Computing Cylindrical Algebraic Decomposition via Triangular Decomposition

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Overview

Cylindrical algebraic decomposition (CAD) is a fundamental tool in real algebraic geometry. For $F_n \subset \mathbb{R}[y_1, \ldots, y_n]$ an F_n-invariant CAD of $\mathbb{R}^n$ is a partition $C_1, \ldots, C_e$ of $\mathbb{R}^n$ together with one *sample point* $S_i \in C_i$, for all $1 \leq i \leq e$, such that the sign of each $f \in F_n$ does not change in C_i and can be determined at S_i.

The original algorithm of Collins and its subsequent ameliorations are based on a projection and lifting scheme which computes from F_n a set $F_{n-1} \subset \mathbb{R}[y_1, \ldots, y_{n-1}]$ such that an F_n-invariant CAD of $\mathbb{R}^n$ results from an F_{n-1}-invariant CAD of $\mathbb{R}^{n-1}$. This construction and the case $n = 1$ rely on real root isolation of univariate polynomials.

We propose a different approach which proceeds by transforming successive partitions of the complex n-dimensional space. Our future goal is to investigate whether fast polynomial arithmetic and modular methods available for triangular decomposition could improve the practical efficiency of CAD implementation.

Our Method

Initial Partition: we decompose $\mathbb{C}^n$ into disjoint constructible sets $C_1, \ldots, C_e$ such that for all $1 \leq i \leq e$, for each $f \in F_n$ either f is identically zero in C_i or f vanishes at no points of C_i.

Make Cylindrical: we transform the initial partition and obtain another partition of $\mathbb{C}^n$ into disjoint constructible sets such that this second decomposition is cylindrical in the following sense: for all $1 \leq j < n$ the projections on the first j coordinates $(y_1, \ldots, y_j)$ of any two constructible sets are either identical or disjoint.

Make Semi-Algebraic: from the previous decomposition we produce an F_n-invariant CAD of $\mathbb{R}^n$.

Zero Separation

We describe our core routine: SeparateZeros. Let $rs = [T, h]$ be a regular system of $\mathbf{k}[y_1 < \cdots < y_n]$. We regard rs as a parametric system in $\mathbf{u} = y_1, \ldots, y_{n-1}$, that we solve via comprehensive triangular decomposition. As a result, we obtain a partition of the projection onto the $\mathbf{u}$-space of zero-set $Z(rs) := W(T) \setminus V(h)$ such that, above each cell R of the partition, $Z(rs)$ equals the union of the zero sets of polynomials $p_1, \ldots, p_r \in \mathbb{R}[y_1, \ldots, y_n]$, where

- the initial of each p_j does not vanish on R,
- the p_j's are squarefree and pairwise coprime at any point of R.

Make Cylindrical

We apply SeparateZeros to the regular systems in the output of InitialPartition recursively. Finally we produce a cylindrical decomposition of $\mathbb{C}^n$, that is, a finite partition of $\mathbb{C}^n$ into constructible sets, called cells and satisfying the following.

- $n = 1$. A cylindrical decomposition of $\mathbb{C}$ is either $\mathbb{C}$ itself or of the form $p_1 = 0, \ldots, p_r = 0, p_1 \cdots p_r \neq 0$ where $p_1, \ldots, p_r$ are nonconstant squarefree and pairwise coprime polynomials of $\mathbf{k}[y_1]$.
- $n > 1$. From a cylindrical decomposition $\mathcal{D}'$ of $\mathbb{C}^{n-1}$, one builds a cylindrical decomposition $\mathcal{D}$ of $\mathbb{C}^n$ as follows. For each cell D_i of $\mathcal{D}'$:

(1) either $D_i \times \mathbb{C}$ is an element of $\mathcal{D}$, or

(2) there exist polynomials $p_{i,1}, \ldots, p_{i,r_i} \in \mathbb{R}[y_1, \ldots, y_n]$ such that

(*a*) the initial of each p_j does not vanish on D_i and,

(*b*) the p_j's are squarefree and pairwise coprime at all $\mathbf{u} \in D_i$,

(*c*) $D_i \times p_1 = 0, \cdots, D_i \times p_r = 0$ and $D_i \times (p_1 \cdots p_r \neq 0)$ are in $\mathcal{D}$.

Make SemiAlgebraic

Let $p \in \mathbb{R}[\mathbf{u}][y_n]$ be nonconstant and R be a region of $\mathbb{R}^{n-1}$. If $\mathrm{lc}(p, y_n)$ does not vanish on R and the number of distinct complex roots of p is invariant on R, then p is delineable on R, that is, $V(p)$ is the union of finitely many disjoint graphs of continuous functions over R.

