Detecting Singularities Using the PowerSeries Library

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Abstract. Local bifurcation analysis of singular smooth maps plays a fundamental role in understanding the dynamics of real world problems. This analysis is accomplished in two steps: first performing the Lyapunov-Schmidt reduction to reduce the dimension of the state variables in the original smooth map and then applying singularity theory techniques to the resulting reduced smooth map. In this paper, we address an important application of the so-called Extended Hensel Construction (EHC) for computing the aforementioned reduced smooth map, which, consequently, leads to detecting the type of singularities of the original smooth map. Our approach is illustrated via two examples displaying pitchfork and winged cusp bifurcations.

Keywords: Singularities · Smooth maps · Lyapunov-Schmidt reduction · Extended Hensel Construction · PowerSeries library.

1 Introduction

Consider the smooth map

$$\Phi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \Phi(\mathbf{x}, \boldsymbol{\alpha}) = \mathbf{0}, \tag{1}$$

where the vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ represent state variables and parameters, respectively. We assume that $\Phi(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. The smooth map Φ is called *singular* when $\det(\mathrm{d}\Phi)_{(\mathbf{0},\mathbf{0})} = \mathbf{0}$. The local zeros of a singular map may experience *qualitative* changes when small perturbations are applied to the parameters $\boldsymbol{\alpha}$. These changes are called *bifurcation.* Local bifurcation analysis of zeros of the singular smooth map (1) plays a pivotal role in exploring the behaviour of many real world problems [4, 7, 8, 10]. *Lyapunov-Schmidt reduction* is a fundamental tool converting the singular map (1) into $g : \mathbb{R}^p \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ with $p = n - \mathrm{rank}(\mathrm{d}\Phi_{\mathbf{0},\mathbf{0}})$. The reduction is achieved through producing an equivalent map to (1) made up of a pair of equations and making use of the Implicit Function Theorem. This solves the $n-1$ variables of $\mathbf{x}$ in the first equation; thereafter, substituting the result into the second one gives an equation for the remaining variable. It is proved that the local zeros of the map g are in one-to-one correspondence with the local zeros of Φ; for more details see [8, Pages

25–34]. Hence, the study of local zeros of (1) is facilitated throught treating their counterparts in g. Singularity theory is an approach providing a comprehensive framework equipped with effective tools for this study. The pioneering work of René Thom established the original ideas of the theory which was then extensively developed by John Mather and V. I. Arnold. The book series [8] written by Marty Golubitsky, Ian Stewart and David G. Schaeffer is a collection of significant contributions of the authors in dealing with a wide range of real world problems using singularity theory techniques as well as explaining the underlying ideas of the theory in ways accessible to applied scientists and mathematicians particularly those dealing with bifurcation problems in the presence of parameters and symmetries. The singularity theory tools are applied to the problems that have emerged as an output of the Lyapunov-Schmidt reduction. Following [8, Page 25], we focus on the reduction when $\operatorname{rank}(\mathrm{d}\Phi_{\mathbf{0},\mathbf{0}}) = n-1$, $m = 1$ and refer to $\alpha_1 = \lambda$ as the bifurcation parameter. In other words, we consider the following map

$$g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \quad g(x, \lambda) = 0. \tag{2}$$

Two smooth maps are regarded as *germ-equivalent* when they are identical on some neighborhood of the origin. In fact, a *germ-equivalence* class of a smooth map is called a *germ*. We denote by $\mathscr{E}_{x,\lambda}$ the space of all scalar smooth germs which is a local ring with $\mathcal{M}_{\mathscr{E}_{x,\lambda}} = \langle x, \lambda\rangle_{\mathscr{E}_{x,\lambda}}$ as the unique maximal ideal; see also [8, Page 56] and [4, Page 3]. Due to the existence of germs with infinite Taylor series and flat germs (whose Taylor series is zero), there does not exist a computational tool to automatically study local bifurcations in $\mathscr{E}_{x,\lambda}$. This has motivated the authors of [4] to propose circumstances under which the computations supporting the bifurcation analysis in $\mathscr{E}_{x,\lambda}$ are transferred to smaller local rings and verify that the corresponding results are valid in $\mathscr{E}_{x,\lambda}$. For instance, the following theorem permits the use of formal power series $K[[x, \lambda]]$ ring as a smaller computational ring in computation of algebraic objects involved in the analysis of bifurcation.

