);
  else {
    int x,y;
    x = cilk_spawn fib(n-1);
    y = fib(n-2);
    cilk_sync;
    return (x+y);
  }
}
```

Example:
`fib(4)`

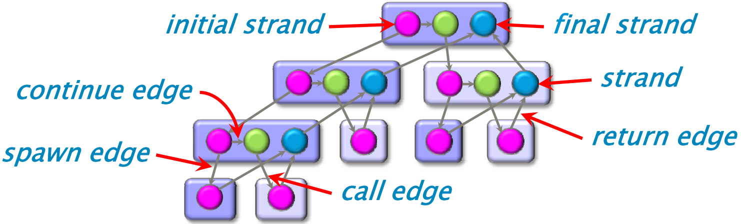


"Processor oblivious"

*The **computation dag** unfolds dynamically.*

We shall also call this model **multithreaded parallelism**.

Terminology



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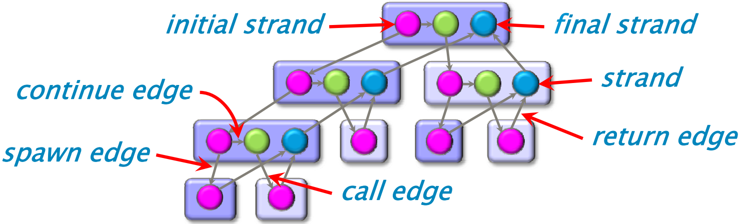

- a **strand** is a maximal sequence of instructions that ends with a `spawn`, `sync`, or `return` (either explicit or implicit) statement.

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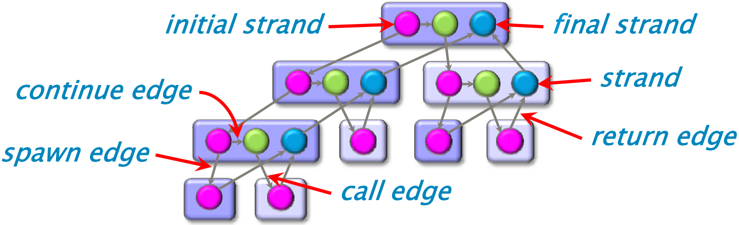
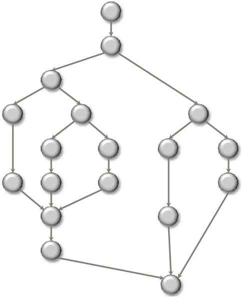


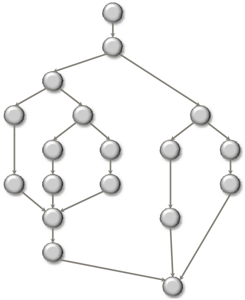
- a **strand** is a maximal sequence of instructions that ends with a `spawn`, `sync`, or `return` (either explicit or implicit) statement.
- At runtime, the *spawn* relation causes procedure instances to be structured as a rooted tree, called **spawn tree** or **parallel instruction stream**, where dependencies among strands form a dag.

Work and span



We define several performance measures. We assume an ideal situation: no cache issues, no interprocessor costs:

Work and span



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T_p is the minimum running time on p processors

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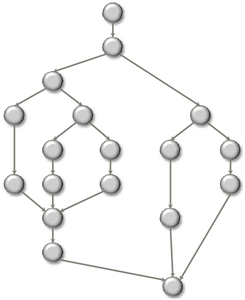

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T_1 is called the **work**, that is, the sum of the number of instructions at each node.

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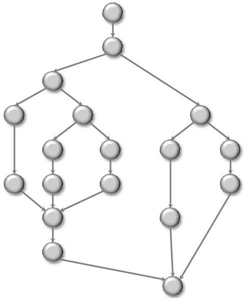

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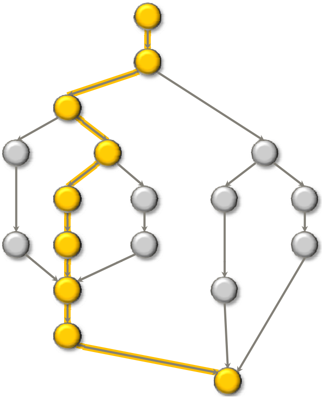
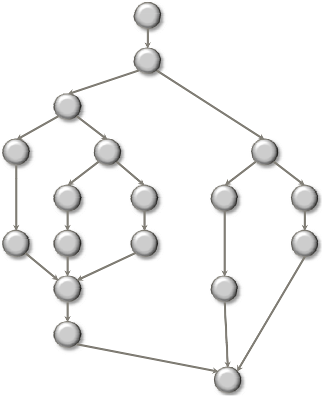
- T_p is the minimum running time on p processors
- T_1 is called the **work**, that is, the sum of the number of instructions at each node.
- T_∞ is the minimum running time with infinitely many processors, called the **span**

The critical path length



Assuming all strands run in unit time, the longest path in the DAG is equal to T_∞. For this reason, T_∞ is also referred to as the **critical path length**.

Work law



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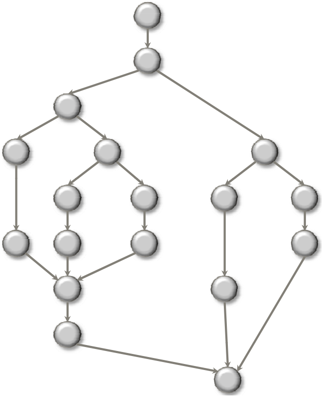
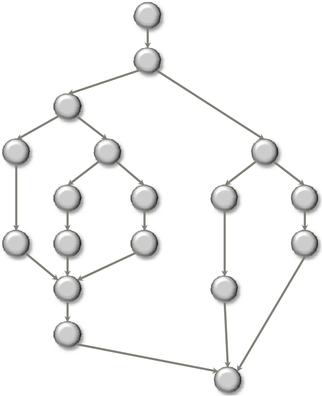


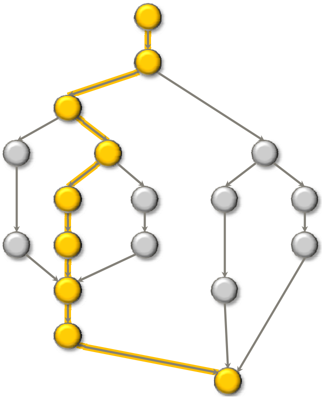
- We have: $T_p \geq T_1/p$.

Work law



- We have: $T_p \geq T_1/p$.
- Indeed, in the best case, p processors can do p works per unit of time.

Span law



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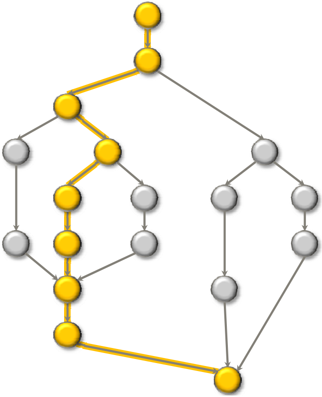
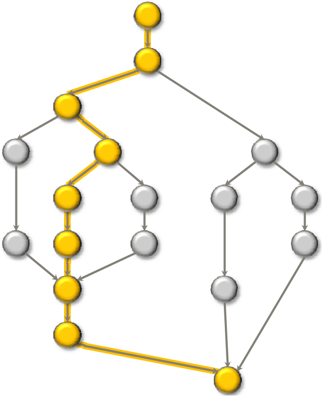


- We have: $T_p \geq T_\infty$.

Span law



- We have: $T_p \geq T_\infty$.
- Indeed, $T_p < T_\infty$ contradicts the definitions of T_p and T_∞.

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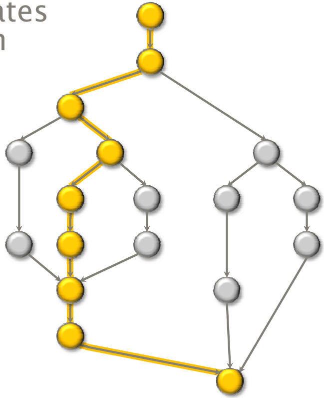
  - ↳ **superlinear speedup**: $T_1/T_P = \omega(p)$ (not possible in this model, though it is possible in others)
  - ↳ **sublinear speedup**: $T_1/T_P = o(p)$

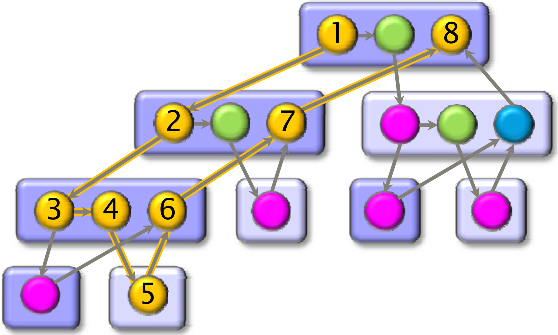
Parallelism

Because the Span Law dictates that $T_P \geq T_\infty$, the maximum possible speedup given T_1 and T_∞ is

$$\begin{aligned} T_1/T_\infty &= \textit{parallelism} \\ &= \text{the average amount of work per step along the span.} \end{aligned}$$



The Fibonacci example (1/2)



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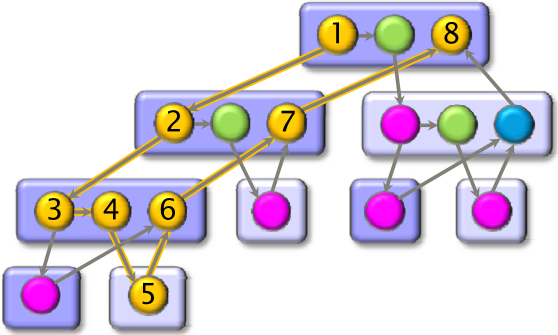

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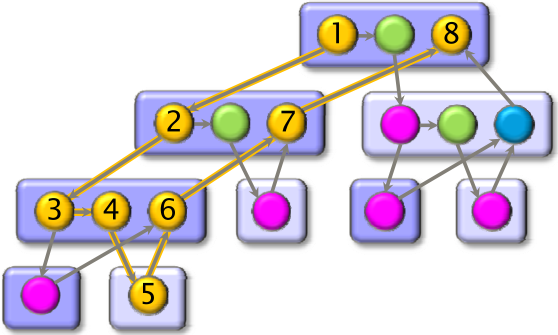

- For $\mathrm{Fib}(4)$, we have $T_1 = 17$ and $T_\infty = 8$ and thus $T_1/T_\infty = 2.125$.
- What about $T_1(\mathrm{Fib}(n))$ and $T_\infty(\mathrm{Fib}(n))$?

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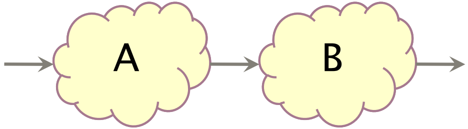
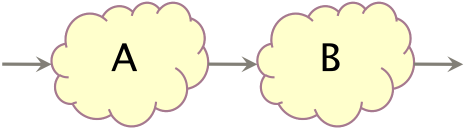
  - ↳ Therefore $T_\infty(n) = \Theta(n)$.
- Consequently the parallelism is $\Theta(\psi^n/n)$.

Series composition



- Work?

Series composition



- Work?
- Span?

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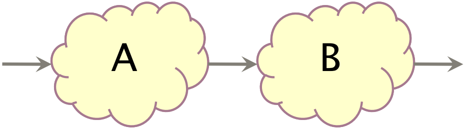

- Work: $T_1(A \cup B) = T_1(A) + T_1(B)$

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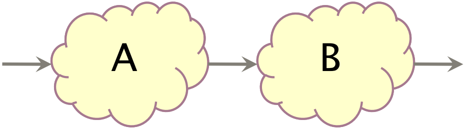

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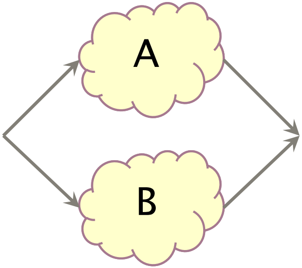

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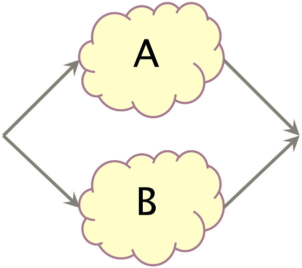

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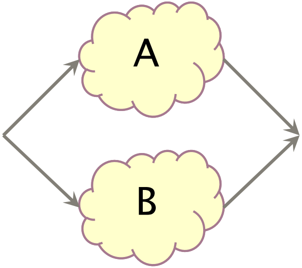

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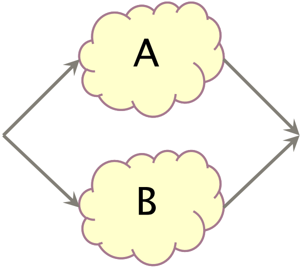

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Some results in the fork-join parallelism model

Algorithm	Work	Span
Merge sort	$\Theta(n \lg n)$	$\Theta(\lg^3 n)$
Matrix multiplication	$\Theta(n^3)$	$\Theta(\lg n)$
Strassen	$\Theta(n^{\lg 7})$	$\Theta(\lg^2 n)$
LU-decomposition	$\Theta(n^3)$	$\Theta(n \lg n)$
Tableau construction	$\Theta(n^2)$	$\Omega(n^{\lg 3})$
FFT	$\Theta(n \lg n)$	$\Theta(\lg^2 n)$
Breadth-first search	$\Theta(E)$	$\Theta(d \lg V)$

We shall prove most of these results in the next sections.

For loop parallelism in Cilk++

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \Rightarrow \underset{A^T}{\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}}$$

