PhD Thesis Lecture: Solving polynomial systems via triangular decomposition

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Gaussian elimination

A set of linear equations

$$F := \begin{cases} x + y + z - 2 &= 0 \\ 2x + 3y + z - 1 &= 0 \\ x - y + z - 1 &= 0 \end{cases}$$

The triangular decomposition

$$T := \begin{cases} 2z - 7 &= 0 \\ 2y - 1 &= 0 \\ x + 2 &= 0 \end{cases}$$

Triangular decomposition

Input system

$$F := \begin{cases} x^2 + y + z - 1 = 0 \\ x + y^2 + z - 1 = 0 \\ x + y + z^2 - 1 = 0 \end{cases}$$

Variable order $z > y > x$.

A triangular decomposition

$$\begin{cases} z - x = 0 \\ y - x = 0 \\ x^2 + 2x - 1 = 0 \end{cases} \quad \begin{cases} z = 0 \\ y = 0 \\ x - 1 = 0 \end{cases} \quad \begin{cases} z = 0 \\ y - 1 = 0 \\ x = 0 \end{cases} \quad \begin{cases} z - 1 = 0 \\ y = 0 \\ x = 0 \end{cases}$$

Polynomial systems with or without parameters

- A polynomial system of $\mathbf{k}[\mathbf{u}, \mathbf{y}]$ consists of equations ($=$).

$$F(s, x, y) := \begin{cases} x(1+y) - s & = & 0 \\ y(1+x) - s & = & 0 \end{cases}, \qquad (1)$$

  the solution set of (1) in $\mathbf{K}^3$ is called an algebraic variety.

- Add inequations ($\neq$) to system (1).

$$\begin{cases} F(s, x, y) \\ x + y - 1 \neq 0 \end{cases}, \qquad (2)$$

  the solution set of (2) in $\mathbf{K}^3$ is called a constructible set.

- Let $\mathbf{k} = \mathbb{Q}$. Add inequalities ($>$, $\geq$, $<$, $\leq$) to system (2).

$$\begin{cases} F(s, x, y) \\ x + y - 1 > 0 \end{cases}, \qquad (3)$$

  the solution set of (3) in $\mathbb{R}^3$ is called a semi-algebraic set.

Outline

Publications

- Comprehensive Triangular Decomposition, joint work with F. Lemaire, O. Golubitsky, M. Moreno Maza and W. Pan. CASC 2007.
- On the Verification of Polynomial System Solvers, joint work with M. Moreno Maza, W. Pan and Y. Xie. AWFS 2007.
- The ConstructibleSetTools and ParametricSystemsTools Modules of the RegularChains Library in Maple, joint work with C. Chen, F. Lemaire, L. Liyun, M. Moreno Maza, W. Pan and Y. Xie. ICCSA 2008.
- Computing Cylindrical Algebraic Decomposition via Triangular Decomposition, joint work with M. Moreno Maza, B. Xia and L. Yang. ISSAC 2009.
- Real Root Isolation of Regular Chains, joint work with F. Boulier, C. Chen, F. Lemaire and M. Moreno Maza. ASCM 2009.
- Triangular decomposition of semi-algebraic systems, joint work with James H. Davenport, John P. May, M. Moreno Maza, Bican Xia and Rong Xiao. ISSAC 2010.
- Computing with Semi-Algebraic Sets Represented by Triangular Decomposition, joint work with James H. Davenport, M. Moreno Maza, Bican Xia and Rong Xiao. ISSAC 2011.
- Algorithms for Computing Triangular Decompositions of Polynomial Systems, joint work with Marc Moreno Maza. ISSAC 2011.
- Semi-algebraic description of the equilibria of dynamical systems, joint work with Marc Moreno Maza. CASC 2011.

Outline

Mad cow disease

http://x-medic.net/infections/
bovine-spongiform-encephalopathy/attachment/mad-cow-disease

A mad cow disease model (M. Laurent, 1996)

Hypothesis: the mad cow disease is spread by prion proteins.

The kinetic scheme

$$\begin{array}{l} \downarrow 1 \\ PrP^C \quad \xrightarrow{3} PrP^{S_C} \xrightarrow{4} \text{Aggregates.} \\ \downarrow 2 \end{array}$$

- PrP^C (resp. PrP^{S_C}) is the normal (resp. infectious) form of prions
- Step 1 (resp. 2) : the synthesis (resp. degradation) of native PrP^C
- Step 3 : the transformation from PrP^C to PrP^{S_C}
- Step 4 : the formation of aggregates

Question: Can a *small amount of* PrP^{S_C} *cause prion disease?*

The dynamical system governing the reaction network

- Let x and y be respectively the concentrations of PrP^C and PrP^{S_C}.
- Let ν_i be the rate of Step i for $i = 1, \ldots, 4$.
- $\nu_1 = k_1$ for some constant k_1.
- $\nu_2 = k_2 x$ and $\nu_4 = k_4 y$.
- $\nu_3 = ax\frac{(1+by^n)}{1+cy^n}$.

$$\begin{array}{l} \downarrow 1 \\ PrP^C \quad \xrightarrow{3} PrP^{S_C} \xrightarrow{4} \text{Aggregates.} \\ \downarrow 2 \end{array}$$

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & \nu_1 - \nu_2 - \nu_3 \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & \nu_3 - \nu_4 \end{array} \right. \quad (4)$$

The simplified dynamical system by experimental values

Experiments (M. Laurent 96) suggest to set $b = 2$, $c = 1/20$, $n = 4$, $a = 1/10$, $k_4 = 50$ and $k_1 = 800$. Now we have:

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & f_1 \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & f_2 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{rcl} f_1 & = & \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4} \\ f_2 & = & \frac{2(x+2xy^4-500y-25y^5)}{20+y^4} \end{array} \right. . \tag{5}$$

- x and y are unknowns and k_2 is the only parameter.
- A constant solution (x_0, y_0) of system (5) is called an equilibrium.
- (x_0, y_0) is called asymptotically stable if the solutions of system (5) starting out close to (x_0, y_0) become arbitrary close to it.
- (x_0, y_0) is called hyperbolic if all the eigenvalues of $\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} \end{pmatrix}$ have nonzero real parts at (x_0, y_0).