Theorem 1. *([4, Theorem 4.3]) Suppose that $\{f_i\}_{i=1}^m \in \mathscr{E}_{x,\lambda}$. For $k, N \in \mathbb{N}$ with $k \leq N$,*

$$\mathcal{M}^k_{K[[x,\lambda]]} \subseteq \langle J^N f_1, \ldots, J^N f_m\rangle_{K[[x,\lambda]]} \quad \textit{iff} \quad \mathcal{M}^k_{\mathscr{E}_{x,\lambda}} \subseteq \langle f_1, \ldots, f_m\rangle_{\mathscr{E}_{x,\lambda}}$$

where $\mathcal{M}^k = \langle x^{\alpha_1}\lambda^{\alpha_2} : \alpha_1 + \alpha_2 = k\rangle$ and $J^N f_i$ is the sum of terms of degree N or less in the Taylor series of f_i.

This, along with other criteria in [4, 6], highlights the importance of alternative rings in performing automatic local bifurcation analysis of scalar and $\mathbb{Z}_2$-equivariant singularities.

The work presented here addresses one of the applications of the so-called Extended Hensel Construction (EHC) invented by Sasaki and Kako, see [12]. We show that the EHC can be used in computing the reduced system $g \in \mathscr{E}_{x,\lambda}$, which, as a result, leads to determining the type of singularity hidden in system (1).

This EHC has been studied and improved by many authors. In particular, the papers [1–3] present algorithmic improvements (where the EHC relies only linear algebra techniques and univariate polynomial arithmetic) together with applications of the EHC in deriving real branches of space curves and consecuently computing limitis of real multivariate rational functions. The same authors implemented their version of the EHC as the `ExtendedHenselConstruction` command of the `PowerSeries` library [1],

The EHC comes into two flavors. In the case of bivariate polynomials it behaves as Newton-Puiseux algorithm while with multivariate polynomials it acts as an effective version of Jung-Abhyankar Theorem. In both cases, it provides a factorization of the input object in the vicinity of the origin. We believe that this capabiliy makes the EHC a desirable tool for an automatic derivation of the zeros of a polynomial system locally near the origin. The rest of this paper is organized as follows. In Section 2, some of the ideas in singularity theory are reviewed. We then discuss the EHC procedure followed by an overview on the `PowerSeries` Library. Finally, our proposed approach is illustrated through two examples revealing pitchfork and winged cusp bifurcations.

2 Background

2.1 Concepts from singularity theory

In this section we explain the materials required for defining recognition problem of a singular germ. These concepts are accompanied by examples. We skip the technical details of singularity theory-related concepts as they are beyond the scope of this paper. The interested readers are referred to [4, 5, 8] for the principal ideas, algebraic formulations and automatic computation of the following objects.

Contact equivalence. We say that two smooth germs $f, g \in \mathscr{E}_{x,\lambda}$ are *contact equivalent* when

$$g(x, \lambda) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda)) \tag{3}$$

is held for a smooth germ $S(x, \lambda) \in \mathscr{E}_{x,\lambda}$ and local diffeomorphisms $((x, \lambda) \longrightarrow (X(x, \lambda), \Lambda(\lambda))) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfying

$$S(x, \lambda), X_x(x, \lambda), \Lambda(\lambda) > 0$$

Normal form. Bifurcation analysis of local zeros of g in 2 requires computing a contact equivalent germ to g which has simpler structure and makes the analysis efficient. Indeed, each step of this analysis, for instance recognition problem, involves normal form computation. To be more precise, the simplest representative of the class of $g \in \mathscr{E}_{x,\lambda}$ under contact equivalence is called a *normal form* of g.

[1] `http://www.regularchains.org/downloads.html`

Example 1. Consider the smooth germ $g(x,\lambda) = \sin(x^3) - \lambda x + \exp(\lambda^3) - 1 \in \mathscr{E}_{x,\lambda}$. Note that $g(0,0) = \frac{\partial}{\partial x} g(0,0) = 0$; therefore, the origin is the singular point of g. The procedure in [4, Section 6] returns $x^3 - \lambda x$ as the normal form of g denoted by $\mathrm{NF}(g)$. The equation $x^3 - \lambda x = 0$ is called the *pitchfork bifurcation problem* and the *bifurcation diagram* for pitchfork is defined by the local variety $\{(x,\lambda) \mid x^3 - x\lambda = 0\}$. When λ smoothly varies around the origin, the number of solutions of the pitchfork bifurcation problem changes from one to three; see Figure 1.