```
cilk_for (int i=1; i<n; ++i) {
    for (int j=0; j<i; ++j) {
        double temp = A[i][j];
        A[i][j] = A[j][i];
        A[j][i] = temp;
    }
}
```

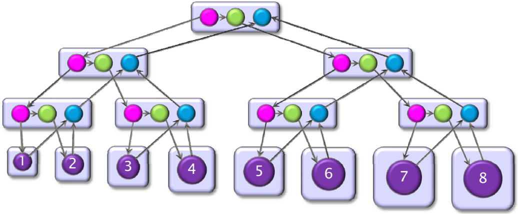
The iterations of a `cilk_for` loop execute in parallel.

Implementation of `for` loops in `Cilk++`

Up to details the previous loop is compiled as follows, using a **divide-and-conquer implementation**:

```
void recur(int lo, int hi) {
    if (hi > lo) { // coarsen
        int mid = lo + (hi - lo)/2;
        cilk_spawn recur(lo, mid);
        recur(mid+1, hi);
        cilk_sync;
    } else
        for (int j=lo; j<hi+1; ++j) {
            double temp = A[hi][j];
            A[hi][j] = A[j][hi];
            A[j][hi] = temp;
        }
    }
}
```

Analysis of parallel `for` loops



Here we do not assume that each strand runs in unit time.

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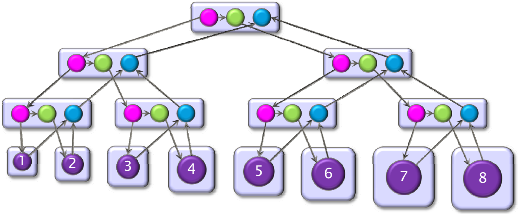

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- **Span of loop control**: $\Theta(\log(n))$

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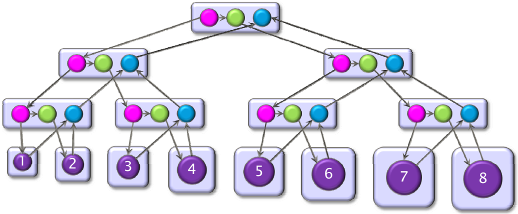

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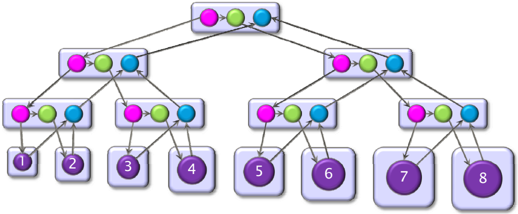

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- **Span of loop control**: $\Theta(\log(n))$
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- **Work**: $\Theta(n^2)$

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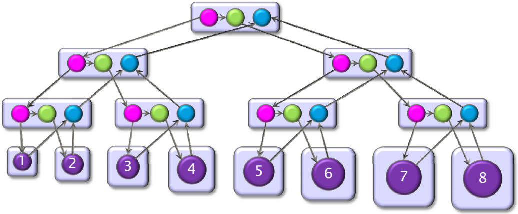

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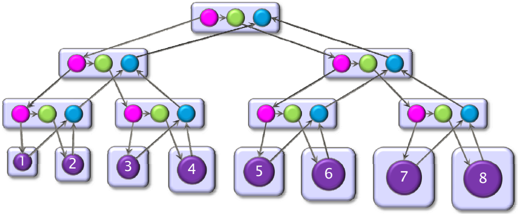

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- **Span of loop control**: $\Theta(\log(n))$
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- **Parallelism**: $\Theta(n)$

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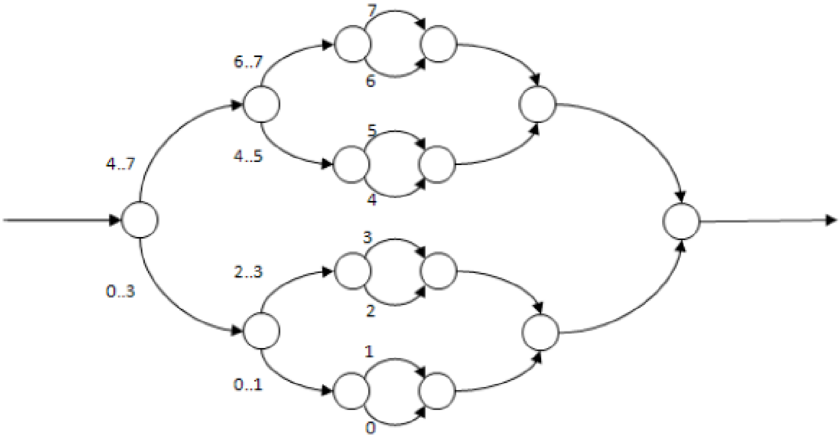

Parallelizing the inner loop

This would yield the following code

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cilk_for (int i=1; i<n; ++i) {
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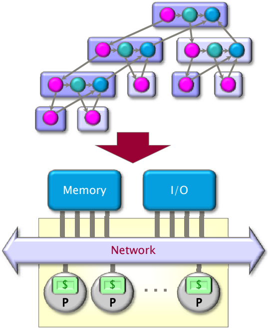
- **Work**: $\Theta(n^2)$
- **Parallelism**: $\Theta(n^2/\log(n))$

In practice, parallelizing the inner loop would increase the memory footprint (allocation of the temporaries) and increase parallelism overheads. So, this is not a good idea.

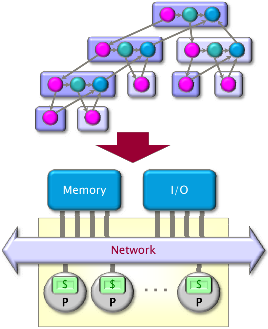
Outline

Scheduling



A **scheduler**'s job is to map a computation to particular processors. Such a mapping is called a **schedule**.

Scheduling



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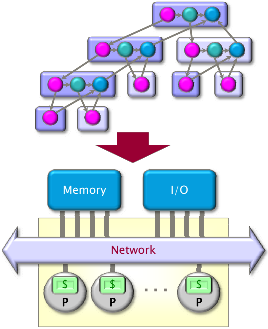
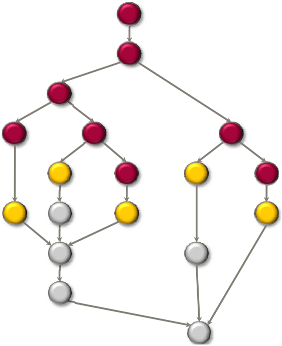


A **scheduler**'s job is to map a computation to particular processors. Such a mapping is called a **schedule**.

- If decisions are made at runtime, the scheduler is *online*, otherwise, it is *offline*
- `Cilk++`'s scheduler maps strands onto processors dynamically at runtime.

Greedy scheduling (1/2)



- A strand is **ready** if all its predecessors have executed

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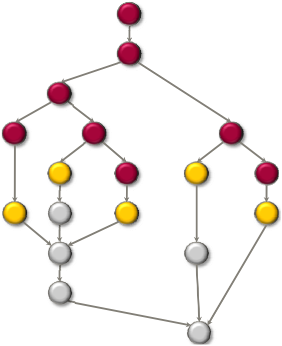
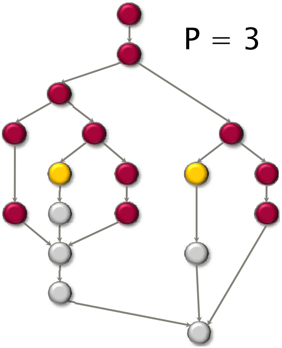


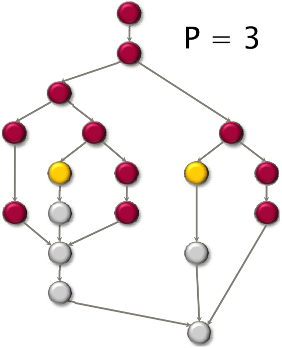