From this theorem of Collins, we derive derive a CAD of $\mathbb{R}^n$ from a cylindrical decomposition of $\mathbb{C}^n$, by means of real root isolation of zero-dimensional regular chains.

Example

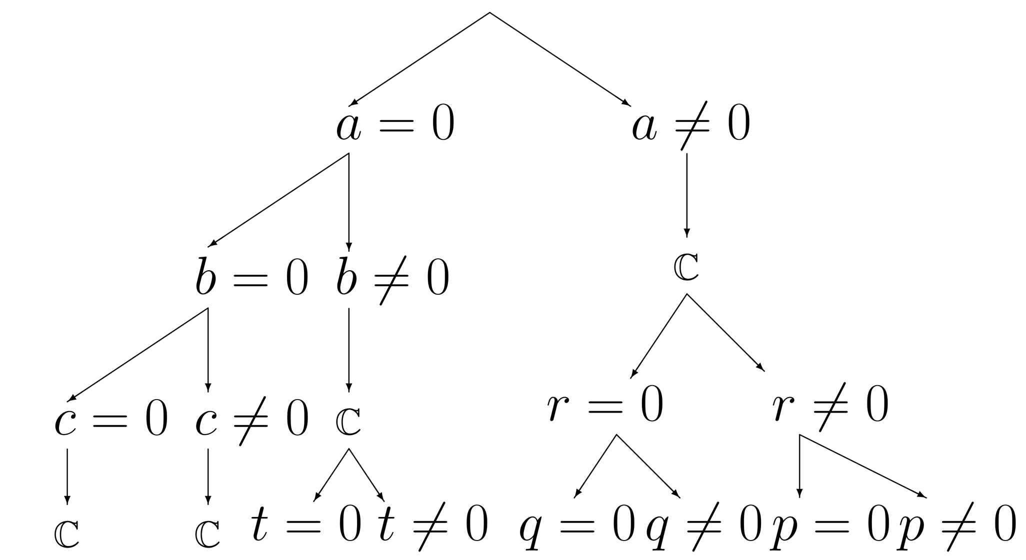
Consider the polynomial $p = ax^2 + bx + c$, where $x > c > b > a$.

The first step, InitialPartition, decomposes $\mathbb{C}^4$ into four pairwise disjoint sets, each of which is the zero set of a regular system.

$$r_1 := \begin{cases} c = 0 \\ b = 0 \\ a = 0 \end{cases}, \quad r_2 := \begin{cases} bx + c = 0 \\ b \neq 0 \\ a = 0 \end{cases}, \quad r_3 := \begin{cases} ax^2 + bx + c = 0 \\ a \neq 0 \end{cases},$$

and $r_4 := \{ax^2 + bx + c \neq 0\}$.

The algorithm MakeCylindrical takes r_1, r_2, r_3 and r_4 as input and outputs a cylindrial decomposition of $\mathbb{C}^4$. Let $t = bx + c$, $q = 2ax + b$, and $r = 4ac - b^2$, the decomposition can be described by a tree.



From the above tree, the algorithm MakeSemiAlgebraic finally produces a CAD of $\mathbb{R}^4$ with 27 cells.

By Collins-Hong or McCallum projection operator, one produces the following polynomials during the projection phase:

$$ax^2 + bx + c, b^2 - 4ac, c, b, a.$$

In the lifting phase, one then obtains a CAD of $\mathbb{R}^4$ with 115 cells!

A CAD with 27 cells is obtained by McCallum-Brown projection operator. However, this operator fails in some (rare) cases.

Concluding Remarks

- Our preliminary implementation, realized with the `RegularChains` library, involves only high-level MAPLE interpreted code. In the table, $N_{\mathbb{R}}$ denotes the number of elements in our CAD.
- Our experimental results show that our method can already process well-known test examples from the literature, see our ISSAC'09 paper for details. Our data also show that polynomial GCDs and resultants modulo regular chains are the dominant cost.
- This suggests that the modular methods and efficient implementation techniques being developed in `RegularChains` library have a large potential for improving our current implementation.

Sys	InitialPartition	MakeCylindrical	MakeSemiAlgebraic	Total	$N_{\mathbb{R}}$
7	2.704	3.600	1.360	7.664	893
8	0.380	1.608	1.196	3.184	365
9	0.288	0.532	0.264	1.084	209
10	5.668	48.079	18.833	72.640	3677
11	0.252	1.192	0.620	2.068	563
12	2.664	135.028	88.142	225.862	20143
13	10.576	35.846	6.905	53.335	4949
14	5.728	71.760	2520.354	2597.878	27547
15	690.731	2513.817	299.250	3503.954	66675
16	895.435	2064.469	> 7200	> 7200	N/A