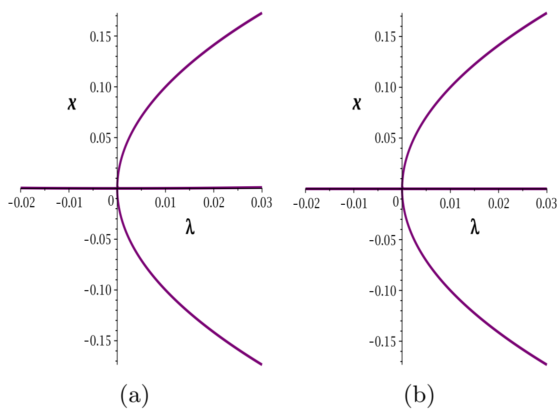

Fig. 1. Figures (a) and (b) depict the bifurcation diagrams of g and $\mathrm{NF}(g)$, respectively.

Now, modulo monomials of degree ≥ 5, we compute the transformation $(X(x,\lambda), S(x,\lambda), \Lambda(\lambda))$ through which g is converted into NF(g) in (3).

$$\begin{aligned}
X(x,\lambda) &:= x + \lambda^2 + \lambda x + \lambda^2 x + \lambda x^2 + x^3, \\
S(x,\lambda) &:= 1 - \lambda + 2\lambda x + x^2, \\
\Lambda(\lambda) &:= \lambda.
\end{aligned}$$

Recognition problem. Let $g \in \mathscr{E}_{x,\lambda}$ be a singular germ. *Recognition problem* for a normal form of g computes a list of zero and non-zero conditions on derivatives of a singular germ $f \in \mathscr{E}_{x,\lambda}$ under which f is contact-equivalent to g. The proposed algorithm in [8, Pages 86–93], divides monomials (in $\mathscr{E}_{x,\lambda}$) into three categories; *low*, *intermediate* and *high order terms*. *Low order terms* refer to the monomials of the form $x^{\alpha_1}\lambda^{\alpha_2}$ that do not participate in the representation of any germ equivalent to g. The *high order terms* consist of the monomials $x^{\alpha_1}\lambda^{\alpha_2}$ which do not change the structure of the local zeros of g when they are present; that is, adding $x^{\alpha_1}\lambda^{\alpha_2}$ to g creates a germ contact equivalent to g. Due to the sophisticated structure of intermediate order terms we skip defining them here and instead introduce *intrinsic* generators $x^{\alpha_1}\lambda^{\alpha_2}$ which contribute to every equivalent germ and provide information about intermediate order terms. Low order terms and intrinsic generators are identified through the following theorem.

Theorem 2. *[8, Theorems 8.3 and 8.4, Page 88] Suppose that $f, g \in \mathscr{E}_{x,\lambda}$ and there exists a positive integer k such that $\mathcal{M}^k_{\mathscr{E}_{x,\lambda}} \subset \langle g, x\frac{\partial}{\partial x}g, \lambda\frac{\partial}{\partial x}g\rangle_{\mathscr{E}_{x,\lambda}}$.*

(a) if f is equivalent to g and $x^{\alpha_1}\lambda^{\alpha_2}$ belongs to low order terms of g then $\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial \lambda^{\alpha_2}}f(0,0) = 0$.
(b) furthermore, assume that $x^{\alpha_1}\lambda^{\alpha_2}$ belongs to intrinsic generators of g. If f is equivalent to g then $\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial \alpha^{\alpha_2}}f(0,0) \neq 0$.