- A strand is **ready** if all its predecessors have executed
- A scheduler is **greedy** if it attempts to do as much work as possible at every step.

Greedy scheduling (2/2)



- In any *greedy schedule*, there are two types of steps:

Greedy scheduling (2/2)



- In any *greedy schedule*, there are two types of steps:
  - ↳ **complete step**: There are at least p strands that are ready to run. The greedy scheduler selects any p of them and runs them.

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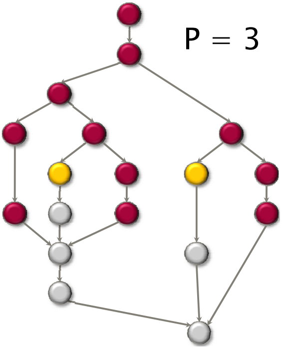
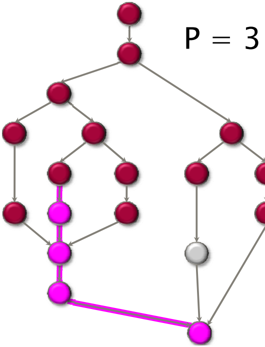


- In any *greedy schedule*, there are two types of steps:
  - ↳ **complete step**: There are at least p strands that are ready to run. The greedy scheduler selects any p of them and runs them.
  - ↳ **incomplete step**: There are strictly less than p strands that are ready to run. The greedy scheduler runs them all.

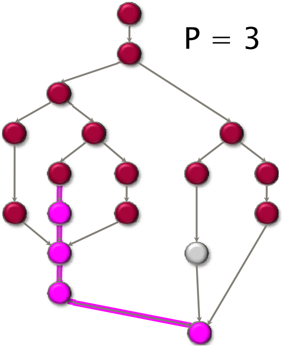
Theorem of Graham and Brent



For any greedy schedule, we have $T_p \;\leq\; T_1/p \;+\; T_\infty$

- $\#\text{complete steps} \leq T_1/p$, by definition of T_1.

Theorem of Graham and Brent



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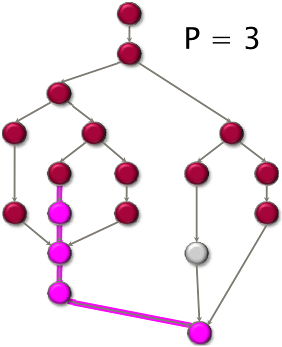

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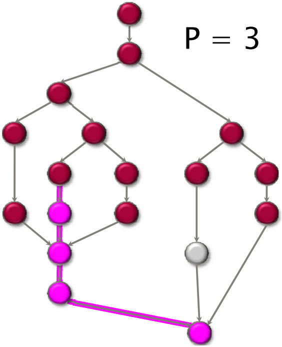

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- A `spawn/return` in `Cilk` is over 100 times faster than a Pthread `create/exit` and less than 3 times slower than an ordinary C function call on a modern Intel processor.

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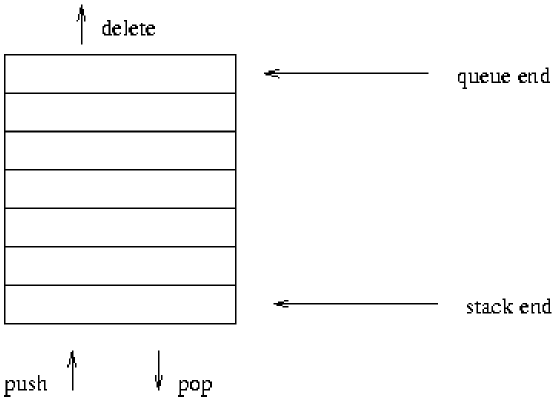

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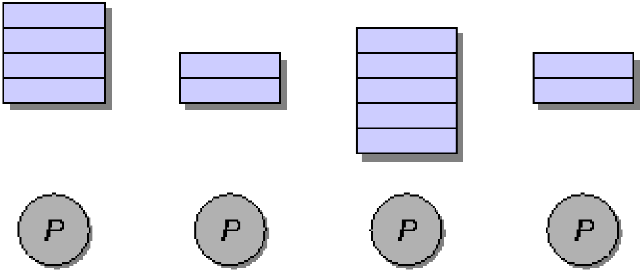

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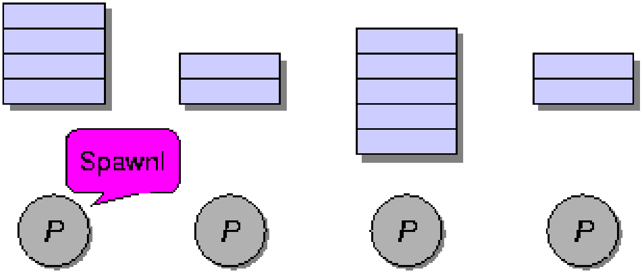

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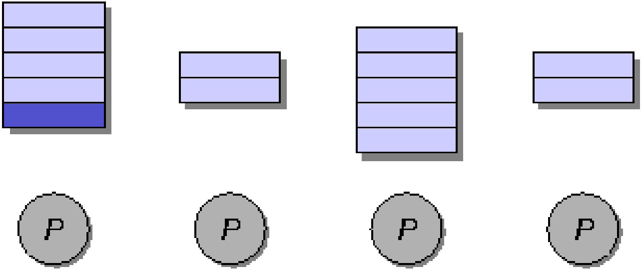

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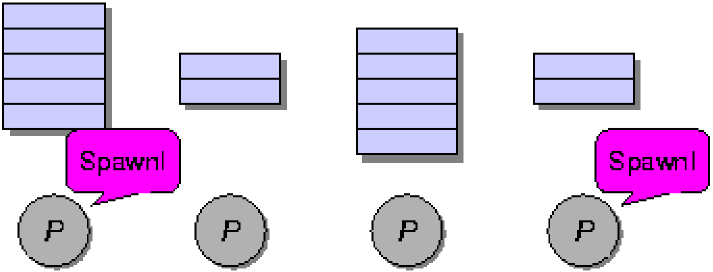

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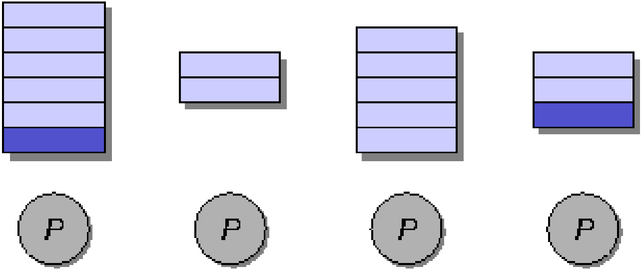

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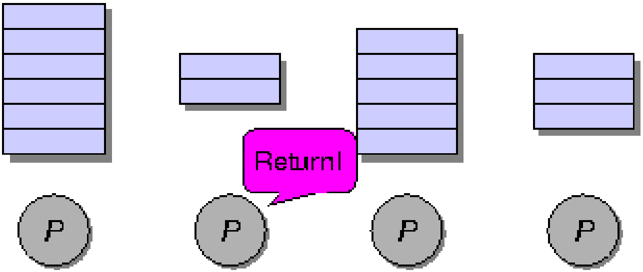

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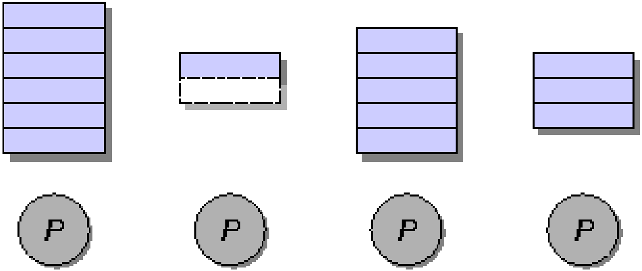

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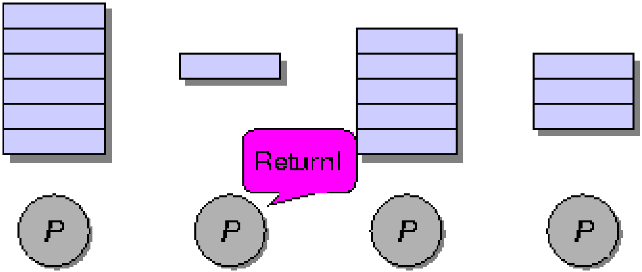

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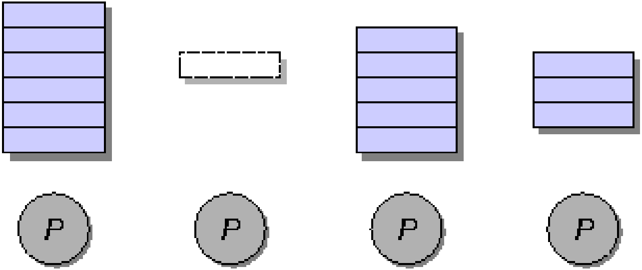

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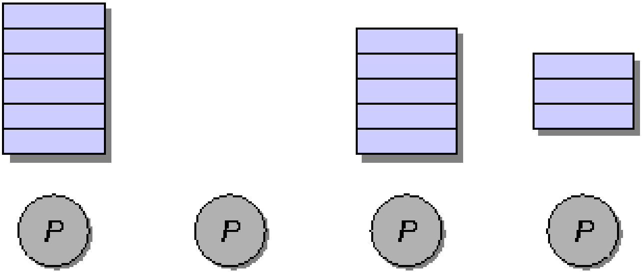

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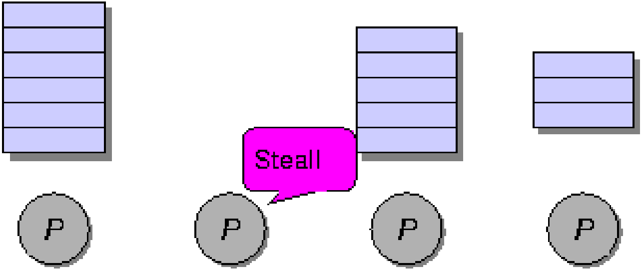

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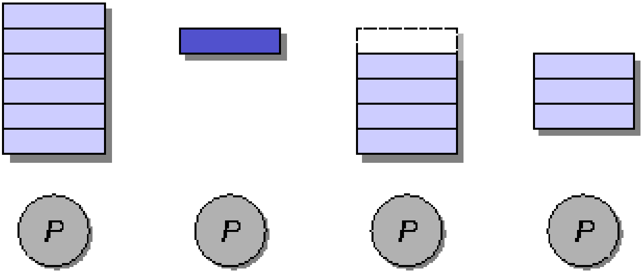

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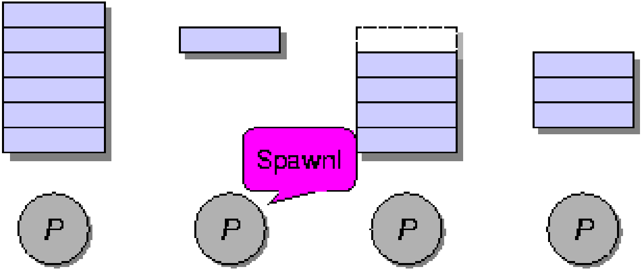

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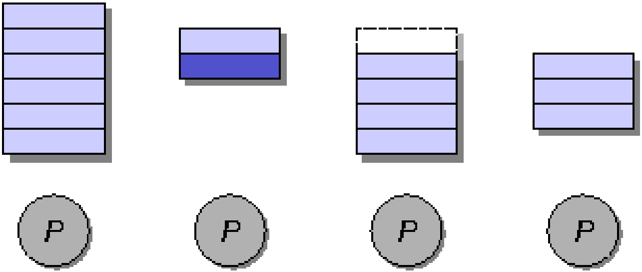

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- Since p processors are working/stealing together, the expected running time

$$T_P = \#\text{steps without steal} + \#\text{steps with steal} = T_1/p + O(pT_\infty)/p.$$