The polynomial system to solve (CASC 2011)

Theorem: Routh-Hurwitz criterion

A hyperbolic equilibrium (x_0, y_0) is asymptotically stable if and only if

$$\Delta_1(x_0, y_0) := -(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}) > 0 \text{ and } \Delta_2(x_0, y_0) := \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial x} > 0.$$

The semi-algebraic systems encoding the equilibria

- Let p_1 (resp. p_2) be the numerator of f_1 (resp. f_2).
- $\mathcal{S}_1 : \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0\}$ encodes the equilibria of (5).
- $\mathcal{S}_2 : \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0, \Delta_1 > 0, \Delta_2 > 0\}$ encodes the asymptotically stable hyperbolic equilibria of (5).

The corresponding constructible systems

- $\mathcal{C}_1 := \{p_1 = 0, p_2 = 0, x \neq 0, y \neq 0, k_2 \neq 0\}$ in $\mathbb{C}^3$.
- $\mathcal{C}_2 := \{p_1 = 0, p_2 = 0, x \neq 0, y \neq 0, k_2 \neq 0, \Delta_1 \neq 0, \Delta_2 \neq 0\}$ in $\mathbb{C}^3$.

Outline

1. Motivation: a biochemical network
2. A new algorithm for computing triangular decompositions
3. Triangular decomposition of semi-algebraic systems
4. A new algorithm for computing cylindrical algebraic decompositions
5. A new tool for solving parametric polynomial systems
6. Study the equilibria of dynamical systems symbolically

A brief historical review of triangular decomposition

Theory

- Characteristic set of a prime ideal (J.F. Ritt, 1930s).
- Characteristic set of a polynomial system (W.T. Wu, 1970s)
- Regular chain (M. Kalkbrener) (L. Yang and J.Z. Zhang), 1990s
- Unification of different concepts (P. Aubry, D. Lazard and M. Moreno Maza, 1999).

Algorithm

(J.F. Ritt 1930s, W.T. Wu, 1970s, M. Kalkbrener 1991, D. Lazard 1991-1992, D.M. Wang 1993-1998-2000, Triade algorithm, M. Moreno Maza 2000)

Software

Epsilon (D.M. Wang), Wsolve (D.K. Wang), RegularChains (initiated by F. Lemaire, M. Moreno Maza and Y. Xie) library in Maple.

Regular chain

Example

$$T := \begin{cases} t_2 = (x_1 + x_2){x_3}^2 + x_3 + 1 \\ t_1 = x_1{x_2}^2 - 2. \end{cases}$$

Under the order $x_3 > x_2 > x_1$,

- Main variable of t_1 and t_2: $\mathrm{mvar}(t_1) = x_2$, $\mathrm{mvar}(t_2) = x_3$
- initial of t_1 and t_2: $\mathrm{init}(t_1) = x_1$, $\mathrm{init}(t_2) = x_1 + x_2$
- $\mathrm{init}(t_2)$ is regular (not zero or zerodivisor) $\mathrm{mod}\ \langle t_1 \rangle : \mathrm{init}(t_1)^\infty$
- $\mathrm{init}(T) := \mathrm{init}(t_1)\mathrm{init}(t_2)$
- $\mathrm{sat}(T) := \langle T \rangle : \mathrm{init}(T)^\infty$
- quasi-component of T: $W(T) = V(T) \setminus V(\mathrm{init}(T))$.

Proposition

Let T be a regular chain. Then $\mathrm{sat}(T)$ is a proper equi-dimensional ideal. Moreover, $V(\mathrm{sat}(T)) = \overline{W(T)}$.

Classification of existing algorithms

By their specification

- represent only the "generic zeros" (Kalkbrener triangular decomposition)

$$V(F) = \cup_{i=1}^{e} \overline{W(T_i)}$$

- encode all the zeros (Lazard-Wu triangular decomposition)

$$V(F) = \cup_{i=1}^{e} W(T_i)$$

By their algorithmic principle

- variable elimination

$$\mathrm{Solve}_n(F \subset \mathbf{k}[x_1, \ldots, x_{n-1}, x_n]) \to \mathrm{Solve}_{n-1}(F' \subset \mathbf{k}[x_1, \ldots, x_{n-1}])$$

- equation elimination (incremental solving)

$$\mathrm{Solve}_m(\{f_1, \ldots, f_{m-1}, f_m\}) \to \mathrm{Solve}_{m-1}(\{f_1, \ldots, f_{m-1}\})$$

Incremental solving

Input system $F := \{p_1, p_2, p_3\}$. Order $z > y > x$.

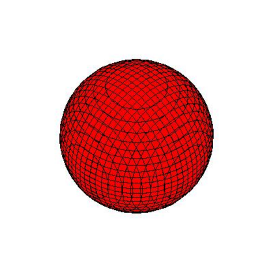
- $p_1 := x^2 + y^2 + z^2 - 4$
- $p_2 := x^2 + y^2 - z^2 - 1$
- $p_3 := z^3 + xy - 1$

$$T_1 := p_1$$

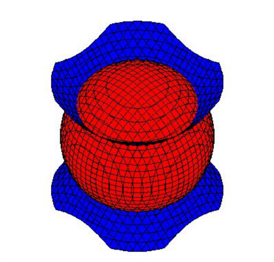
$$T_2 := \begin{cases} 2z^2 - 3 \\ 2y^2 + 2x^2 - 5 \end{cases}$$

$$T_3 := \begin{cases} 3z + 2xy - 2 \\ 16xy + 8x^4 - 20x^2 + 19 \\ 64x^8 - 320x^6 + 960x^4 - 1400x^2 + 361 \end{cases}$$