Example 2. For the smooth germ g given by Example 1, we deduce the vector space $\mathbb{R}\{1, \lambda, x, x^2\}$ as low order terms. It follows from Theorem 2(a) that any germ f equivalent to g satisfies

$$f(0,0) = \frac{\partial}{\partial \lambda}f(0,0) = \frac{\partial}{\partial x}f(0,0) = \frac{\partial^2}{\partial x^2}f(0,0) = 0 \tag{4}$$

Moreover, the higher order terms of g are determinded by the ideal

$$\langle x^4, \lambda^4, x^3\lambda, x\lambda^3, x^2\lambda^2\rangle_{\mathscr{E}_{x,\lambda}} + \langle x^2\lambda, \lambda^3, x\lambda^2\rangle_{\mathscr{E}_{x,\lambda}} + \langle \lambda^2\rangle_{\mathscr{E}_{x,\lambda}}$$

which means that adding/removing any monomial, taken from this ideal, to/from g gives a new germ equivalent to g. Finally, the corresponding intrinsic generators of g are described via $\{x^3, \lambda x\}$ verified by Theorem 2(b) that for any germ f equivalent to g the following is valid

$$\frac{\partial^3}{\partial x^3}f(0,0) \neq 0, \quad \frac{\partial}{\partial \lambda}\frac{\partial}{\partial x}f(0,0) \neq 0 \tag{5}$$

To sum up, the recognition problem for a normal form of g is characterized by (4) and (5).

2.2 The Extended Hensel Construction

This part is summarized from [1].

Notation 1 Suppose that $\mathbb{K}$ is an algebraic number field whose algebraic closure is denoted by $\overline{\mathbb{K}}$. Assume that $F(X,Y) \in \mathbb{K}[X,Y]$ is a bivariate polynomial with complex number coefficients. Let also F be a univariate polynomial in X which is monic and square-free. The partial degree of F w.r.t. X is represented by d. We denote by $\mathbb{K}[[U^*]] = \bigcup_{\ell=1}^{\infty} \mathbb{K}[[U^{\frac{1}{\ell}}]]$ the ring of *formal Puiseux series*. Hence, given $\varphi \in \mathbb{K}[[U^*]]$, there exists $\ell \in \mathbb{N}_{>0}$ such that $\varphi \in \mathbb{K}[[U^{\frac{1}{\ell}}]]$ holds. Thus, we can write $\varphi = \sum_{m=0}^{\infty} a_m U^{\frac{m}{\ell}}$, for some $a_0, \ldots, a_m, \ldots \in \mathbb{K}$. We denote by $\mathbb{K}((U^*))$ the quotient field of $\mathbb{K}[[U^*]]$. Let $\varphi \in \mathbb{K}[[U^*]]$ and $\ell \in \mathbb{N}$ such that $\varphi = f(U^{\frac{1}{\ell}})$ holds for some $f \in \mathbb{K}[[U]]$. We say that the Puiseux series φ is *convergent* if we have $f \in \mathbb{K}\langle U\rangle$. We recall Puiseux's theorem: if $\mathbb{K}$ is an algebraically closed field of characteristic zero, the field $\mathbb{K}((U^*))$ of formal Puiseux series over $\mathbb{K}$ is the algebraic closure of the field of formal Laurent series over $\mathbb{K}$; moreover, if $\mathbb{K} = \mathbb{C}$, then the field $\mathbb{C}(\langle Y^*\rangle)$ of convergent Puiseux series over $\mathbb{C}$ is algebraically closed as well.

The purpose of the EHC is to factorize $F(X,Y)$ as $F(X,Y) = G_1(X,Y) \cdots G_r(X,Y)$, with $G_i(X,Y) \in \overline{\mathbb{K}}(\langle Y^* \rangle)[X]$ and $\deg_X(G_i) = m_i$, for $1 \leq i \leq r$. Thus, the EHC factorizes $F(X,Y)$ over $\overline{\mathbb{K}}(\langle Y^* \rangle)$, thus over $\mathbb{C}(\langle Y^* \rangle)$.

Newton line. We plot each non-zero term $c\,X^{e_x}Y^{e_y}$ of $F(X,Y)$ to the point of coordinates (e_x, e_y) in the Euclidean plane equipped with Cartesian coordinates. We call *Newton Line* the straight line L passing through the point $(d,0)$ and another point, such that no other points lie below L. The equation of L is $e_x/d + e_y/\delta = 1$ for some $\delta \in \mathbb{Q}$. We define $\widehat{\delta}, \widehat{d} \in \mathbb{Z}^{>0}$ such that $\widehat{\delta}/\widehat{d} = \delta/d$ and $\gcd \widehat{\delta}, \widehat{d} = 1$ both hold.