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- We say that $g(n)$ is an **asymptotic lower bound** of $f(n)$ and we write $f(n) \in \Omega(g(n))$ if there exists a strictly positive constants c_1 such that for n big enough we have

$$0 \;\leq\; c_1 \; g(n) \;\leq\; f(n). \tag{14}$$

Examples

- With $f(n) = \frac{1}{2}n^2 - 3n$ and $g(n) = n^2$ we have $f(n) \in \Theta(g(n))$. Indeed we have
$$c_1 \, n^2 \quad \leq \quad \frac{1}{2}n^2 - 3n \quad \leq \quad c_2 \, n^2. \tag{15}$$
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- Assume a and b are positive real constants. Then we have
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Indeed for $n \geq a$ we have
$$0 \quad \leq \quad n^b \quad \leq \quad (n + a)^b \quad \leq \quad (2n)^b. \tag{19}$$
Hence we can choose $c_1 = 1$ and $c_2 = 2^b$.

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In practice $\in$ is replaced by $=$ in each of the expressions $f(n) \in \Theta(g(n))$, $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$. Hence, the following

$$f(n) = h(n) + \Theta(g(n)) \tag{21}$$

means

$$f(n) - h(n) \in \Theta(g(n)). \tag{22}$$

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Exercise: Prove the following

$$\Sigma_{k=1}^{k=n} k \ \in \ \Theta(n^2). \tag{23}$$

Outline

Divide-and-Conquer Algorithms

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Equation satisfied by $T(n)$.

- Assume that the size of the input problem increases with an integer n.
- Let $T(n)$ be the time complexity of a divide-and-conquer algorithm to solve this problem.
- Then $T(n)$ satisfies an equation of the form:

$$T(n) = a\,T(n/b) + f(n). \tag{24}$$

where $f(n)$ is the cost of the combine-part, $a \geq 1$ is the number of recursively calls and n/b with $b > 1$ is the size of a sub-problem.

Tree associated with a divide-and-conquer recurrence

Labeled tree associated with the equation. Assume n is a power of b, say $n = b^p$.

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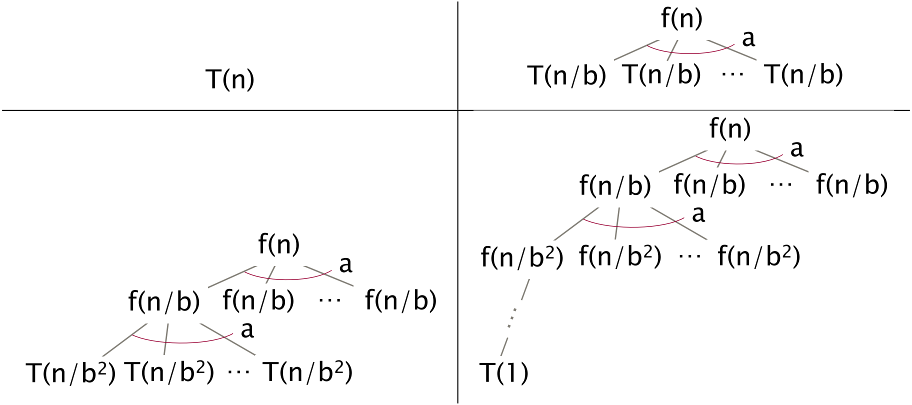
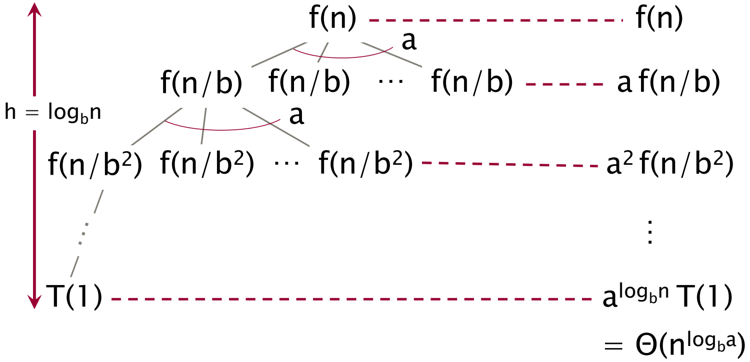
The labeled tree $\mathcal{A}(n)$ associated with $T(n) = a\,T(n/b) + f(n)$ has height $p + 1$. Moreover the sum of its labels is $T(n)$.

Solving divide-and-conquer recurrences (1/2)

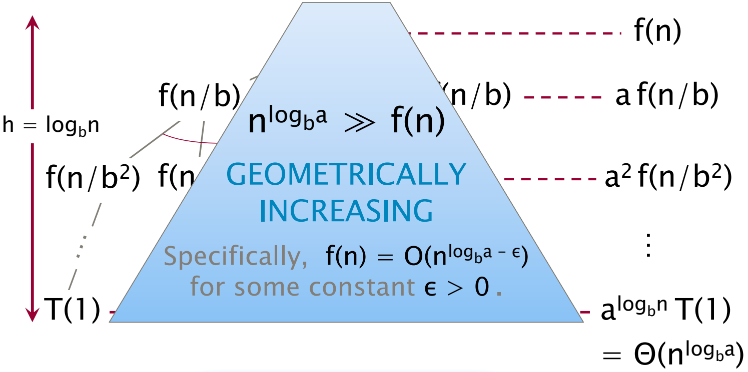


Solving divide-and-conquer recurrences (2/2)



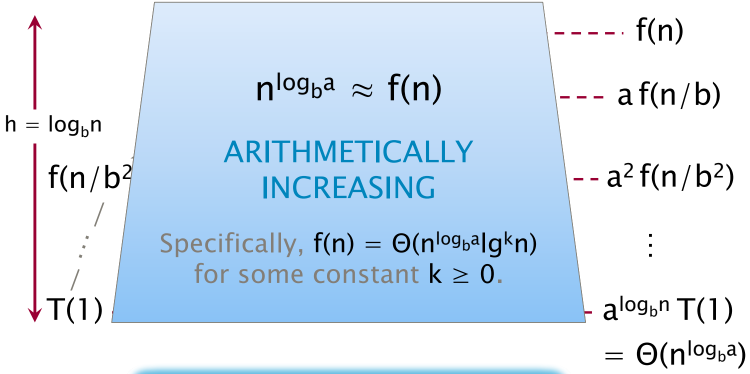
IDEA: Compare $n^{\log_b a}$ with $f(n)$.

Master Theorem: case $n^{\log_b a} \gg f(n)$



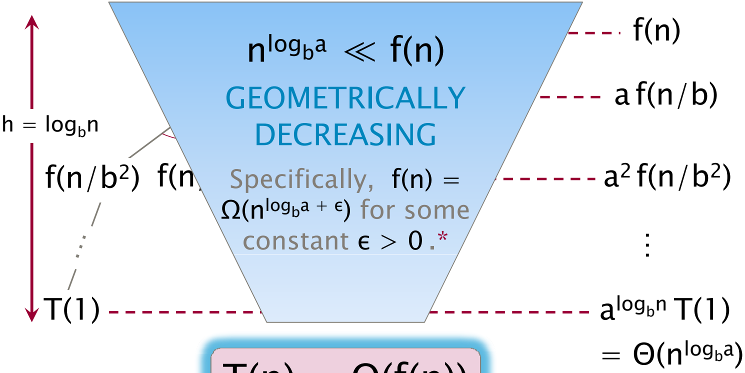
$$T(n) = \Theta(n^{\log_b a})$$

Master Theorem: case $f(n) \in \Theta(n^{\log_b a} \log^k n)$



$$T(n) = \Theta(n^{\log_b a} \lg^{k+1} n))$$

Master Theorem: case where $f(n) \gg n^{\log_b a}$



$$\boxed{T(n) = \Theta(f(n))}$$

*and $f(n)$ satisfies the ***regularity condition*** that $a\, f(n/b) \leq c\, f(n)$ for some constant $c < 1$.

More examples

- Consider the relation:

$$T(n) = 2\,T(n/2) + n^2. \tag{25}$$

We obtain:

$$T(n) = n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \frac{n^2}{8} + \cdots + \frac{n^2}{2^p} + n\,T(1). \tag{26}$$

Hence we have:

$$T(n) \in \Theta(n^2). \tag{27}$$

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- Consider the relation:

$$T(n) = 3T(n/3) + n. \tag{28}$$

We obtain:

$$T(n) \in \Theta(\log_3(n)n). \tag{29}$$

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$$T(n) \leq (2 - 2/n)\, f(n) + T(1) \in \mathcal{O}(f(n)), \tag{30}$$

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(3) if $a \geq 3$ then

$$T(n) \leq \frac{2}{a-2} \left(n^{\log_2(a)-1} - 1 \right) f(n) + T(1) n^{\log_2(a)} \in \mathcal{O}(f(n) n^{\log_2(a)-1}). \tag{32}$$

Master Theorem when $b = 2$

Indeed

$$\begin{array}{rcl} T(2^p) & \leq & a\,T(2^{p-1}) + f(2^p) \\ & \leq & a\left[a\,T(2^{p-2}) + f(2^{p-1})\right] + f(2^p) \\ & = & a^2\,T(2^{p-2}) + a\,f(2^{p-1}) + f(2^p) \\ & \leq & a^2\left[a\,T(2^{p-3}) + f(2^{p-2})\right] + a\,f(2^{p-1}) + f(2^p) \\ & = & a^3\,T(2^{p-3}) + a^2\,f(2^{p-2}) + a\,f(2^{p-1}) + f(2^p) \\ & \leq & a^p\,T(s1) + \sigma_{j=0}^{j=p-1}\;a^j\,f(2^{p-j}) \end{array} \tag{33}$$

Master Theorem when $b = 2$

Moreover

$$\begin{array}{rcl} f(2^p) & \geq & 2\, f(2^{p-1}) \\ f(2^p) & \geq & 2^2\, f(2^{p-2}) \\ \vdots & \vdots & \vdots \\ f(2^p) & \geq & 2^j\, f(2^{p-j}) \end{array} \tag{34}$$

Thus

$$\Sigma_{j=0}^{j=p-1}\, a^j\, f(2^{p-j}) \;\leq\; f(2^p)\, \Sigma_{j=0}^{j=p-1} \left(\frac{a}{2}\right)^j. \tag{35}$$

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Hence

$$T(2^p) \ \leq \ a^p\, T(1) + f(2^p)\, \Sigma_{j=0}^{j=p-1} \left(\frac{a}{2}\right)^j. \tag{36}$$

For $a = 1$ we obtain

$$\begin{array}{rcl} T(2^p) & \leq & T(1) + f(2^p)\, \Sigma_{j=0}^{j=p-1} \left(\frac{1}{2}\right)^j \\ & = & T(1) + f(2^p)\, \frac{\frac{1}{2^p} - 1}{\frac{1}{2} - 1} \\ & = & T(1) + f(n)\,(2 - 2/n). \end{array} \tag{37}$$

For $a = 2$ we obtain

$$\begin{array}{rcl} T(2^p) & \leq & 2^p\, T(1) + f(2^p)\, p \\ & = & n\, T(1) + f(n) \log_2(n). \end{array} \tag{38}$$

Master Theorem cheat sheet

For $a \geq 1$ and $b > 1$, consider again the equation

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Master Theorem quizz!

- $T(n) = 4T(n/2) + n$

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- $T(n) = 4T(n/2) + n^2/\log n$

Outline

Matrix multiplication

$$
\underbrace{\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}}_{B}
$$

We will study three approaches:

Matrix multiplication

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We will study three approaches:

- a naive and iterative one

Matrix multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$C \qquad\qquad\qquad A \qquad\qquad\qquad B$$

We will study three approaches:

- a naive and iterative one
- a divide-and-conquer one

Matrix multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$\qquad C \qquad\qquad\qquad\qquad A \qquad\qquad\qquad\qquad B$$

We will study three approaches:

- a naive and iterative one
- a divide-and-conquer one
- a divide-and-conquer one with memory management consideration

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

- **Work**: ?

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

- **Work**: ?
- **Span**: ?

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

- **Work**: ?
- **Span**: ?
- **Parallelism**: ?

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

- **Work**: $\Theta(n^3)$

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

- **Work**: $\Theta(n^3)$
- **Span**: $\Theta(n)$

Naive iterative matrix multiplication

