Jordan Canonical Form with Parameters From Frobenius Form with Parameters*

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Abstract. The Jordan canonical form (JCF) of a square matrix is a foundational tool in matrix analysis. If the matrix A is known exactly, symbolic computation of the JCF is possible though expensive. When the matrix contains parameters, exact computation requires either a potentially very expensive case discussion, significant expression swell or both. For this reason, no current computer algebra system (CAS) of which we are aware will compute a case discussion for the JCF of a matrix $A(\alpha)$ where α is a (vector of) parameter(s). This problem is extremely difficult in general, even though the JCF is encountered early in most curricula. In this paper we make some progress towards a practical solution. We base our computation of the JCF of $A(\alpha)$ on the theory of regular chains and present an implementation built on the `RegularChains` library of the MAPLE CAS. Our algorithm takes as input a matrix in Frobenius (rational) canonical form where the entries are (multivariate) polynomials in the parameter(s). We do not solve the problem in full generality, but our approach is useful for solving some examples of interest.

Keywords: Jordan form; rational canonical form; parametric linear algebra; regular chains; triangular decomposition

1 Introduction

The Jordan canonical form (JCF) of a matrix and its close cousin the Weyr canonical form are foundational tools in the analysis of eigenvalue problems and dynamical systems. For a summary of theory, see for instance Chapter 6 in The Handbook of Linear Algebra [19]; for the Weyr form, see [28].

The first use usually seen for the JCF is as a canonical form for matrix similarity: two matrices are similar if and only if they have identical (sets of, up to ordering) Jordan canonical forms [20]. Of course, there are other (often better) canonical forms for similarity such as the Frobenius (rational) canonical form, or the rational Jordan form [13, 21].

The JCF is well known to be discontinuous with respect to changes in the entries if the base field $\mathbb{K}$ has nonempty open sets. We typically take $\mathbb{K} = \mathbb{C}$, the field of complex numbers. Therefore, the JCF cannot be computed numerically

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with small forward error, even when using a numerically stable algorithm. This has forced the development of alternatives to the JCF, such as the Schur form, which is numerically stable and useful in the computation of matrix functions via the Parlett recurrences, for instance [18]. Consider the computation of the matrix exponential. First computing the JCF is one of the famous "Nineteen Dubious Ways to Compute the Exponential of a Matrix" [18, 26]; computing the matrix exponential remains of serious interest today (or perhaps is even of increased interest) because of new methods for "geometric" numerical integration of large systems [11, 17, 23].

Analysis of small systems containing symbolic parameters is also of great interest, in mathematical biology especially (models of disease dynamics in populations and in individual hosts, evolutionary or ecological models) but also in many other dynamical systems applications such as fluid-structure interactions, robot kinematics, and electrical networks. The algorithmic situation for systems containing parameters is much less well-developed than is the corresponding situation for numerical systems. Although alternatives to the JCF exist for the analysis of these systems, the JCF has become a standard tool with implementations available in every major CAS.

The current situation in MAPLE is that explicit computation of the JCF of a matrix containing parameters of dimension 5 or more may fail in some simple cases. For example, MAPLE simply does not provide a result for the JCF of the Frobenius companion matrix of $p(x) = x^5 + x^4 + x^3 + x^2 + x + a$. Similar failures occur for the `MatrixFunction` and `MatrixExponential` procedures. Wolfram Alpha gives the generic answers, but fails to give non-generic ones. Computing matrix functions may succeed in cases where computing the JCF does not because the JCF need not be used (an interpolation algorithm can be used instead); see for instance Definition 1.4 in [18].

Most computer algebra systems (CAS) have adopted some variation of the definition of algebraic functions as implicit roots of their defining polynomials. In MAPLE, the syntax uses `RootOf`; together with an `alias` facility. This gives a useful way to encode the mathematical statement (for instance) "Let α be a root of the polynomial $x^5 + \varepsilon x + 1 = 0$".

```
> alias(alpha = RootOf(x^5+eps*x+1,x)):
```

This should, in theory, allow symbolic computation of the JCF of (small) matrices, even ones containing parameters. To date in practice it has not.

In this paper, we offer some progress, although we note that combinatorial growth of the resulting expressions remains a difficulty. However, the tools we provide here are already useful for some example applications and go some way towards filling a scientific and engineering need. We aim to minimize unnecessary growth throughout the computation. The tools we use here include provisos [8] and comprehensive square-free factorization with the `RegularChains` package.