Newton polynomial. The sum of all the terms of $F(X,Y)$ which are plotted on the Newton line of F is called the *Newton polynomial* of F. We denote it by $F^{(0)}$. Observe that the Newton polynomial is a homogeneous polynomial in $(X, Y^{\delta/d})$. Let $\zeta_1, \ldots, \zeta_r \in \overline{\mathbb{K}}$ be the distinct roots of $F^{(0)}(X,1)$, for some $r \geq 2$. Hence we have $\zeta_i \neq \zeta_j$ for all $1 \leq i < j \leq r$ and there exist positive integers $m_1 \leq m_2 \leq \cdots \leq m_r$ such that, using the homogeneity of $F^{(0)}(X,Y)$, we have

$$F^{(0)}(X,Y) = (X - \zeta_1 Y^{\delta/d})^{m_1} \cdots (X - \zeta_r Y^{\delta/d})^{m_r}.$$

The *initial factors* of $F^{(0)}(X,Y)$ are $G_i^{(0)}(X,Y) := (X - \zeta_i Y^{\delta/d})^{m_i}$, for $1 \leq i \leq r$. For simplicity, we put $\widehat{Y} = Y^{\widehat{\delta}/\widehat{d}}$.

Theorem 3 (Extended Hensel Construction). *We define the ideal*

$$S_k = \langle X^d Y^{(k+0)/\widehat{d}}, X^{d-1} Y^{(k+\widehat{\delta})/\widehat{d}}, \ldots, X^0 Y^{(k+d\widehat{\delta})/\widehat{d}} \rangle, \tag{6}$$

for $k = 1, 2, \ldots$. Then, for all integer $k > 0$, we can construct $G_i^{(k)}(X,Y) \in \mathbb{C}\langle Y^{1/\widehat{d}} \rangle[X]$, for $i = 1, \ldots, r$, satisfying

$$F(X,Y) = G_1^{(k)}(X,Y) \cdots G_r^{(k)}(X,Y) \mod S_{k+1}, \tag{7}$$

and $G_i^{(k)}(X,Y) \equiv G_i^{(0)}(X,Y) \mod S_1$, for all $i = 1, \ldots, r$.

2.3 The `PowerSeries` library

The `PowerSeries` library consists of two modules, dedicated respectively to multivariate power series over the algebraic closure of $\mathbb{Q}$, and univariate polynomials with multivariate power series coefficients. Figure 2 illustrates *Weiertrass Preparation Factorization.* The command `PolynomialPart` displays all the terms of a power series (or a univariate polynomial over power series) up to a specified degree. In fact, each power series is represented by its terms that have been computed so far together with a program for computing the next ones. A command like `WeiertrassPreparation` computes the terms of the factors p and α up to the specified degree; moreover, the encoding of p and α contains a program for computing their terms in higher degree. Figures 3 and 4 illustrate the *Extended*

```
> PS := PowerSeries([X, Y]) :
  with(PS) :
  UPoPS := UnivariatePolynomialOverPowerSeries([X, Y], Z) :
  with(UPoPS) :
  u := UPoPS:-FromListOfPolynomials([Y, 1, X + 1]);
  UPoPS:-PolynomialPart(u, 2);
  (p, alpha) := UPoPS:-WeierstrassPreparation(u, 2);
  UPoPS:-PolynomialPart(p, 2);
  UPoPS:-PolynomialPart(alpha, 2);
```

$$u := polynomial\_over\_power\_series$$

$$Y + Z + (X + 1)\, Z^2$$

$$p, \alpha := polynomial\_over\_power\_series,\ polynomial\_over\_power\_series$$

$$Y^2 + Y + Z$$

$$-X\, Y - Y^2 - Y + 1 + (X + 1)\, Z$$

Fig. 2. Weierstrass Preparation Factorization for a univariate polynomial with multivariate power series coefficients.

```
> P := PowerSeries([y, z]) :
  U := UnivariatePolynomialOverPowerSeries([y, z], x) :
  poly := y·x^3 + (-2 · y + z + 1)·x + y:
  U:-ExtendedHenselConstruction(poly, [0, 0], 3);
```

$$\left[\left[x = \frac{-RootOf(\_Z^2 + y) + RootOf(\_Z^2 + y)\, y - \frac{1}{2}\, RootOf(\_Z^2 + y)\, z + \frac{1}{2}\, y^2}{y}\right],\right.$$

$$\left[x = \frac{RootOf(\_Z^2 + y) - RootOf(\_Z^2 + y)\, y + \frac{1}{2}\, RootOf(\_Z^2 + y)\, z + \frac{1}{2}\, y^2}{y}\right],$$

$$\left.[x = -y]\right]$$

Fig. 3. Extended Hensel construction applied to a trivariate polynomial for computing its absolute factorization.