```
cilk_for (int i=1; i<n; ++i) {
    cilk_for (int j=0; j<n; ++j) {
        for (int k=0; k<n; ++k {
            C[i][j] += A[i][k] * B[k][j];
  }
}
```

- **Work**: $\Theta(n^3)$
- **Span**: $\Theta(n)$
- **Parallelism**: $\Theta(n^2)$

Matrix multiplication based on block decomposition

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix} + \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

The divide-and-conquer approach is simply the one based on blocking, presented in the previous lecture.

Divide-and-conquer matrix multiplication

```
// C <- C + A * B
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync;
  MAdd(C, D, n, size); // C += D;
  delete[] D;
}
```

Work ? **Span** ? **Parallelism** ?

Divide-and-conquer matrix multiplication

```
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync; MAdd(C, D, n, size); // C += D;
  delete[] D;  }
```

- $A_p(n)$ and $M_p(n)$: times on p proc. for $n \times n$ $\textsc{Add}$ and $\textsc{Mult}$.

Divide-and-conquer matrix multiplication

```
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync; MAdd(C, D, n, size); // C += D;
  delete[] D;  }
```

- $A_p(n)$ and $M_p(n)$: times on p proc. for $n \times n$ $\textsc{Add}$ and $\textsc{Mult}$.
- $A_1(n) = 4A_1(n/2) + \Theta(1) = \Theta(n^2)$

Divide-and-conquer matrix multiplication

```
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync; MAdd(C, D, n, size); // C += D;
  delete[] D;  }
```

- $A_p(n)$ and $M_p(n)$: times on p proc. for $n \times n$ $\textsc{Add}$ and $\textsc{Mult}$.
- $A_1(n) = 4A_1(n/2) + \Theta(1) = \Theta(n^2)$
- $A_\infty(n) = A_\infty(n/2) + \Theta(1) = \Theta(\lg n)$

Divide-and-conquer matrix multiplication

```
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync; MAdd(C, D, n, size); // C += D;
  delete[] D;  }
```

- $A_p(n)$ and $M_p(n)$: times on p proc. for $n \times n$ ADD and MULT.
- $A_1(n) = 4A_1(n/2) + \Theta(1) = \Theta(n^2)$
- $A_\infty(n) = A_\infty(n/2) + \Theta(1) = \Theta(\lg n)$
- $M_1(n) = 8M_1(n/2) + A_1(n) = 8M_1(n/2) + \Theta(n^2) = \Theta(n^3)$

Divide-and-conquer matrix multiplication

```
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync; MAdd(C, D, n, size); // C += D;
  delete[] D;  }
```

- $A_p(n)$ and $M_p(n)$: times on p proc. for $n \times n$ ADD and MULT.
- $A_1(n) = 4A_1(n/2) + \Theta(1) = \Theta(n^2)$
- $A_\infty(n) = A_\infty(n/2) + \Theta(1) = \Theta(\lg n)$
- $M_1(n) = 8M_1(n/2) + A_1(n) = 8M_1(n/2) + \Theta(n^2) = \Theta(n^3)$
- $M_\infty(n) = M_\infty(n/2) + \Theta(\lg n) = \Theta(\lg^2 n)$

Divide-and-conquer matrix multiplication

```
void MMult(T *C, T *A, T *B, int n, int size) {
  T *D = new T[n*n];
  //base case & partition matrices
  cilk_spawn MMult(C11, A11, B11, n/2, size);
  cilk_spawn MMult(C12, A11, B12, n/2, size);
  cilk_spawn MMult(C22, A21, B12, n/2, size);
  cilk_spawn MMult(C21, A21, B11, n/2, size);
  cilk_spawn MMult(D11, A12, B21, n/2, size);
  cilk_spawn MMult(D12, A12, B22, n/2, size);
  cilk_spawn MMult(D22, A22, B22, n/2, size);
             MMult(D21, A22, B21, n/2, size);
  cilk_sync; MAdd(C, D, n, size); // C += D;
  delete[] D;  }
```

- $A_p(n)$ and $M_p(n)$: times on p proc. for $n \times n$ ADD and MULT.
- $A_1(n) = 4A_1(n/2) + \Theta(1) = \Theta(n^2)$
- $A_\infty(n) = A_\infty(n/2) + \Theta(1) = \Theta(\lg n)$
- $M_1(n) = 8M_1(n/2) + A_1(n) = 8M_1(n/2) + \Theta(n^2) = \Theta(n^3)$
- $M_\infty(n) = M_\infty(n/2) + \Theta(\lg n) = \Theta(\lg^2 n)$
- $M_1(n)/M_\infty(n) = \Theta(n^3/\lg^2 n)$

Divide-and-conquer matrix multiplication: No temporaries!