Consider, for example, the Jacobian matrix in [35]. MAPLE's built-in `JordanForm` command returns a diagonal matrix where the eigenvalues are large nested radical expressions as a result of explicitly solving the characteristic polynomial. In contrast, our `ComprehensiveJordanForm` method gives a full case discussion.

> $J := \begin{bmatrix} 0 & 2\rho & 0 \\ a & 2\beta & 2\nu \\ b & -2\nu & 2\beta \end{bmatrix}$:

> $R := PolynomialRing([a, b, \rho, \beta, \nu])$:

> $JCF := ComprehensiveJordanForm(J, R, 'output'='lazard')$:

> $Display(JCF[1], R), Display(JCF[25], R)$;

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} a = 0 \\ \beta = 0 \\ \nu = 0 \\ b \neq 0 \\ \rho \neq 0 \end{cases}\right], \left[\begin{bmatrix} 2\beta & 0 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{bmatrix}, \begin{cases} 2\rho a + \beta^2 = 0 \\ \nu = 0 \\ 2\rho \neq 0 \\ \beta \neq 0 \end{cases}\right]$$

Fig. 1. Our implementation provides a full case discussion of the JCF of a matrix with 5 parameters. Two non-trivial cases are shown.

Two interesting cases where the JCF is non-trivial are shown in Figure 1. Further details of this example are given in section 6.4.

In Section 5, we present an algorithm for computing the JCF of a matrix in Frobenius form where the entries are multivariate polynomials whose indeterminants are regarded as parameters. Our approach uses comprehensive square-free factorization to provide a complete case discussion. Classical approaches for computing the JCF rely on elementary row and column operations that maintain a similarity relation at each step [3, 14, 29]. Because the entries of the matrices we are considering are multivariate polynomials, row and column operations lead to significant expression growth that can be difficult to control. Additionally, this would require us to work over matrices of multivariate rational functions in the parameters, again making it difficult to control expression growth. By instead computing fraction free square-free factorizations, we are able to maintain better control over expression growth. Because our implementation does not use elementary row and column operations, we do not compute the similarity transformation matrix Q such that $J = Q^{-1}AQ$ gives the Jordan form J of A. We leave this problem for future work.

We present an implementation of our algorithm in Section 6 and use it to solve several problems taken from the literature. These examples are not in Frobenius form and we do not discuss in detail how we obtain the Frobenius form. The Frobenius form implementation uses standard algorithms based on GCD computations of parametric polynomials to find the Smith form of $A - xI$ and the relation between this and the Frobenius form of A [21].

Section 4 presents a new approach for computing the JCF of a non-parametric matrix in Frobenius form over the splitting field of the characteristic polynomial. Our discussion is based on the theory of regular chains. We do not apply this splitting field approach in the parametric case because the square-free factorization approach we use gives the complete structure of the JCF. Constructing the splitting field would be vastly more expensive than the approach of Section 5.

2 Some Prior Work

As previously mentioned, the JCF of a matrix $A \in \mathbb{C}^{n\times n}$ as a function of the entries of A has discontinuities. These discontinuities are often precisely what is important in applications. This also means that even numerically stable algorithms can sometimes give results with $\mathcal{O}(1)$ forward error. This is often also stated by saying that "computing the JCF is an ill-posed problem" [3]. This has not prevented people from trying to compute the JCF numerically anyway (see [3] and the references therein), but in general such efforts cannot always be satisfactory: discontinuous is ill-posed, and without regularization such efforts are (sometimes) doomed. There have been at least three responses in the literature.

One is to find other ways to solve your problem, i.e. compute matrix functions such as A^n and $\exp(tA)$, without first computing the JCF, and the invention of the numerically stable Schur factoring and the Parlett recurrences for instance has allowed significant success [18].

The second response is to find a canonical form that explicitly preserves the continuity or smoothness of the matrix; the versal forms of [1] do this. Incidentally, the Frobenius form with parameters is an example of a versal form (Arnol'd calls this a Sylvester family), but there are others. The paper [7] uses Carleman linearization to do something similar.

The third response is to assume exact input and try to do exact or symbolic computation of the JCF. Early attempts, e.g. [14], had high complexity: $\mathcal{O}(n^8)$ [29] in the dimension n and with expression growth $\mathcal{O}(2^{n^2})$. A key step is the computation of the Frobenius form, and the current best complexity algorithm is $\mathcal{O}(n^3)$ field operations, and keeps expression swell to a minimum [31]. Boolean circuit complexity results can be found in [21].

