Solving Parametric Polynomial Systems with the RegularChains Library in Maple

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What does “solving parametric systems” mean?

For a parametric polynomial system $F \subset k[u][x]$, the following problems are of interest:

1. compute the values $u$ of the parameters for which $F(u)$ has solutions, or has finitely many solutions.
2. compute the solutions of $F$ as continuous functions of the parameters.
3. provide an automatic case analysis for the number (dimension) of solutions depending on the parameter values.
Related work ($C^3$)

- (Comprehensive) Gröbner bases (CGB): (V. Weispfenning, 92, 02), (D. Kapur 93), (A. Montes, 02), (A. Suzuki & Y. Sato, 03, 06), (D. Lazard & F. Rouillier, 07), (Guillaume Moroz, 06) and others.

- Cylindrical algebraic decompositions (CAD): (G.E. Collins 75), (G.E. Collins, H. Hong 91), (H. Hong 92), (S. McCallum 98), (A. Strzeboński 00), (C.W. Brown 01) and others.

Outline

- comprehensive triangular decomposition of parametric constructible sets
- complex root classification
- real root classification
- border polynomial and related notions
- Maple demo
Triangular decompositions of a constructible set

A pair \( R = [T, h] \) is called a regular system if \( T \) is a regular chain, and \( h \) is a polynomial which is regular w.r.t \( \text{sat}(T) \).

**Theorem (CGLMP, CASC2007)**

*Every constructible set can be written as a finite union of the zero sets of regular systems.*

The constructible set

\[
\begin{align*}
  x(1 + y) - s &= 0 \\
y(1 + x) - s &= 0 \\
x + y - 1 &\neq 0
\end{align*}
\]

(1)

can be represented by two regular systems

\[
R_1 : \begin{cases} 
  T_1 = \begin{cases} 
    (y + 1)x - s \\
y^2 + y - s
  \end{cases} \\
h_1 = y - 2s + 1
\end{cases} \quad R_2 : \begin{cases} 
  T_2 = \begin{cases} 
    x + 1 \\
y + 1
  \end{cases} \\
h_2 = 1
\end{cases}
\]
Specialization

Definition
A regular system $R := [T, h]$ specializes well at $u \in \mathbb{K}^d$ if $[T(u), h(u)]$ is a regular system of $\mathbb{K}[x]$ after specialization and no initials of polynomials in $T$ vanish during the specialization.

Example

\[
R_1 : \begin{cases}
T_1 = \begin{cases}
(y + 1)x - s \\
y^2 + y - s
\end{cases} \\
h_1 = y - 2s + 1
\end{cases}
\]

does not specializes well at $s = 0$ or at $s = \frac{3}{4}

\[
R_1(0) : \begin{cases}
T_1(0) = \begin{cases}
(y + 1)x \\
(y + 1)y
\end{cases} \\
h_1(0) = y + 1
\end{cases}
\]

\[
R_1\left(\frac{3}{4}\right) : \begin{cases}
T_1\left(\frac{3}{4}\right) = \begin{cases}
(y + 1)x - \frac{3}{4} \\
(y - \frac{1}{2})(y + \frac{3}{2})
\end{cases} \\
h_1\left(\frac{3}{4}\right) = y - \frac{1}{2}
\end{cases}
\]
Comprehensive Triangular Decomposition (CTD)

Definition
Let $CS$ be a constructible set of $k[u, x]$. A comprehensive triangular decomposition of $CS$ is given by:

1. a finite partition $C$ of the parameter space $K^d$,
2. for each $C \in C$ a set of regular systems $R_C$ s.t. for $u \in C$
   2.1 each of the regular systems $R \in R_C$ specializes well at $u$
   2.2 and we have
   $$CS(u) = \bigcup_{R \in R_C} Z(R(u)).$$

Let $F := \{x(1 + y) - s = 0, y(1 + x) - s = 0, x + y - s \neq 0\}$.

A CTD of $F$ is as follows:

1. $s \neq 0$ and $s \neq \frac{3}{4} \rightarrow \{R_1\}$
   \[R_1: \begin{align*} T_1 &= \begin{cases} (y + 1)x - s \\ y^2 + y - s \end{cases} \\ h_1 &= y - 2s + 1 \end{align*}\]
2. $s = 0 \rightarrow \{R_2, R_3\}$
   \[R_2: \begin{align*} T_2 &= \begin{cases} x + 1 \\ y + 1 \end{cases} \\ h_2 &= 1 \end{align*}\]
   \[R_3: \begin{align*} T_3 &= \begin{cases} x \\ y \end{cases} \\ h_3 &= 1 \end{align*}\]
3. $s = \frac{3}{4} \rightarrow \{R_4\}$
   \[R_4: \begin{align*} T_4 &= \begin{cases} 2x + 3 \\ 2y + 3 \\ 4s - 3 \end{cases} \\ h_4 &= 1 \end{align*}\]
Separation