Hensel Construction (EHC)[2] For the case of an input bivariate polynomial, see Figure 4, this coincides with the Newton-Puiseux algorithm, thus computing the Puiseux parametrizations of a plane curve about a point; this functionality is at the core of the `LimitPoints` command. For the case of a univariate polynomial with multivariate polynomial coefficients, the EHC is a weak version of Jung-Abhyankar Theorem.

3 Applications

In this section we are concerned with two smooth maps $\Phi, \Psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ whose state variables and bifurcation parameter are denoted by (x, y) and λ, respectively. Since the Jacobian matrix of each map is not full rank at the origin, the Implicit Function Theorem fails at solving (x, y) as a function of λ locally around the origin. This causes bifurcations to reside in local zeros of each singular smooth map. We recall that these bifurcations are treated via first applying the Lyapunov-Schmidt reduction to a singular smooth map ending up with a

[2] The factorization based on Hensel Lemma is in fact a weaker construction since: (1) the input polynomial must be monic and (2) the output factors may not be linear.

```
> P := PowerSeries([y]):
> U := UnivariatePolynomialOverPowerSeries([y], x):
> poly := y·x^3 + (-2·y + 1)·x + y:
> OutputFlag :: name :='parametric':
> parametricVar :: name := T:
> iter := 3:
> verificationFlag :: boolean := true:
> U:-ExtendedHenselConstruction(poly, 0, iter, OutputFlag, parametricVar, verificationFlag);
```

$$\left[\left[y = T^2, x = -T^3\right], \left[y = T^2, x = \frac{RootOf(\_Z^2 + 1)\; T - T^3\; RootOf(\_Z^2 + 1) + \frac{1}{2}\; T^4}{T}\right], \left[y = T^2, x = \frac{-RootOf(\_Z^2 + 1)\; T + T^3\; RootOf(\_Z^2 + 1) + \frac{1}{2}\; T^4}{T}\right]\right]$$

Fig. 4. Extended Hensel construction applied to a bivariate polynomial for computing its Puiseux parametrizations around the origin.

reduced map of the form (2) and then passing the result through singularity theory techniques. Here, we follow the same approach except that we employ the `ExtendedHenselConstruction` command to compute the reduced map. The latter factorizes one of the equations around the origin and the resulting real branches that go through the origin are plugged in the other one to obtain the desired map (2). Once the map is computed we use the concept of recognition problem to identify the type of singularity.

3.1 The pitchfork bifurcation

In spite of simple structure, the pitchfork bifurcation is highly observed in physical phenomena mostly in the presence of symmetry breaking. For instance, [9] reports on *spontaneous mirror-symmetry breaking through a pitchfork bifurcation in a photonic molecule made up of two coupled photonic-crystal nanolasers.* Furthemore, authors in [11] study the pitchfork bifurcation arising in LugiatoLefever (LL) equation which is a model for a *passive Kerr resonator in an optical fiber ring cavity.* Finally, [6, Example 4.1] captures pitchfork bifurcation while analysing the local bifurcations of Chua's circuit. Here, we consider the exercise 3.2 on [8, Page 34]. Suppose that $\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is defined by $\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ where

$$\begin{aligned} \Phi_1(x, y, \lambda) &= 2x - 2y + 2x^2 + 2y^2 - \lambda x \\ \Phi_2(x, y, \lambda) &= x - y + xy + y^2 - 3\lambda x. \end{aligned} \tag{8}$$