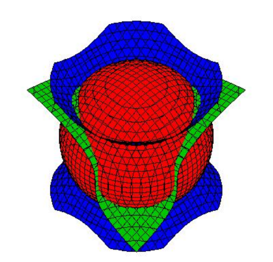
$W(T_1) := V(p_1)$



$W(T_2) := V(p_2) \cap W(T_1)$



$W(T_3) := V(p_3) \cap W(T_2)$



Intersect operation and the incremental algorithm

Intersect operation

- Let $R = \mathbf{k}[x_1 < \cdots < x_n]$.
- Let p be a polynomial and T be a regular chain in R.
- Intersect(p, T) returns regular chains $T_1, \ldots, T_e \subset R$ such that

$$V(p) \cap W(T) \subseteq W(T_1) \cup \cdots \cup W(T_e) \subseteq V(p) \cap \overline{W(T)}.$$

Triangularize(F)

- **if** $F = \{\ \}$ **then** return $\{\varnothing\}$
- Choose a polynomial $p \in F$ with maximal rank
- **for** $T \in$ Triangularize$(F \setminus \{p\}, R)$ **do**
  output Intersect(p, T)
  end

Main ideas for improving the Intersect operation

Save on the algebraic complexity

- identify operations whose specifications can be weakened or whose algorithm cost can be reduced
- redesign the algorithm to implement these modifications
- **Weakened usage of the notion of regular GCD**
- **Avoiding unnecessary regularity test**

Extract common works of computations

- Identify (potential) common expensive computations
- Compute them once and reuse the results
- **Recycling the computation of subresultant chains**

Regular GCD

Definition of regular GCD (M. Moreno Maza, 2000)

- $\mathbb{A}$: be a commutative ring with unity.
- $p, t \in \mathbb{A}[y] \setminus \mathbb{A}$.

We say that $g \in \mathbb{A}[y]$ is a *regular GCD* of p, t if:

(R_1) $\mathrm{lc}(g, y)$ is a regular element in $\mathbb{A}$;

(R_2) $g \in \langle p, t \rangle$ in $\mathbb{A}[y]$;

(R_3) if $\deg(g, y) > 0$, then $\mathrm{prem}(p, g) = \mathrm{prem}(t, g) = 0$.

Remark

- If $\mathbb{A}$ is a field, the definition coincides with the usual notion of a GCD.
- Let $R = \mathbf{k}[x_1, \ldots, x_{k-1}]$ and let T be a regular chain of R.

- In Triade algorithm and (Li-MMM-Pan, 2009), $\mathbb{A} = R/\mathrm{sat}(T)$.
- In this study, $\mathbb{A} = R/\sqrt{\mathrm{sat}(T)}$.

Regular GCD and the Intersect operation

- Let $R := \mathbf{k}[x_1, \ldots, x_{k-1}]$, where $1 \leq k \leq n$.
- Let $T \subset \mathbf{k}[x_1, \ldots, x_{k-1}]$ be a regular chain.
- Let $p, t \in R[x_k]$ be polynomials with main variable x_k.
- Assume that $T \cup t$ is also a regular chain.

Recycling Theorem

One can compute finitely many regular chains $T_1 \cup g_1, \ldots, T_e \cup g_e$ s.t.

- g_i is a regular GCD of p and t in $R[x_k]/\sqrt{\mathrm{sat}(T_i)}$, and
- $\mathsf{Intersect}(p, T \cup t) = \{T_1 \cup g_1, \ldots, T_e \cup g_e\}$.

Corollary

The regular GCDs g_i can be computed by the same subresultant chain of p and t.

Benchmark I: versus Triade algorithm implemented in Maple 13

TK16	Current Triangularize in Kalkbrener's sense	TK13	Version of TK in Maple 13
TL16	Current Triangularize in Lazard-Wu's sense.	TL13	Version of TL in Maple 13
STK16	TK16 with squarefree option.	STL16	TL16 with squarefree option

	sys	TK13	TK16	TL13	TL16	STK16	STL16
1	4corps-1parameter-homog	$>$ 1h	36.9	$>$ 1h	$>$ 1h	62.8	$>$ 1h
2	8-3-config-Li	8.7	5.9	29.7	25.8	6.0	26.6
3	Alonso-Li	0.3	0.4	14.0	2.1	0.4	2.2
4	Bezier	$>$ 1h	88.2	$>$ 1h	$>$ 1h	$>$ 1h	$>$ 1h
7	childDraw-2	$>$ 1h	$>$ 1h	$>$ 1h	$>$ 1h	1326.8	1437.1
8	Cinquin-Demongeot-3-3	3.2	0.6	$>$ 1h	7.1	0.7	8.8
9	Cinquin-Demongeot-3-4	166.1	3.1	$>$ 1h	$>$ 1h	3.3	$>$ 1h
11	f-744	$>$ 1h	12.7	$>$ 1h	14.8	12.9	15.1
12	Haas5	452.3	0.3	$>$ 1h	$>$ 1h	0.3	$>$ 1h
14	Lichtblau	0.7	0.3	801.7	143.5	0.3	531.3
16	Liu-Lorenz	0.4	0.4	4.7	2.3	0.4	4.4
17	Mehta2	$>$ 1h	2.2	$>$ 1h	4.5	2.2	6.2
18	Mehta3	$>$ 1h	14.4	$>$ 1h	51.1	14.5	63.1
19	Mehta4	$>$ 1h	859.4	$>$ 1h	1756.3	859.2	1761.8
21	p3p-isosceles	1.2	0.3	$>$ 1h	352.5	0.3	$>$ 1h
22	p3p	168.8	0.3	$>$ 1h	$>$ 1h	0.3	$>$ 1h
23	Pavelle	0.8	0.5	$>$ 1h	7.0	0.4	12.6
25	Wang93	0.5	0.7	0.6	0.8	0.8	0.9