Inclusion of symbolic parameters makes things much more complicated and expensive, of course. Early work by Guoting Chen, who used with a single parameter [6] does not seem to have been improved upon. Some modern computer algebra systems simply give up when asked to compute the JCF of a matrix bigger than 5×5 that contains a parameter as we showed in Section 1.

There has been a significant body of relevant computational algebraic work, in computing the Frobenius form, the Zigzag form, and the Smith form [31, 32] but relatively few works [1, 6] on matrices with parameters. The difficulty appears to be combinatorial growth in the number of possible different cases. In the context of solving parametric linear systems, not eigenvalues, a significant amount of work has been done [2, 4, 9, 10, 22, 30]. Parametric nonlinear systems are studied in [25, 27, 36] and the references therein.

3 Preliminaries

Sections 3.1 and 3.2 gather the basic concepts and results from polynomial algebra that are needed in this paper. Meanwhile, Sections 3.3 and 3.4 review the notions of the Frobenius canonical form and the Jordan canonical form.

3.1 Regular chain theory

Let $\mathbb{K}$ be a field and $\overline{\mathbb{K}}$ its algebraic closure. Let $X_1 < \cdots < X_s$ be $s \geq 1$ ordered variables. We denote by $\mathbb{K}[X]$ the ring of polynomials in the variables $X = X_1, \ldots, X_s$ with coefficients in $\mathbb{K}$. For $F \subset \mathbb{K}[X]$, we denote by $\langle F \rangle$ and $V(F)$, the ideal generated by F in $\mathbb{K}[X]$ and the algebraic set of $\overline{\mathbb{K}}^s$ consisting of the common roots of the polynomials of F. For a non-constant polynomial $p \in \mathbb{K}[X]$, the greatest variable of p is called the *main variable* of p and denoted $\mathrm{mvar}(p)$, and the leading coefficient of p w.r.t. $\mathrm{mvar}(p)$ is called the *initial* of p, denoted by $\mathrm{init}(p)$. The Zariski closure of $W \subseteq \overline{\mathbb{K}}^s$, denoted by $\overline{W}$, is the intersection of all algebraic sets $V \subseteq \overline{\mathbb{K}}^s$ such that $W \subseteq V$ holds.

A set $T \subset \mathbb{K}[X] \setminus \mathbb{K}$ is *triangular* if $\mathrm{mvar}(t) \neq \mathrm{mvar}(t')$ holds for all $t \neq t'$ in T. Let h_T be the product of the initials of the polynomials in T. We denote by $\mathrm{sat}(T)$ the *saturated ideal* of T; if T is empty, then $\mathrm{sat}(T)$ is defined as the trivial ideal $\langle 0 \rangle$, otherwise it is the ideal $\langle T \rangle : h_T^{\infty}$. The *quasi-component* $W(T)$ of T is defined as $V(T) \setminus V(h_T)$. The following property holds: $\overline{W(T)} = V(\mathrm{sat}(T))$.

A triangular set $T \subset \mathbb{K}[X]$ is a *regular chain* if either T is empty, or the set T' is a regular chain, and the initial of p is regular (that is, neither zero nor zero divisor) modulo $\mathrm{sat}(T')$, where p is the polynomial of T with largest main variable, and $T' := T \setminus \{p\}$. Let $T \subset \mathbb{K}[X]$ be a regular chain. If T contains s polynomials $t_1(X_1), t_2(X_1, X_2), \ldots, t_s(X_1, \ldots, X_s)$, then T generates a zero-dimensional ideal which is equal to $\mathrm{sat}(T)$. If, in addition, the ideal $\mathrm{sat}(T)$ is prime (and, thus maximal in this case), then T is an encoding of the field extension $\mathbb{L} := \mathbb{K}[X]/\langle T \rangle$. Let $H \subset \mathbb{K}[X]$. The pair $[T, H]$ is a *regular system* if each polynomial in H is regular modulo $\mathrm{sat}(T)$; the zero set of $[T, H]$, denoted by $Z(T, H)$, consists of all points of $\mathbb{K}^s$ satisfying $t = 0$ for all $t \in T$, $h \neq 0$ for all $h \in H \cup \{h_T\}$. A regular chain T, or a regular system $[T, H]$, is *square-free* if for all $t \in T$, the polynomial $\mathrm{der}(t)$ is regular w.r.t. $\mathrm{sat}(T)$, where $\mathrm{der}(t) = \frac{\partial t}{\partial v}$ and $v = \mathrm{mvar}(t)$.