Definition
A squarefree regular system $R := [T, h]$ separates well at $u \in K^d$ if: $R$ specializes well at $u$ and $R(u)$ is a squarefree regular system of $K[x]$.

\[
R_1 : 
\begin{align*}
  T_1 &= \begin{cases} 
    (y + 1)x - s \\
    y^2 + y - s 
  \end{cases} \\
  h_1 &= y - 2s + 1
\end{align*}
\]

specializes well but not separates well at $s = -\frac{1}{4}$:

\[
R_1(-\frac{1}{4}) : 
\begin{align*}
  T_1(-\frac{1}{4}) &= \begin{cases} 
    (y + 1)x + \frac{1}{4} \\
    (y + \frac{1}{2})^2 
  \end{cases} \\
  h_1(-\frac{1}{4}) &= y + \frac{3}{2}
\end{align*}
\]

where the second polynomial of $T_1(-\frac{1}{4})$ is not squarefree.
Disjoint Squarefree Comprehensive Triangular Decomposition (DSCTD)

Definition
A disjoint squarefree comprehensive triangular decomposition of a constructible set \( CS \) is given by:

1. a finite partition \( \mathcal{C} \) of the parameter space \( \mathbb{K}^d \),
2. for each \( C \in \mathcal{C} \) a set of squarefree regular systems \( \mathcal{R}_C \) such that for each \( u \in C \):
   2.1 each \( R \in \mathcal{R}_C \) separates well at \( u \),
   2.2 the zero sets \( Z(R(u)) \), for \( R \in \mathcal{R}_C \), are pairwise disjoint and

\[
CS(u) = \bigcup_{R \in \mathcal{R}_C} Z(R(u)).
\]
DSCTD and complex root counting

Let \( F := \{ x(1+y) - s = 0, y(1+x) - s = 0, x + y - s \neq 0 \} \). A DSCTD of \( F \) is as follows:

1. \( s \neq 0, s \neq -\frac{1}{4}, s \neq \frac{3}{4} \rightarrow \{ R_1 \} \)
2. \( s = 0 \rightarrow \{ R_2, R_3 \} \)
3. \( s = \frac{3}{4} \rightarrow \{ R_4 \} \)
4. \( s = -\frac{1}{4} \rightarrow \{ R_5 \} \)

where

\[
R_1: \quad T_1 = \begin{cases} 
(y + 1)x - s & \\
y^2 + y - s & \\
h_1 = y - 2s + 1
\end{cases} \quad R_2: \quad T_2 = \begin{cases} 
x + 1 & \\
y + 1 & \\
h_2 = 1
\end{cases} \quad R_3: \quad T_3 = \begin{cases} 
x & \\
y & \\
h_3 = 1
\end{cases} \\
R_4: \quad T_4 = \begin{cases} 
2x + 3 & \\
2y + 3 & \\
h_4 = 1
\end{cases} \quad R_5: \quad T_5 = \begin{cases} 
2x + 1 & \\
2y + 1 & \\
h_5 = 1
\end{cases}
\]

Therefore, we conclude that: if \((s + \frac{1}{4})(s - \frac{3}{4}) = 0\), system (1) has 1 complex root; otherwise system (1) has 2 complex roots.
Real root classification of semi-algebraic system

A parametric semi-algebraic system (SAS) is of the following form:

\[
\begin{align*}
\begin{cases}
p_1(u, x) = 0, & \ldots, p_r(u, x) = 0, \\
g_1(u, x) \geq 0, & \ldots, g_k(u, x) \geq 0, \\
g_{k+1}(u, x) > 0, & \ldots, g_t(u, x) > 0, \\
h_1(u, x) \neq 0, & \ldots, h_s(u, x) \neq 0,
\end{cases}
\end{align*}
\]

(2)

Here all polynomials are with rational number coefficients. The system is denoted by a quadruple \([F, N, P, H]\), where

- \(F = [p_1, \ldots, p_r]\)
- \(N = [g_1, \ldots, g_k]\), \(P = [g_{k+1}, \ldots, g_t]\), \(H = [h_1, \ldots, h_s]\)

For an integer \(n\), the problem of real root classification is to provide conditions on parameters s.t. the system has exactly \(n\) distinct real solutions.
Delayed computation and border polynomial

In (L. Yang, X.R. Hou & B.C. Xia, 01), the author proposed the idea of delayed computation for solving semi-algebraic systems:

- Decompose original system into a family of triangular systems
- Solve each sub-system forgetting degenerate parametric values
- Wrap all the degenerate values as a border polynomial
- Add the border polynomial into the input system and start over

Definition

Let $S(u, x)$ be a parametric semi-algebraic system. A polynomial $R(u)$ is called a border polynomial of $S$ if

(a) $S(u)$ has only finitely many real solutions when $R(u) \neq 0$

(b) the number of distinct real solutions of $S$ is constant in each connected component of $R(u) \neq 0$ in $\mathbb{R}^d$
Output of RealRootClassification

The output of RealRootClassification \([F, N, P, H, n]\) is a pair 
\([\Phi(u), bp(u)]\) interpreted by the following theorem.