Benchmark II: versus other solvers

TK16	Current Triangularize in Kalkbrener's sense	GL	Groebner:-Basis (plex order) in MAPLE 15
TL16	Current Triangularize in Lazard-Wu's sense	GS	Groebner:-Solve in MAPLE 15.
GD	Groebner:-Basis (tdeg order)	WS	Command wsolve of the package `Wsolve`

	sys	GL	TK16	GS	WS	TL16
1	4corps-1parameter-homog	> 1h	36.9	> 1h	> 1h	> 1h
2	8-3-config-Li	108.7	5.9	> 1h	27.8	25.8
3	Alonso-Li	3.4	0.4	> 1h	7.9	2.1
4	Bezier	> 1h	88.2	> 1h	> 1h	> 1h
7	childDraw-2	19.3	> 1h	> 1h	> 1h	> 1h
8	Cinquin-Demongeot-3-3	63.6	0.6	> 1h	> 1h	7.1
9	Cinquin-Demongeot-3-4	> 1h	3.1	> 1h	> 1h	> 1h
11	f-744	30.8	12.7	> 1h	> 1h	14.8
12	Haas5	> 1h	0.3	> 1h	> 1h	> 1h
14	Lichtblau	125.9	0.3	> 1h	> 1h	143.5
16	Liu-Lorenz	3.2	0.4	2160.1	40.2	2.3
17	Mehta2	> 1h	2.2	> 1h	5.7	4.5
18	Mehta3	> 1h	14.4	> 1h	> 1h	51.1
19	Mehta4	> 1h	859.4	> 1h	> 1h	1756.3
21	p3p-isosceles	6.2	0.3	> 1h	792.8	352.5
22	p3p	33.6	0.3	> 1h	> 1h	> 1h
23	Pavelle	1.8	0.5	> 1h	> 1h	7.0
25	Wang93	0.2	0.7	1580.0	0.8	0.8

Benchmark III: output size of different solvers

TK16	Current Triangularize in Kalkbrener's sense	GL	Groebner:-Basis (plex order) in MAPLE 15
TL16	Current Triangularize in Lazard-Wu's sense	GS	Groebner:-Solve in MAPLE 15.
GD	Groebner:-Basis (tdeg order)	WS	Command wsolve of the package Wsolve

	sys	GL	TK16	GS	GD	TL16
1	4corps-1parameter-homog	-	30738	-	21863	-
2	8-3-config-Li	67965	1384	-	72698	7538
3	Alonso-Li	1270	374	-	614	2050
4	Bezier	-	114109	-	32054	-
7	childDraw-2	938846	-	-	157765	-
8	Cinquin-Demongeot-3-3	1652062	895	-	680	2065
9	Cinquin-Demongeot-3-4	-	2322	-	690	-
11	f-744	102082	4510	-	83559	4509
12	Haas5	-	548	-	28	-
14	Lichtblau	6600095	5243	-	224647	110332
16	Liu-Lorenz	47688	938	123965	712	2339
17	Mehta2	-	5097	-	1374931	5347
18	Mehta3	-	25537	-	-	25951
19	Mehta4	-	71239	-	-	71675
21	p3p-isosceles	56701	840	-	1453	9253
22	p3p	160567	1712	-	1768	-
23	Pavelle	17990	1086	-	1552	3351
25	Wang93	2772	391	56383	1377	1016

Outline

Regular semi-algebraic system

Notation

- Let T be a regular chain of $\mathbb{Q}[\mathbf{x}]$, where $\mathbf{x} = x_1 < \cdots < x_n$
- Let $\mathbf{y} = \{\mathrm{mvar}(t) \mid t \in T\}$, called the algebraic variables of T
- Let $\mathbf{u} = \mathbf{x} \setminus \mathbf{y}$, called the free variables of T, renamed as $\mathbf{u} = u_1 < \ldots < u_d$
- Let P be a finite set of polynomials, s.t. each polynomial in P is regular modulo $\mathrm{sat}(T)$
- Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{u}]$

Definition

We say that $R := [\mathcal{Q}, T, P_>]$ is a regular semi-algebraic system if:

(i) $\mathcal{Q}$ defines a non-empty open semi-algebraic set S in $\mathbb{R}^d$,

(ii) the regular system $[T, P]$ specializes well at every point u of S,

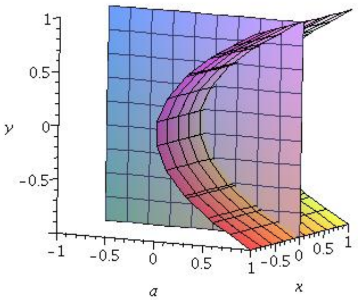
(iii) at each point u of S, the specialized system $[T(u), P(u)_>]$ has at least one real zero.

Example

The system $[\mathcal{Q}, T, P_>]$, where

$$\mathcal{Q} := a > 0,\ T := \begin{cases} y^2 - a = 0 \\ x = 0 \end{cases},\ P_> := \{y > 0\}$$

is a regular semi-algebraic system.



Triangular decompositions of semi-algebraic systems

Proposition

Let $R := [\mathcal{Q}, T, P_>]$ be a regular semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Let d be the number of free variables of T. Then the zero set of R is a nonempty semi-algebraic set of dimension d.

Theorem

Given any semi-algebraic system $\mathfrak{S}$, there exists a finite family of regular semi-algebraic systems, such that the union of their zero sets is the zero set of $\mathfrak{S}$. We call such a family of regular semi-algebraic systems a (full) triangular decomposition of $\mathfrak{S}$.

Theorem

Under generic assumptions, a lazy version of $\mathfrak{S}$ can be computed in singly exponential time w.r.t. the number of variables in $\mathfrak{S}$.

```
> with(RegularChains): R := PolynomialRing([x, y, z]):
> Limao := x^2-y^3*z^3; Zylinder := y^2+z^2-1;
```

$$Limao := x^2 - y^3 z^3$$

$$Zylinder := y^2 + z^2 - 1 \qquad (1)$$

```
> Triangularize([Zylinder, Limao], R, output=lazard): Display(%[1], R);
```

$$\begin{cases} x^2 + (z^5 - z^3)\, y = 0 \\ y^2 + z^2 - 1 = 0 \end{cases} \qquad (2)$$

```
> RealTriangularize([Zylinder, Limao], R):Display(%, R);
```

$$\left[\begin{cases} x^2 + (z^5 - z^3)\, y = 0 \\ y^2 + z^2 - 1 = 0 \\ z \neq 0 \text{ and } z < 1 \text{ and } z + 1 > 0 \end{cases}, \begin{cases} x = 0 \\ y + 1 = 0 \\ z = 0 \end{cases}, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases}, \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0 \end{cases}, \begin{cases} x = 0 \\ y = 0 \\ z + 1 = 0 \end{cases}\right] \qquad (3)$$

```
> SamplePoints([Zylinder, Limao], R, output=record);
```