```
template <typename T>
void MMult2(T *C, T *A, T *B, int n, int size) {
  //base case & partition matrices
  cilk_spawn MMult2(C11, A11, B11, n/2, size);
  cilk_spawn MMult2(C12, A11, B12, n/2, size);
  cilk_spawn MMult2(C22, A21, B12, n/2, size);
             MMult2(C21, A21, B11, n/2, size);
  cilk_sync;
  cilk_spawn MMult2(C11, A12, B21, n/2, size);
  cilk_spawn MMult2(C12, A12, B22, n/2, size);
  cilk_spawn MMult2(C22, A22, B22, n/2, size);
             MMult2(C21, A22, B21, n/2, size);
  cilk_sync;  }
```

Work ? **Span** ? **Parallelism** ?

Divide-and-conquer matrix multiplication: No temporaries!

```
template <typename T>
void MMult2(T *C, T *A, T *B, int n, int size) {
  //base case & partition matrices
  cilk_spawn MMult2(C11, A11, B11, n/2, size);
  cilk_spawn MMult2(C12, A11, B12, n/2, size);
  cilk_spawn MMult2(C22, A21, B12, n/2, size);
             MMult2(C21, A21, B11, n/2, size);
  cilk_sync;
  cilk_spawn MMult2(C11, A12, B21, n/2, size);
  cilk_spawn MMult2(C12, A12, B22, n/2, size);
  cilk_spawn MMult2(C22, A22, B22, n/2, size);
             MMult2(C21, A22, B21, n/2, size);
  cilk_sync;  }
```

- $MA_p(n)$: time on p proc. for $n \times n$ $\textsc{Mult-Add}$.

Divide-and-conquer matrix multiplication: No temporaries!

```
template <typename T>
void MMult2(T *C, T *A, T *B, int n, int size) {
  //base case & partition matrices
  cilk_spawn MMult2(C11, A11, B11, n/2, size);
  cilk_spawn MMult2(C12, A11, B12, n/2, size);
  cilk_spawn MMult2(C22, A21, B12, n/2, size);
             MMult2(C21, A21, B11, n/2, size);
  cilk_sync;
  cilk_spawn MMult2(C11, A12, B21, n/2, size);
  cilk_spawn MMult2(C12, A12, B22, n/2, size);
  cilk_spawn MMult2(C22, A22, B22, n/2, size);
             MMult2(C21, A22, B21, n/2, size);
  cilk_sync;  }
```

- $MA_p(n)$: time on p proc. for $n \times n$ MULT-ADD.
- $MA_1(n) = \Theta(n^3)$

Divide-and-conquer matrix multiplication: No temporaries!

```
template <typename T>
void MMult2(T *C, T *A, T *B, int n, int size) {
  //base case & partition matrices
  cilk_spawn MMult2(C11, A11, B11, n/2, size);
  cilk_spawn MMult2(C12, A11, B12, n/2, size);
  cilk_spawn MMult2(C22, A21, B12, n/2, size);
             MMult2(C21, A21, B11, n/2, size);
  cilk_sync;
  cilk_spawn MMult2(C11, A12, B21, n/2, size);
  cilk_spawn MMult2(C12, A12, B22, n/2, size);
  cilk_spawn MMult2(C22, A22, B22, n/2, size);
             MMult2(C21, A22, B21, n/2, size);
  cilk_sync;  }
```

- $MA_p(n)$: time on p proc. for $n \times n$ MULT-ADD.
- $MA_1(n) = \Theta(n^3)$
- $MA_\infty(n) = 2MA_\infty(n/2) + \Theta(1) = \Theta(n)$

Divide-and-conquer matrix multiplication: No temporaries!

```
template <typename T>
void MMult2(T *C, T *A, T *B, int n, int size) {
  //base case & partition matrices
  cilk_spawn MMult2(C11, A11, B11, n/2, size);
  cilk_spawn MMult2(C12, A11, B12, n/2, size);
  cilk_spawn MMult2(C22, A21, B12, n/2, size);
             MMult2(C21, A21, B11, n/2, size);
  cilk_sync;
  cilk_spawn MMult2(C11, A12, B21, n/2, size);
  cilk_spawn MMult2(C12, A12, B22, n/2, size);
  cilk_spawn MMult2(C22, A22, B22, n/2, size);
             MMult2(C21, A22, B21, n/2, size);
  cilk_sync;  }
```

- $MA_p(n)$: time on p proc. for $n \times n$ MULT-ADD.
- $MA_1(n) = \Theta(n^3)$
- $MA_\infty(n) = 2MA_\infty(n/2) + \Theta(1) = \Theta(n)$
- $MA_1(n)/MA_\infty(n) = \Theta(n^2)$

Divide-and-conquer matrix multiplication: No temporaries!

```
template <typename T>
void MMult2(T *C, T *A, T *B, int n, int size) {
  //base case & partition matrices
  cilk_spawn MMult2(C11, A11, B11, n/2, size);
  cilk_spawn MMult2(C12, A11, B12, n/2, size);
  cilk_spawn MMult2(C22, A21, B12, n/2, size);
             MMult2(C21, A21, B11, n/2, size);
  cilk_sync;
  cilk_spawn MMult2(C11, A12, B21, n/2, size);
  cilk_spawn MMult2(C12, A12, B22, n/2, size);
  cilk_spawn MMult2(C22, A22, B22, n/2, size);
             MMult2(C21, A22, B21, n/2, size);
  cilk_sync;  }
```

- $MA_p(n)$: time on p proc. for $n \times n$ MULT-ADD.
- $MA_1(n) = \Theta(n^3)$
- $MA_\infty(n) = 2MA_\infty(n/2) + \Theta(1) = \Theta(n)$
- $MA_1(n)/MA_\infty(n) = \Theta(n^2)$
- Besides, saving space often saves time due to hierarchical memory.

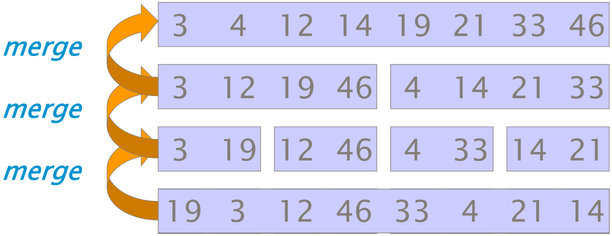
Outline

Merging two sorted arrays

```
void Merge(T *C, T *A, T *B, int na, int nb) {
  while (na>0 && nb>0) {
    if (*A <= *B) {
      *C++ = *A++; na--;
    } else {
      *C++ = *B++; nb--;
    }
  }
  while (na>0) {
    *C++ = *A++; na--;
  }
  while (nb>0) {
    *C++ = *B++; nb--;
  }
}
```

Time for merging n elements is $\Theta(n)$.

Merge sort



merge
merge
merge
3 4 12 14 19 21 33 46
3 12 19 46
4 14 21 33
3 19
12 46
4 33
14 21
19 3 12 46 33 4 21 14

Parallel merge sort with serial merge

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C[n];
    cilk_spawn MergeSort(C, A, n/2);
               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
```

Parallel merge sort with serial merge

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C[n];
    cilk_spawn MergeSort(C, A, n/2);
               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
```

- **Work**?

Parallel merge sort with serial merge

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C[n];
    cilk_spawn MergeSort(C, A, n/2);
               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
```

- **Work**?
- **Span**?

Parallel merge sort with serial merge

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C[n];
    cilk_spawn MergeSort(C, A, n/2);
               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
```

Parallel merge sort with serial merge

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C[n];
    cilk_spawn MergeSort(C, A, n/2);
               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
```

- $T_1(n) = 2T_1(n/2) + \Theta(n)$ thus $T_1(n) = \Theta(n \lg n)$.

Parallel merge sort with serial merge

```
template <typename T>
void MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
  } else {
    T* C[n];
    cilk_spawn MergeSort(C, A, n/2);
               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
  }
```

- $T_1(n) = 2T_1(n/2) + \Theta(n)$ thus $T_1(n) = \Theta(n \lg n)$.
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Parallel merge sort with serial merge

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template <typename T>
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    T* C[n];
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               MergeSort(C+n/2, A+n/2, n-n/2);
    cilk_sync;
    Merge(B, C, C+n/2, n/2, n-n/2);
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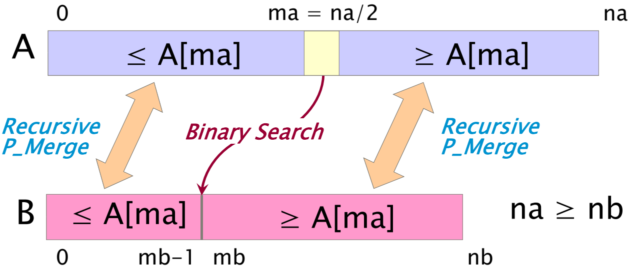
- $T_1(n)/T_\infty(n) = \Theta(\lg n)$. **Puny parallelism!**
- We need to parallelize the merge!