**Theorem**

*Provided the border polynomial \(bp(u)\) does not vanish, the system has \(n\) distinct real solutions if and only if a logic formula \(\Phi(u)\) holds.*

Here the logic formula \(\Phi(u)\) can be

- a quantifier free formula, like

\[
(a > 0 \land a + b < 0) \lor (a - b^2 < 0)
\]

- a quantifier free formula plus a description of which root of a regular chain, like \(a > 0\) and \((b, c)\) is the first root of the regular chain \([c - b, b^2 - 1]\).

Such a description can be encoded as a regular semi-algebraic set.
Regular semi-algebraic set

Let \([Q, T, L]\) be a triple of \(\mathbb{Q}[w, y]\) where

- \(Q(w)\) is a quantifier-free formula defining a nonempty set \(S\),
- \(T = \{t_1, \ldots, t_m\}\) is a squarefree regular chain, with main variables \(y = y_1, \ldots, y_m\) and free variables \(w = w_1, \ldots, w_s\)
- for each point \(\alpha\) of \(S\), \(T\) separates well at \(\alpha\) and \(T(\alpha)\) has at least one real solution,
- \(L\) is a list of root indices of \(T\)

The zero set of \([Q, T, L]\) is defined as the set of \((\alpha, \beta) \in \mathbb{R}^{s+m}\) s.t.

- \(\alpha\) satisfies \(Q\) and \(\beta = (\beta_1, \ldots, \beta_m)\) is a real solution of \(T(\alpha, y)\)
- if \(L\) is empty, then \(\beta\) is any real solution of \(T(\alpha, y)\); otherwise there exists an \(L_j\) in \(L\) s.t. each \(\beta_i\) is the \(L_j, i\)-th real solution of \(t_i(\alpha, \beta_1, \ldots, \beta_{i-1}, y_i)\) w.r.t \(y_i\).

The zero set of \([Q, T, L]\) as defined above is called a regular semi-algebraic set.
Theorem
A regular semi-algebraic set is not empty and every semi-algebraic set can be decomposed as a finite union of regular semi-algebraic sets.

Algorithm: RealRootClassification

Input: A parametric semi-algebraic system $S$ and a solution number to query

Output: Necessary and sufficient conditions on the parameters for the system to have a given number of solutions provided its border polynomial does not vanish. The conditions are encoded by a list of regular semi-algebraic sets.
Example

Given a system

\[
F := \begin{cases}
  x(y + 1) - s &= 0 \\
  y(x + 1) - s &= 0 \\
  x + y - 1 &> 0
\end{cases}
\]

By RealRootClassification, we get the following theorem

**Theorem**

*Provided the border polynomial \( bp = s(4s + 1)(4s - 3) \neq 0 \), the system \( F \) has real solutions if and only if \( 4s - 3 > 0 \).*

Adding \( bp = 0 \) into \( F \), one finds that the system has no real solutions. So the final conclusion is that

**Corollary**

*The system \( F \) has real solutions if and only if \( 4s - 3 > 0 \).*
Let \( R = [T, h] \) be a squarefree regular system of \( \mathbb{Q}[u, x] \), where \( x = \text{mvar}(T) \). Denote

\[
bp := \text{sqrres}(\text{sep}(T)h, T).
\]

**Theorem**

Considering the real solutions of \( R \), then the polynomial \( bp \) is a border polynomial of the semi-algebraic system \( R \).

**Theorem**

Considering the complex solutions of \( R \), then the variety of \( bp \) is the minimal discriminant variety of the constructible set \( Z(R) \).
Border polynomial and related notions (II)

Theorem
Let $CS$ be a parametric constructible set of $\mathbb{Q}[u,x]$. Let $d$ be the number of parameters. Assume that for almost all parameter values $u$, $CS(u)$ has finitely many complex solutions. Then one could compute a disjoint squarefree CTD of $CS$ s.t.

- there exists one and only one cell $C$, whose complement in $\mathbb{C}^d$ is a hypersurface
- the hypersurface is a discriminant variety of $CS$
- let $bp(u)$ be the squarefree polynomial defining the hypersurface, then $bp$ is a border polynomial of the semi-algebraic set $CS \cap \mathbb{R}^d$