$$\begin{cases} x = \left[-\frac{73}{256}, -\frac{9}{32}\right] \\ y = \left[-\frac{111}{128}, -\frac{55}{64}\right] \\ z = -\frac{1}{2} \end{cases}, \begin{cases} x = \left[\frac{9}{32}, \frac{73}{256}\right] \\ y = \left[-\frac{111}{128}, -\frac{55}{64}\right] \\ z = -\frac{1}{2} \end{cases}, \begin{cases} x = \left[-\frac{37}{128}, -\frac{9}{32}\right] \\ y = \left[\frac{443}{512}, \frac{887}{1024}\right] \\ z = \frac{1}{2} \end{cases}, \begin{cases} x = \left[\frac{9}{32}, \frac{37}{128}\right] \\ y = \left[\frac{443}{512}, \frac{887}{1024}\right] \\ z = \frac{1}{2} \end{cases}, \begin{cases} x = 0 \\ y = -1 \\ z = 0 \end{cases}, \begin{cases} x = 0 \\ y = 1 \\ z = 0 \end{cases}, \qquad (4)$$

$$\begin{cases} x = 0 \\ y = 0 \\ z = 1 \end{cases}, \begin{cases} x = 0 \\ y = 0 \\ z = -1 \end{cases}$$

```
> with(RegularChains):
> R := PolynomialRing([z, y, x]);
```

$$R := polynomial\_ring \qquad (1)$$

```
> LazyRealTriangularize([z^3-y^2, z^2-x], R, output=piecewise);
```

$$\begin{cases} [[x z - y^2 = 0, y^4 - x^3 = 0]] & 0 < x \\ \%LazyRealTriangularize([x = 0, y^4 - x^3 = 0, x z - y^2 = 0, z^2 - x = 0, z^3 - y^2 = 0], polynomial\_ring) & x = 0 \\ [\,] & otherwise \end{cases} \qquad (2)$$

Experimental results

#e	number of equations	#v	number of variables	d	total degree of equations
T	Triangularize	ST	squarefree Triangularize	Q	QEPCAD B 1.61
LR	LazyRealTriangularize	R	RealTriangularize	S	SamplePoints

system	#v/#e/d	T	ST	LR	R	S	Q
BM05-1	4/2/3	0.28	0.28	0.65	1.15	1.19	8.16
BM05-2	4/2/4	0.29	0.29	3.50	$> 1h$	$> 1h$	FAIL
Solotareff-4b	5/4/3	0.91	0.93	1.98	881.15	14.42	$> 1h$
Solotareff-4a	5/4/3	0.71	0.74	1.63	4.00	3.12	FAIL
putnam	6/4/2	0.27	0.30	0.76	1.65	1.70	$> 1h$
MPV89	6/3/4	0.23	0.29	0.89	2.75	2.42	$> 1h$
IBVP	8/5/2	0.58	0.62	1.26	14.23	13.89	$> 1h$
Lafferriere37	3/3/4	0.33	0.38	0.69	0.72	0.62	2.3
Xia	6/3/4	0.46	0.46	2.20	209.65	168.49	$> 1h$
SEIT	11/4/3	0.70	0.71	32.67	$> 1h$	1355.81	$> 1h$
p3p-isosceles	7/3/3	0.35	0.35	$> 1h$	$> 1h$	$> 1h$	$> 1h$
p3p	8/3/3	0.37	0.40	$> 1h$	$> 1h$	$> 1h$	FAIL
Ellipse	6/1/3	0.18	0.19	0.96	$> 1h$	$> 1h$	$> 1h$

Maple SamplePoints **vs** **Mathematica** SemialgebraicComponentInstances

system	Maple SP		Mathematica SP	
	time (sec)	#sample points	time (sec)	#sample points
BM05-2	0.120	10	0.040	27
IBVP	4.770	30	-	-
Jirstrand42	0.629	9	0.040	15
Lafferriere37	0.099	3	0.210	3
MPV89	0.360	3	0.060	7
p3p-isosceles	3.540	44	1.820	1876
p3p	5.019	128	6.590	8452
putnam	0.340	16	0.010	16
SEIT	6.050	1	0.000	2
Solotareff-4a	0.970	1	-	-
Xia	0.419	1	11.430	97
8-3-config-Li	311.39	81	-	-
Cheaters-homotopy-easy	0.020	1	0.000	1
Cinquin_Demongeot-3-3	0.520	24	88.330	223
collins-jsc02	0.590	19	-	-
dgp6	4.410	3	-	-
DonatiTraverso-rev	0.159	6	-	-
Hairer-2-BGK	2.689	36	-	-
hereman-8	1.140	21	-	-
Leykin-1	1.520	20	-	-
Lichtblau	0.010	1	0.010	1
L	27.850	198	-	-

Outline

Cylindrical algebraic decomposition (CAD)

Theory and algorithm

- In 1973, G.E. Collins introduced the theory of CAD together with an algorithm for computing it.
- CAD has been studied and improved by many other researchers: (D. Arnon, G.E. Collins and S. McCallum 84), (H. Hong 90), (G.E. Collins and H. Hong 91), (H. Hong 92), (S. McCallum 98), (A. Strzeboński 00), (C. Brown 01), (G.E. Collins and J. Johnson and W. Krandick 02), (A. Dolzmann, A. Seidl and T. Sturm 04), (C. Brown and J. Davenport 07) and others.

Software

QEPCAD (H. Hong, C. Brown et al.), Mathematica CAD (A. Strzeboński), Redlog (A. Dolzmann and T. Sturm) and SNCAD (H. Iwane, H. Yanami, H. Anai and K. Yokoyama).

Cylindrical algebraic decomposition

A cylindrical algebraic decomposition of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$, where

- all the cells in the partition are cylindrically arranged, that is for all $1 \leq j < n$ the projections on the first j coordinates $(y_1, \ldots, y_j)$ of any two cells are either identical or disjoint.
- each cell is a connected semi-algebraic subset of $\mathbb{R}^n$.