The zero set S of an arbitrary system of polynomial equations and inequations is called a *constructible set* and can be decomposed as the union of the zero sets of finitely many square-free regular systems $[T_1, H_1], \ldots, [T_e, H_e]$. When this holds we have $S = Z(T_1, H_1) \cup \cdots \cup Z(T_e, H_e)$ and we say that $[T_1, H_1], \ldots, [T_e, H_e]$ is a *triangular decomposition* of S.

We specify below a core routine thanks to which triangular decompositions can be computed. For more details about the theory of regular chains and its algorithmic aspects, we refer to [5].

Notation 1 *The function* Squarefree_RC(p, T, H) *computes a set of triples* $((b_{i,1}, \ldots, b_{i,\ell_i}), T_i, H_i)$ *with* $1 \leq i \leq e$*, such that* $[T_1, H_1], \ldots, [T_e, H_e]$ *are regular systems forming a triangular decomposition of* $Z(T, H)$*, and for all* $1 \leq i \leq e$*:*

1. *$b_{i,1}, \ldots, b_{i,\ell_i}$ are polynomials with the same main variable $v = \mathrm{mvar}(p)$ such that we have $p \equiv \prod_{j=1}^{\ell_i} b_{i,j}^j \mod \mathrm{sat}(T_i)$,*
2. *all discriminants $\mathrm{discr}(b_{i,j}, v)$ and all resultants $\mathrm{res}(b_{i,j}, b_{i,k}, v)$ are regular modulo $\mathrm{sat}(T_i)$, thus $\prod_{j=1}^{\ell_i} b_{i,j}^j$ is a square-free factorization of p modulo $\mathrm{sat}(T_i)$.*

3.2 Regular chain representation of a splitting field

Let $p(x) \in K[x]$ be a monic univariate polynomial. The *splitting field* of $p(x)$ over $\mathbb{K}$ is the smallest field extension of $\mathbb{K}$ over which $p(x)$ splits into linear factors,

$$p(x) = \prod_{i=1}^{\ell} (x - r_i)^{m_i} \ . \tag{1}$$

The set $\{r_1, \ldots, r_\ell\}$ generates $\mathbb{L}$ over $\mathbb{K}$. That is, $\mathbb{L} = \mathbb{K}(r_1, \ldots, r_\ell)$.

Assume that $p(x)$ is an irreducible, monic polynomial in $\mathbb{K}[x]$ of degree $n \geq 2$. To construct the splitting field $\mathbb{L}$ of $p(x)$ and compute the factorization of $p(x)$ into linear factors over $\mathbb{L}$, we proceed as follows.

1. Initialize $i := 1$, $y_i := x$, $\mathbb{L} := \mathbb{K}$, $T := \{\}$, $\mathcal{P} := \{\}$ and $\mathcal{F} := \{p\}$; the set $\mathcal{F}$ is assumed to maintain a list of univariate polynomials in y_i, irreducible over the current value of $\mathbb{L}$ and, of degree at least two,
2. While $\mathcal{F}$ is not empty do
   - (S1) pick a polynomial $f(y_i) \in \mathcal{F}$ over $\mathbb{L}$,
   - (S2) let α_i be a root of $f(y_i)$ (in the algebraic closure of $\mathbb{K}$),
   - (S3) replace $\mathbb{L}$ by $\mathbb{L}(\alpha_i)$, that is, by adjoining α_i to $\mathbb{L}$,
   - (S4) replace T by $T \cup \{t_i(y_1, \ldots, y_i)\}$, where the multivariate $t_i(y_1, \ldots, y_i)$ is obtained from $f(y_i)$ after replacing the algebraic numbers $\alpha_1, \ldots, \alpha_{i-1}$ with the variables $y_1, \ldots, y_{i-1}$,
   - (S5) replace $\mathcal{P}$ by $\mathcal{P} \cup \{x - y_i\}$,
   - (S6) factor $f(y_i)$ into irreducible factors over $\mathbb{L}$, then add the obtained factors of degree 1 (resp. greater than 1) to $\mathcal{P}$ (resp. $\mathcal{F}$); when adding a factor to $\mathcal{P}$, replace $\alpha_1, \ldots, \alpha_{i-1}$ with $y_1, \ldots, y_{i-1}$; when adding a factor to $\mathcal{F}$, replace y_i with y_{i+1}.
   - (S7) if $\mathcal{F}$ is not empty then $i := i + 1$.
3. Set $s := i$ and return $(s, T, \mathcal{P})$.