To obtain the reduced system g in (2) we pass Φ_1 to the `ExtendedHenselConstruction` giving rise to the branches in Figure 5. Note that the second branch is not of interest as it does not pass the origin. Substituting the first branch into Φ_2,

```
> P := PowerSeries([y, lambda]) :
> U := UnivariatePolynomialOverPowerSeries([y, lambda], x) :
> poly := 2·x − 2·y + 2·x^2 + 2·y^2 − lambda·x :
> U:-ExtendedHenselConstruction(poly, [0, 0], 4);
```

$$\left[\left[y=0, \lambda=0, x=-2y^2+\frac{1}{2}y\lambda+y+4y^3-2y^2\lambda+\frac{1}{4}y\lambda^2+\frac{1}{8}y\lambda^3+8y^3\lambda-\frac{7}{4}y^2\lambda^2-12y^4\right], \left[y=0, \lambda=0, x=2y^2-\frac{1}{2}y\lambda-y+\frac{1}{2}\lambda-1-4y^3+2y^2\lambda-\frac{1}{4}y\lambda^2-\frac{1}{8}y\lambda^3-8y^3\lambda+\frac{7}{4}y^2\lambda^2+12y^4\right]\right]$$

Fig. 5. EHC applied to $\Phi_1(x, y, \lambda)$.

modulo monomials of degree ≥ 4, results in

$$g(y, \lambda) = 2y^3 - \frac{5}{2}y\lambda + \frac{9}{2}y^2\lambda - \frac{5}{4}y\lambda^2. \tag{9}$$

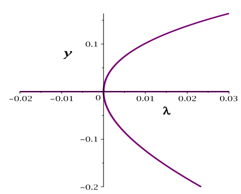

Fig. 6. Pitchfork bifurcation diagram associated with g in Equation (9).

Given g in (9), the low order terms and the intrinsic generators are determined by $\mathbb{R}\{1, y, \lambda, y^2\}$ and $\{y\lambda, y^3\}$, respectively. Thus, Theorem 2 implies that g satisfies the recognition problem for pitchfork

$$f(0,0) = \frac{\partial}{\partial y}f(0,0) = \frac{\partial}{\partial \lambda}f(0,0) = \frac{\partial^2}{\partial y^2}f(0,0) = 0$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial \lambda}f(0,0) \neq 0, \quad \frac{\partial^3}{\partial y^3}f(0,0) \neq 0$$

This proves that the original system Φ has pitchfork singularity located at the origin.

3.2 The winged cusp bifurcation

The *winged cusp bifurcation problem* is defined by the equation $x^3+\lambda^2 = 0$ and its corresponding bifurcation diagram $\{(x, \lambda) \mid x^3 + \lambda^2 = 0\}$ is exhibited via Figure

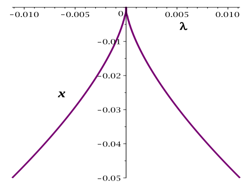

Fig. 7. The winged cusp bifurcation diagram.

7. Singularity theory tools have been utilized in the area of chemical engineering with the aim of studying the solutions of the continuous flow stirred tank reactor (CSTR) model. This study proves that the winged cusp bifurcation is the normal form for describing the *organizing center* of the bifurcation diagrams of the model produced by numerical methods. It, further, unravels more bifurcation diagrams that have not been reported through these numerical methods; see [7, 8, 13, 14]. Now assume that $\Psi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is given by $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ where

$$\begin{aligned} \Psi_1(x, y, \lambda) &= -2x + 3y + \lambda^2 + y^3 + x^4 \\ \Psi_2(x, y, \lambda) &= 2x - 3y + y^2\lambda + x^3. \end{aligned} \tag{10}$$

Applying the `ExtendedHenselConstruction` to Ψ_2 leads to the branches in Figure 8.

```
> P := PowerSeries([y, lambda]) :
> U := UnivariatePolynomialOverPowerSeries([y, lambda], x) :
> poly := 2·x - 3·y + y^2·lambda + x^3 :
> U:-ExtendedHenselConstruction(poly, [0, 0], 3);
```

$$\left[\left[y = 0, \lambda = 0, x = \frac{3}{2} y - \frac{1}{2} y^2 \lambda - \frac{27}{16} y^3\right], \left[y = 0, \lambda = 0, x = -RootOf(\_Z^2 + 2) - \frac{3}{4} y - \frac{27}{64} y^2 RootOf(\_Z^2 + 2) + \frac{1}{4} y^2 \lambda + \frac{27}{32} y^3\right], \left[y = 0, \lambda = 0, x = RootOf(\_Z^2 + 2) - \frac{3}{4} y + \frac{27}{64} y^2 RootOf(\_Z^2 + 2) + \frac{1}{4} y^2 \lambda + \frac{27}{32} y^3\right]\right]$$

Fig. 8. EHC applied to $\Psi_2(x, y, \lambda)$.