Let $F \subset \mathbb{Q}[y_1, \ldots, y_n]$. A CAD of $\mathbb{R}^n$ is F-invariant if above each cell of it, the sign of each polynomial in F is constant.

Scheme proposed by Collins for computing CAD

- Projection. Starting from the input $F_n \subset \mathbb{Q}[y_1, \ldots, y_n]$, repeatedly apply a projection operator to eliminate the variables one by one until a set of univariate polynomials are obtained

$$F_n \to F_{n-1} \to \cdots \to F_1.$$

- Lifting.
  - The real roots of polynomial in F_1 plus the open intervals between them form an F_1-invariant CAD of $\mathbb{R}^1$.
  - For each cell C of the F_{k-1} invariant CAD of $\mathbb{R}^{k-1}$, isolate the polynomials of F_k at a sample point of C, which produces all the cells of the F_k-invariant CAD of $\mathbb{R}^k$ above C.

Main idea of our algorithm.

F: a set of polynomials of $\mathbb{Q}[y_1, \ldots, y_n]$.

The whole algorithm consists of three main steps

- Initial Partition: we decompose $\mathbb{C}^n$ into disjoint constructible sets $C_1, \ldots, C_e$ such that each $f \in F$ is identically zero in C_i or vanishes at no points of C_i.
- Make Cylindrical: we transform the initial partition into a cylindrically arranged one in $\mathbb{C}^n$.

  We call the output of the above two steps an F-invariant Cylindrical decomposition of $\mathbb{C}^n$.

- Make Semi-Algebraic: for each cell C in the above decomposition, we decompose $C \cap \mathbb{R}^n$ into connected cylindrically arranged semi-algebraic subsets of $\mathbb{R}^n$.

Timing and number of cells for TCAD and QEPCAD B

	Sys	TCAD		QEPCAD B	
1	Parabola	0.144	**27**	0.02	**115**
2	Whitney-umbrella	5.088	895	0.048	895
3	Quartic	8.220	233	0.052	223 (with error)
4	Sphere-catastrophe	2.716	**421**	0.048	**509**
5	Arnon-84	0.184	55	0.024	55
6	Arnon-84-2	0.384	41	0.02	41
7	Real-implicitization	7.664	893	0.052	889 (with warning)
8	Ball-cylindar	3.184	365	0.068	365
9	Termination-term-rewrite	1.084	209	0.02	207
10	Collins-Johnson	72.640	3677	0.32	3673
11	Range-lower-bounds	2.068	**563**	0.184	**4199** (with warning)
12	X-axis-ellipse	225.862	**20143**	3.156	**64625** (with warning)
13	Davenport-Heintz	53.335	4949	0.148	4949
14	Hong-90	2597.878	**27547**	13.852	**79289** (with warning)
15	Solotareff-3	3503.954	66675	4.188	66675
16	Collision	-	-	2.076	45979
17	McCallum-random	-	-	21.797	877
18	Ellipse-cad	-	-	-	-

Experimentation by research group of University of Bath

Table: Extract from Table 8-11 of Nalina Phisanbut's PhD thesis.

Rank according to QEPCAD cell count			
Variable order	QEPCAD cells	TCAD cells	ratio
$v > y > u > x$ (Good)	785	673	1.16
$y > v > x > u$ (Good)	785	673	1.16
$y > v > u > x$ (Good)	901	557	1.62
$v > y > x > u$ (Good)	901	557	1.62
$y > x > u > v$	2049	989	2.07
$y > u > x > v$	2049	1869	1.10
$u > x > v > y$	5985	557	10.74
$x > u > y > v$	5985	557	10.74
$x > u > v > y$	6597	673	9.80
$u > x > y > v$	6597	673	9.80
$u > v > y > x$	9101	989	9.20
$x > y > v > u$	9101	989	9.20
$u > y > v > x$	28821	1781	16.18
$x > v > y > u$	28821	1781	16.18
$u > v > x > y$	37957	989	36.38
$x > y > u > v$	37957	989	36.38
$x > v > u > y$	92829	1781	52.12
$u > y > x > v$	92829	1781	52.12

(G) = recommended by A. Dolzmann et al.'s Greedy Algorithm

Outline

Objectives

For a parametric polynomial system $F \subset \mathbf{k}[\mathbf{u}][\mathbf{x}]$, the following problems are of interest:

1. compute the values u of the parameters for which $F(u)$ has solutions, or has finitely many solutions.
2. compute the solutions of F as continuous functions of the parameters.
3. provide an automatic case analysis for the number (dimension) of solutions depending on the parameter values.

Related work

- (Comprehensive) Gröbner bases: (V. Weispfenning, 92, 02), (D. Kapur 93), (A. Montes, 02), (M. Manubens & A. Montes, 02), (A. Suzuki & Y. Sato, 03, 06), (D. Lazard & F. Rouillier, 07), (Y. Sun, D. Kapur & D. Wang, 10) and others.
- Triangular decompositions: (S.C. Chou & X.S. Gao 92), (X.S. Gao & D.K. Wang 03), (D. Kapur 93), (D.M. Wang 05), (L. Yang, X.R. Hou & B.C. Xia, 01), (R. Xiao, 09) and others.
- Cylindrical algebraic decompositions: (G.E. Collins 75), (H. Hong 90), (G.E. Collins, H. Hong 91), (S. McCallum 98), (A. Strzeboński 00), (C.W. Brown 01) and others.

Main Results

- We investigated the specialization property of regular chains
- We introduced the concept of comprehensive triangular decomposition (CTD) of an algebraic variety and extend it to:
- CTD of a parametric constructible set with application to complex root classification
- CTD of a parametric semi-algebraic system with application to real root classification

Specialization

Definition

A (squarefree) regular chain T of $\mathbf{k}[\mathbf{u}, \mathbf{y}]$ specializes well at $u \in \mathbf{K}^d$ if $T(u)$ is a (squarefree) regular chain of $\mathbf{K}[\mathbf{y}]$ and $\mathrm{init}(T)(u) \neq 0$.