Parallel merge



Idea: if the total number of elements to be sorted in $n = n_a + n_b$ then the maximum number of elements in any of the two merges is at most $3n/4$.

Parallel merge

```
template <typename T>
void P_Merge(T *C, T *A, T *B, int na, int nb) {
  if (na < nb) {
    P_Merge(C, B, A, nb, na);
  } else if (na==0) {
      return;
  } else {
    int ma = na/2;
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- One should coarsen the base case for efficiency.

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- One should coarsen the base case for efficiency.
- **Work**? **Span**?

Parallel merge

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- Let $PM_p(n)$ be the *p-processor* running time of $\mathrm{P\text{-}M{\small ERGE}}$.

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- Let $PM_p(n)$ be the p-*processor* running time of $\text{P-M}\textsc{erge}$.
- In the worst case, the span of $\text{P-M}\textsc{erge}$ is

$$PM_\infty(n) \le PM_\infty(3n/4) + \Theta(\lg n) = O(\lg^2 n)$$

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- Let $PM_p(n)$ be the *p-processor* running time of P-MERGE.
- In the worst case, the span of P-MERGE is

$$PM_\infty(n) \leq PM_\infty(3n/4) + \Theta(\lg n) = O(\lg^2 n)$$

- The worst-case work of P-MERGE satisfies the recurrence

$$PM_1(n) \leq PM_1(\alpha n) + PM_1((1-\alpha)n) + \Theta(\lg n)$$

Analyzing parallel merge

- Recall $PM_1(n) \leq PM_1(\alpha n) + PM_1((1-\alpha)n) + \Theta(\lg n)$ for some $1/4 \leq \alpha \leq 3/4$.

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- Then pick a large enough to satisfy the base conditions, leading to $PM_1(n) \in O(n)$.
- Since we clearly have $PM_1(n) \in \Omega(n)$ (because n array elements are accessed anyway), we finally have $PM_1(n) = \Theta(n)$.

Parallel merge sort with parallel merge

```
template <typename T>
void P_MergeSort(T *B, T *A, int n) {
  if (n==1) {
    B[0] = A[0];
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    T C[n];
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    cilk_sync;
P_Merge(B, C, C+n/2, n/2, n-n/2);
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- **Work**?

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- **Work**?
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```

- The work satisfies $T_1(n) = 2T_1(n/2) + \Theta(n)$ (as usual) and we have $T_1(n) = \Theta(n\log(n))$.

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$$T_\infty(n) = T_\infty(n/2) + \Theta(\lg^2 n) = \Theta(\lg^3 n)$$

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.

- The parallelism is now $\Theta(n \lg n)/\Theta(\lg^3 n) = \Theta(n/\lg^2 n)$.

Outline

Tableau construction

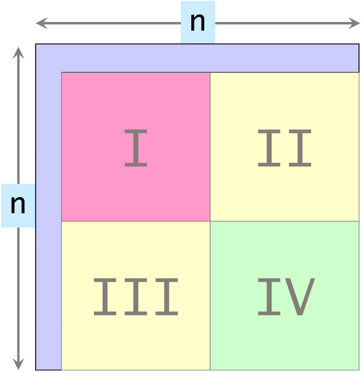
00	01	02	03	04	05	06	07
10	11	12	13	14	15	16	17
20	21	22	23	24	25	26	27
30	31	32	33	34	35	36	37
40	41	42	43	44	45	46	47
50	51	52	53	54	55	56	57
60	61	62	63	64	65	66	67
70	71	72	73	74	75	76	77

Constructing a tableau A satisfying a relation of the form:

$$A[i,j] = R(A[i-1,j], A[i-1,j-1], A[i,j-1]). \tag{43}$$

The work is $\Theta(n^2)$.

Recursive construction



Parallel code

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I;
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III;
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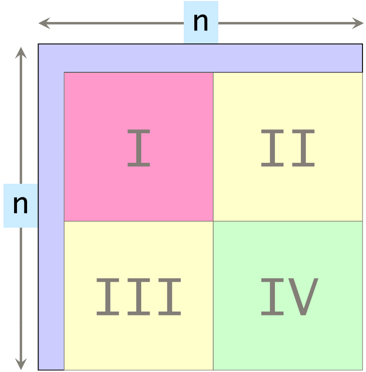
IV;
```

Recursive construction



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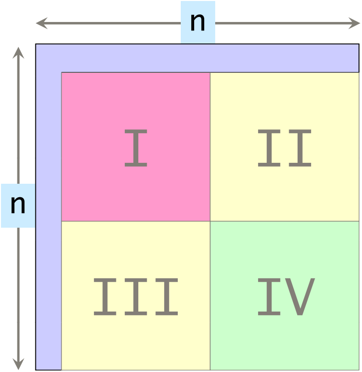

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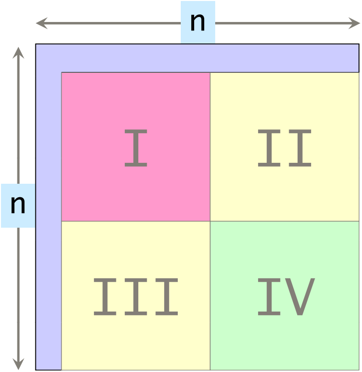

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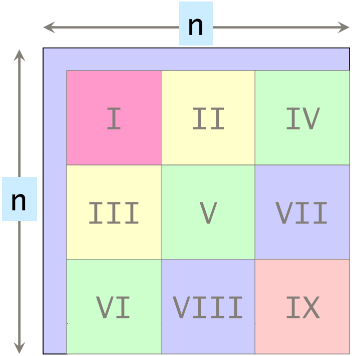
- $T_\infty(n) = 3T_\infty(n/2) + \Theta(1)$, thus $T_\infty(n) = \Theta(n^{\log_2 3})$.
- **Parallelism**: $\Theta(n^{2-\log_2 3}) = \Omega(n^{0.41})$.

A more parallel construction

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I;
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cilk_spawn IV;
cilk_spawn V;
VI;
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cilk_spawn VII;
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```

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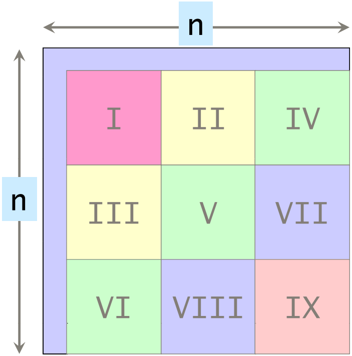


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- $T_1(n) = 9T_1(n/3) + \Theta(1)$, thus $T_1(n) = \Theta(n^2)$.

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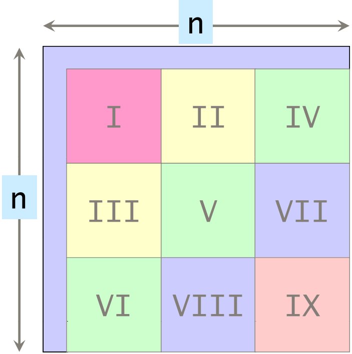


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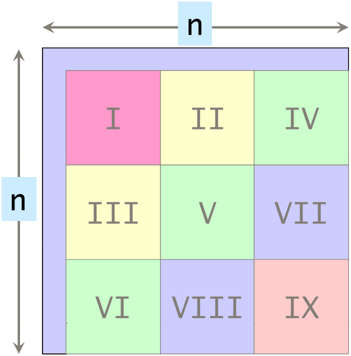


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- **Parallelism**: $\Theta(n^{2-\log_3 5}) = \Omega(n^{0.53})$.

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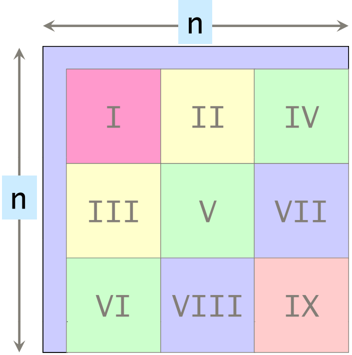


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- **Parallelism**: $\Theta(n^{2-\log_3 5}) = \Omega(n^{0.53})$.
- This nine-way d-n-c has more parallelism than the four way but exhibits more cache complexity.

Acknowledgements

- Charles E. Leiserson (MIT) and Matteo Frigo (Oracle) for providing me with the sources of their lecture notes.
- My former students Yuzhen Xie and Liyun Li or generating the experimental data.

References

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