Substituting the first branch into Ψ_1, modulo monomials of degree ≥ 4, yields

$$g(y, \lambda) = \frac{35}{8} y^3 + \lambda^2 + y^2\lambda. \tag{11}$$

As $\{1, y, \lambda, y^2, y\lambda\}$ spans the space of low order terms and intrinsic generators are $\{\lambda^2, y^3\}$, Theorem 2 guarantees that g satisfies the recognition problem for the winged cusp

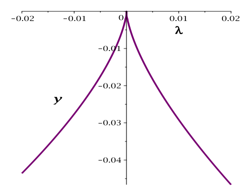

Fig. 9. Bifurcation diagram associated with g in 11.

$$f(0,0)=\frac{\partial}{\partial y}f(0,0)=\frac{\partial}{\partial\lambda}f(0,0)=\frac{\partial^2}{\partial y^2}f(0,0)=\frac{\partial}{\partial y}\frac{\partial}{\partial\lambda}f(0,0)=0$$

$$\frac{\partial^2}{\partial\lambda^2}f(0,0)\neq 0,\quad \frac{\partial^3}{\partial y^3}f(0,0)\neq 0$$

References

1. Alvandi, P., Ataei, M., Kazemi, M., Moreno Maza, M.: On the Extended Hensel Construction and its Application to the Computation of Limit Points. To appear in Journal of Symbolic Computation, (2019)
2. Alvandi, P., Kazemi, M., Moreno Maza, M.: Computing Limits of Real Multivariate Rational Functions. In Proceedings of ISSAC'16, pp. 39–46. ACM, New York, USA (2016)
3. Alvandi, P., Ataei, M., Moreno Maza, M.: On the Extended Hensel Construction and its Application to the Computation of Limit Points. In Proceedings of ISSAC'17, pp. 13–20. ACM, New York, USA (2017)
4. Gazor, M., Kazemi, M.: Symbolic Local Bifurcation Analysis of Scalar Smooth Maps. ArXiv:1507.06168, (2016)
5. Gazor, M., Kazemi, M.: A User Guide for Singularity. ArXiv:1601.00268, (2017)
6. Gazor, M., Kazemi, M.: Normal Form Analysis of $\mathbb{Z}_2$-equivariant Singularities. International Journal of Bifurcation and Chaos, **29**(2), pp. 1950015-1–1950015-20 (2019)
7. Golubitsky, M., Keyfitz, B. L.: A Qualitative Study of the Steady-state Solutions for a Continuous Flow Stirred Tank Chemical Reactor. SIAM J. Math. Anal, **11**(2), pp. 316–339, (1980)
8. Golubitsky, M., Stewart, I., Schaeffer, D. G.: Singularities and Groups in Bifurcation Theory. Volumes 1-2, Springer, New York 1985 and 1988
9. Hamel, P., Haddadi, S., Raineri, F., Monnier, P., Beaudoin, G., Sagnes, I., Levenson, A., Yacomotti, A. M.: Spontaneous Mirror-symmetry breaking in coupled photonic-crystal nanolasers. Nature Photonics, **9**, pp. 311–315, (2015)
10. Labouriau, I.: Applications of Singularity Theory to Neurobiology. PhD Thesis, Warwick University, (1984)
11. Rossi, J., Carretero-González, R., Kevrekidis, P. G., Haragus, M.: On the Spontaneous Time-reversal Symmetry Breaking in Synchronously-pumped Passive Kerr Resonators. Journal of Physics A: Mathematical and Theoretical, **49**(45), pp. 455201–455221, (2016)

12. Sasaki, T., and Kako, F.: Solving Multivariate Algebraic Equation by Hensel Construction. Japan J. Indust. and Appl. Math, pp. 257–285, (1999)
13. Uppal, A., Ray, W. H., Poore, A. B.: The Classification of the Dynamic Behavior of Continuous Stirred Tank Reactorsinfluence of Reactor Residence Time. Journal of Chemical Engineering Science, **31**(3), pp. 205–214, (1976)
14. Zeldovich, Y. V., Zisin, U. A.: On the Theory of Thermal Stress. Flow in an exothermic stirred reactor, II.Study of Heat Loss in a Flow Reactor. Journal of Technical Physics, **11**(6), pp. 501–508, (Russian), (1941)