Example

$$T = \begin{cases} (s+1)z \\ (x+1)y + s \\ x^2 + x + s \end{cases}$$

does not specialize well at $s = 0$ or $s = -1$

$$T(0) = \begin{cases} z \\ (x+1)y \\ (x+1)x \end{cases} \quad T(1) = \begin{cases} 0z \\ (x+1)y - 1 \\ x^2 + x - 1 \end{cases}$$

Comprehensive Triangular Decomposition (CTD)

Let $F \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$. A CTD of $V(F)$ is given by :

- a finite partition $\mathcal{C}$ of the parameter space,
- above each $C \in \mathcal{C}$, there is a set of regular chains $\mathcal{T}_C$ such that
  - each regular chain $T \in \mathcal{T}_C$ specializes well at any $u \in C$ and
  - for any $u \in C$, $V(F(u)) = \bigcup_{T \in \mathcal{T}_C} W(T(u))$.

Example

A CTD of $F := \{x^2(1+y) - s, y^2(1+x) - s\}$ is as follows:

1. $s \neq 0 \longrightarrow \{T_1, T_2\}$
2. $s = 0 \longrightarrow \{T_2, T_3\}$

where

$$T_1 = \begin{cases} x^2y + x^2 - s \\ x^3 + x^2 - s \end{cases} \quad T_2 = \begin{cases} (x+1)y + x \\ x^2 - sx - s \end{cases} \quad T_3 = \begin{cases} y + 1 \\ x + 1 \\ s \end{cases}$$

Disjoint squarefree comprehensive triangular decomposition (DSCTD)

Let $F \subset \mathbf{k}[\mathbf{u}, \mathbf{y}]$. A DSCTD of $V(F)$ is given by :

- a finite partition $\mathcal{C}$ of the parameter space,
- each cell $C \in \mathcal{C}$ is associated with a set of squarefree regular chains $\mathcal{T}_C$ such that
  - each squarefree regular chain $T \in \mathcal{T}_C$ specializes well at any $u \in C$ and
  - for any $u \in C$, $V(F(u)) = \dot{\cup}_{T \in \mathcal{T}_C} W(T(u))$. ($\dot{\cup}$ denotes disjoint union)

Example

1. $s \neq 0$, $s \neq 4/27$ and $s \neq -4 \longrightarrow \{T_1, T_2\}$
2. $s = -4 \longrightarrow \{T_1\}$
3. $s = 0 \longrightarrow \{T_3, T_4\}$
4. $s = 4/27 \longrightarrow \{T_2, T_5, T_6\}$

$$T_4 = \begin{cases} y \\ x \\ s \end{cases} \quad T_5 = \begin{cases} 3y - 1 \\ 3x - 1 \\ 27s - 4 \end{cases} \quad T_6 = \begin{cases} 3y + 2 \\ 3x + 2 \\ 27s - 4 \end{cases}$$

Properties of CTD

Properties of CTD

Above each cell,

- either there are either no solutions
- or finitely many solutions and the solutions are continuous functions of parameters
- or infinitely many solutions, but the dimension of the solution set is invariant

Extra properties of DSCTD

Above each cell, where the system has finitely many solutions

- the graphs of functions are disjoint
- the number of distinct complex solutions is constant

Comprehensive triangular decomposition of semi-algebraic systems (RCTD)

Encouraged by

- Cylindrical algebraic decomposition (CAD by G.E. Collins 75)
- Border polynomial (BP by L. Yang, X.R. Hou & B.C. Xia, 01)
- Discriminant variety (DV by D. Lazard & F. Rouillier, 07)

and motivated by applications such as studying of the equilibria of dynamical systems.

- Input: a parametric semi-algebraic system S
- Output: a partition of the whole parameter space such that above each cell
  - either the corresponding constructible system of S has infinitely many complex solutions,
  - or S has no real solutions
  - or S has finitely many real solutions which are continuous functions of parameters with disjoint graphs

  and a description of the solutions of S as functions of parameters by triangular systems in case of finitely many real solutions.

How to compute a RCTD?

Specifications

- Input: a parametric semi-algebraic system S
- Output: a RCTD of S

Algorithm

For simplicity, we assume S consists of only equations.

(1) Compute a DSCTD $(\mathcal{C}, (\mathcal{T}_C, C \in \mathcal{C}))$ of S.

(2) Refine each constructible set cell $C \in \mathcal{C}$ into connected semi-algebraic sets by CAD.

(3) Let C be a connected cell above which S has finitely many complex solutions. Compute the number of real solutions of $T \in \mathcal{T}_C$ at a sample point u of C and remove those Ts which have no real solutions at u.

Outline

Equilibria of mad cow disease model

Recall the dynamical system

$$\left\{\begin{array}{lcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & f_1 \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & f_2 \end{array}\right. \quad \text{with} \quad \left\{\begin{array}{lcl} f_1 & = & \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4} \\ f_2 & = & \frac{2(x+2xy^4-500y-25y^5)}{20+y^4} \end{array}\right. .$$

Let p_1 (resp. p_2) be the numerator of f_1 (resp. f_2).

$$\begin{array}{lcl} p_1 & := & (-20k_2 - k_2y^4 - 2 - 4y^4)x + 16000 + 800y^4 \\ p_2 & := & (2y^4+1)x - 500y - 25y^5 \end{array}$$

The system $\mathcal{S}_1 : \{p_1 = p_2 = 0, x > 0, y > 0, k_2 > 0\}$ encode the equilibria.