At the end of this procedure, the set T is a regular chain in the polynomial ring $\mathbb{K}[y_1, \ldots, y_s]$ generating a maximal ideal such that $\mathbb{K}[y_1, \ldots, y_s]/\langle T \rangle$ is isomorphic to the splitting field $\mathbb{K}(p)$ of $p(x)$. This procedure can be derived from S. Landau's paper [24]; note that the factorization at Step (S6) can be performed, for instance, by the algorithm of B. Trager [33]. Example: with $p(x) = x^3 - 2$, one can find $T = \{y_1^3 - 2, y_2^2 + y_1 y_2 + y_1^2\}$ and $\mathcal{P} = \{x - y_1, x - y_2, x + y_2 + y_1\}$.

3.3 The Frobenius canonical form

Throughout the sequel of this section, we denote by A a square matrix of dimension n with entries in a field $\mathbb{K}$.

Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a monic polynomial in $\mathbb{K}[x]$. The *Frobenius companion matrix*[1] of $p(x)$ is a square $n \times n$ matrix of the form

$$C(p(x)) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}. \tag{2}$$

A matrix $F \in \mathbb{K}^{n\times n}$ is said to be in *Frobenius (rational) canonical form* if it is a block diagonal matrix where the blocks are companion matrices of monic polynomials $\psi_i(x) \in \mathbb{K}[x]$

$$F = \bigoplus_{i=1}^{m} C(\psi_i(x)) \tag{3}$$

such that $\psi_{i-1} \mid \psi_i$ for $i = 1, \ldots, m-1$. The polynomials ψ_i are the *invariant factors* of F. We recall a few properties below, see [12, 15, 21] for details:

1. Every companion matrix is in Frobenius canonical form.
2. For all $i = 1, \ldots, m$, the companion matrix $C(\psi_i)$ is non-derogatory[2].
3. There exists a nonsingular matrix $Q \in \mathbb{K}^{n\times n}$ such that $F := Q^{-1}AQ$ is in Frobenius canonical form. The matrix F is called the *Frobenius canonical form* of A and the matrices A and F are said to be *similar*. We note that A and F have the same invariant factors.
4. The polynomial ψ_1 is the *minimal polynomial* of F and the product $\prod \psi_i$ is the *characteristic polynomial* of F.

3.4 The Jordan canonical form

An element $\lambda \in \overline{\mathbb{K}}$ is an *eigenvalue* of A if it satisfies $\det(A - \lambda I_n) = 0$ where I_n is the identity matrix of dimension n. The *algebraic multiplicity* of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial of A, and its *geometric multiplicity* is the dimension of the null space of $A - \lambda I_n$.

Let $F = \mathrm{diag}(C(\psi_1), C(\psi_2), \ldots, C(\psi_m))$ be the Frobenius form of A where $C(\psi_i)$ is the companion matrix of the ith invariant factor ψ_i of A. We note that the geometric multiplicity of an eigenvalue λ of A is the number of invariant factors that λ is a solution for. Thus, the Frobenius form of A tells us both the algebraic and geometric multiplicities of all eigenvalues of A.

A matrix is called a *Jordan block* of dimension n if it is zero everywhere except for ones along its superdiagonal, and a single value λ along its main diagonal. A Jordan block has one eigenvalue λ with geometric multiplicity 1 and algebraic multiplicity n. We use the notation $\mathrm{JBM}_n(\lambda)$ to denote a Jordan block of dimension n with eigenvalue λ.

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[1] There are many other companion matrices, but in this paper a "companion matrix" is a Frobenius companion matrix.

[2] The characteristic polynomial and the minimal polynomial coincide up to a factor of ± 1.

Let F be a matrix in Frobenius form as in Equation (3). The *Jordan canonical form* of F is given by

$$J = \bigoplus_{i=1}^{m} \mathrm{JCF}(C(\psi_i(x))) \tag{4}$$

where $\mathrm{JCF}(C(\psi(x)))$ is the Jordan form of a companion matrix of $\psi(x)$, see Chapter VI, §6 of [12] for a proof.