RCTD of $\mathcal{S}_1$

Let $0 < \alpha_1 < \alpha_2$ be the two positive real roots of the following polynomial

$$\begin{aligned} r &:= 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 \\ &- 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056. \end{aligned}$$

The isolating intervals for α_1 and α_2 are respectively $[3.175933838, 3.175941467]$ and $[14.49724579, 14.49725342]$.
A RCTD of S_1 is as follows.

$$\begin{cases} \{\ \} & k_2 \leq 0 \\ \{B_1\} & 0 < k_2 < \alpha_1 \\ \{B_2\} & k_2 = \alpha_1 \\ \{B_1\} & \alpha_1 < k_2 < \alpha_2 \\ \{B_2\} & k_2 = \alpha_2 \\ \{B_1\} & k_2 > \alpha_2 \end{cases} \qquad \begin{cases} 0 & k_2 \leq 0 \\ 1 & 0 < k_2 < \alpha_1 \\ 2 & k_2 = \alpha_1 \\ 3 & \alpha_1 < k_2 < \alpha_2 \\ 2 & k_2 = \alpha_2 \\ 1 & k_2 > \alpha_2 \end{cases}$$

Theorem

If $0 < k_2 < \alpha_1$ or $k_2 > \alpha_2$, then the dynamical system has 1 equilibrium;
if $k_2 = \alpha_1$ or $k_2 = \alpha_2$, then the dynamical system has 2 equilibria;
if $\alpha_1 < k_2 < \alpha_2$, then dynamical system has 3 equilibria.

Stability and bifurcation analysis of mad cow disease model

Combining several RCTDs

- RCTD($\mathcal{S}_1$) : equilibria.
- RCTD($\mathcal{S}_1, \Delta_1 = \Delta_2 = 0$), RCTD($\mathcal{S}_1, \Delta_1 \neq 0, \Delta_2 = 0$), and RCTD($\mathcal{S}_1, \Delta_1 = 0, \Delta_2 > 0$): nonhyperbolic equilibria.
- RCTD($\mathcal{S}_1, \Delta_1 > 0, \Delta_2 > 0$) : asymptotically stable hyperbolic equilibria.

Theorem

- $0 < k_2 < \alpha_1$ or $k_2 > \alpha_2 \longrightarrow$ the system has 1 equilibrium, which is hyperbolic and asymptotically stable
- $k_2 = \alpha_1$ or $k_2 = \alpha_2 \longrightarrow$ the system has 2 equilibria, one is nonhyperbolic, another one is hyperbolic and asymptotically stable
- $\alpha_1 < k_2 < \alpha_2 \longrightarrow$ the system has 3 equilibria, two are hyperbolic and asymptotically stable, one is hyperbolic and nonstable.
- the system experiences a bifurcation at $k_2 = \alpha_1$ or $k_2 = \alpha_2$

Can a small amount of PrP^{Sc} cause prion disease? (I)

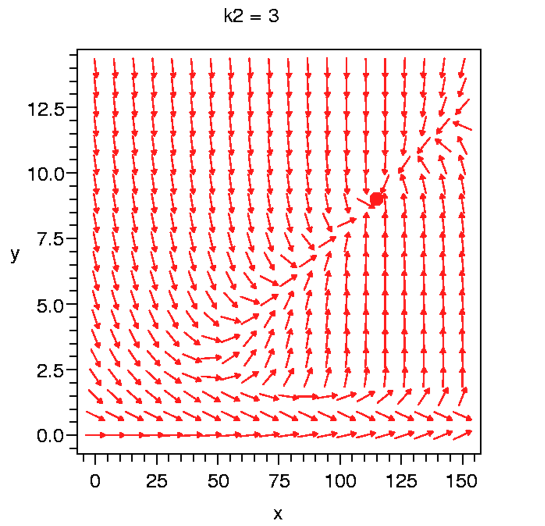


Figure: Vector field for $k_2 = 3$ ($x : PrP^C$, $y : PrP^{Sc}$)

Can a small amount of PrP^{Sc} cause prion disease? (II)

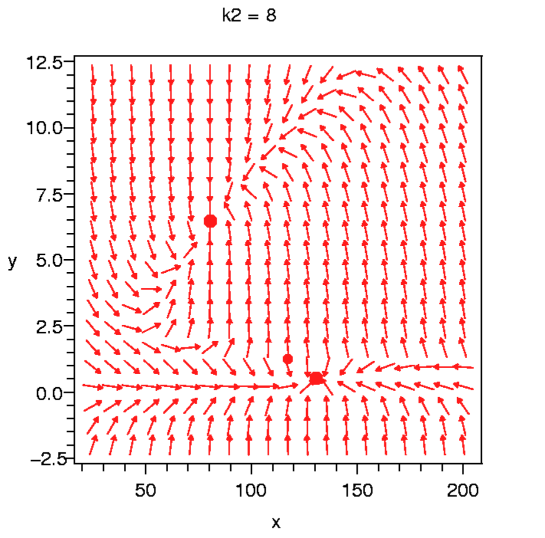


Figure: Vector field for $k_2 = 8$ ($x : PrP^C$, $y : PrP^{Sc}$)

Can a small amount of PrP^{Sc} cause prion disease? (III)

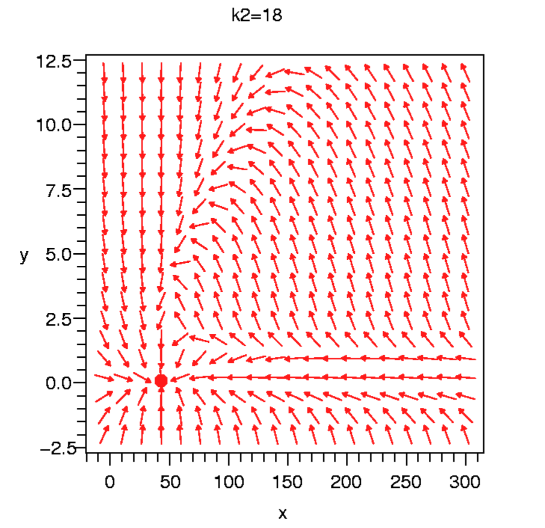


Figure: Vector field for $k_2 = 18$ ($x : PrP^C$, $y : PrP^{Sc}$)

Conclusion

1. We propose a new and much more efficient algorithm for computing triangular decompositions (TD).
2. We present a brand-new algorithm for computing cylindrical algebraic decompositions (CAD).
3. We adapt the concept of triangular decomposition to the real space (RTD) and obtain an efficient algorithm to solve arbitrary semi-algebraic systems. In many cases, it can play the role of CAD. Under generic assumptions, computing its lazy version runs in singly exponential time.
4. We introduce a new tool, called comprehensive triangular decomposition (CTD), to solve parametric polynomial systems, both over the complex numbers and over the reals.

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