4 JCF Over a Splitting Field

4.1 Jordan form of a companion matrix

Let $\psi(x) \in \mathbb{K}[x]$ be a univariate monic polynomial of degree n. Let $\mathbb{L}$ be the splitting field of $\psi(x)$ over $\mathbb{K}$. Let $C = C(\psi(x))$ be the companion matrix of $\psi(x)$. Assume that the complete factorization into linear factors of $\psi(x)$ writes

$$\psi(x) = \prod_{i=1}^{\ell} (x - r_i)^{m_i} \tag{5}$$

where $r_i \in \mathbb{L}$ for $i = 1 \ldots \ell$ and $r_i \neq r_j$ for $i \neq j$. Then, the Jordan form of C is given by

$$J = \bigoplus_{i=1}^{\ell} \mathrm{JBM}_{m_i}(r_i) \tag{6}$$

where the entries of J are in $\mathbb{L}$. Thus, once the splitting field of $\psi(x)$ is computed, the Jordan canonical form of the companion matrix of $\psi(x)$ can be constructed.

Using the algorithm described in Section 3.2, the roots $r_1, \ldots, r_\ell$ of $\psi(x)$ are represented by the residue classes of multivariate polynomials $r_1(y_1, \ldots, y_s), \ldots, r_\ell(y_1, \ldots, y_s)$ modulo $\langle T \rangle$, since the regular chain $T = t_1(y_1), \ldots, t_s(y_1, \ldots, y_s)$ encodes the splitting field $\mathbb{K}(\psi)$ of $\psi(x)$ in the sense that this field is isomorphic to $\mathbb{K}[y_1, \ldots, y_i]/\langle T \rangle$. Therefore, the Jordan form of C is given by

$$\bigoplus_{i=1}^{\ell} \mathrm{JBM}_{m_i}(r_i(y_1, \ldots, y_s)) \tag{7}$$

together with the regular chain T.

4.2 Frobenius form to Jordan form

Let $F \in \mathbb{K}^{n \times n}$ be in Frobenius form, with $F = \mathrm{diag}(C(\psi_1), C(\psi_2), \ldots, C(\psi_m))$, where the polynomials ψ_i are the invariant factors of F. By Equation (4), the Jordan form of F is given by $J = \bigoplus_{i=1}^{m} \mathrm{JCF}(C(\psi_i))$ and a regular chain T defining the splitting field of ψ_1. This is, indeed, sufficient to compute all the entries of the JCF of F, since every subsequent polynomial ψ_i divides ψ_1.

4.3 Example

Let $\psi(x) = (x^3 + x^2 + x - 1)(x^2 + x + 1)^2$, where the coefficients are in $\mathbb{Q}$. Let C be the companion matrix of ψ. The JCF of C over the splitting field $\mathbb{L}$ of ψ over $\mathbb{Q}$ is

$$\begin{bmatrix} y_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 - y_1 - y_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 - y_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & & -1 - y_2 \end{bmatrix}$$

where (y_1, y_2, y_3) are any point in the zero set $V(T)$ where T is

$$T = \{y_1^2 + (1 + y_3)y_1 + y_3^2 + y_3 + 1, y_2^2 + y_2 + 1, y_3^3 + y_3^2 + y_3 - 1\}.$$

5 JCF of a Matrix with Parameters

In this section we show how to compute a complete case discussion for the JCF of a matrix F in Frobenius form where the entries are polynomials in $\mathbb{K}[\alpha_1, \ldots, \alpha_s]$. Note that, as Arnol'd points out in [1], a parametric Frobenius form is continuous in its parameters, though its Jordan form may not be. Throughout this section, $T \subset \mathbb{K}[\alpha]$ will be a regular chain and $H \subset \mathbb{K}[\alpha]$ a set of polynomial inequations such that $[T, H]$ forms a regular system.

5.1 Square-free factorization of a parametric polynomial

Let $\alpha_1 < \cdots < \alpha_s$ be $s \geq 1$ ordered variables. Let $\mathbb{K}[\alpha] = \mathbb{K}[\alpha_1, \ldots, \alpha_s]$ be the ring of polynomials in the variables $\alpha = \alpha_1, \ldots, \alpha_s$. Let x be a variable. Let $\mathbb{K}[x]$ (resp. $\mathbb{K}[\alpha][x]$) be the ring of polynomials in x with coefficients in $\mathbb{K}$ (resp. $\mathbb{K}[\alpha]$). A polynomial $p(x; \alpha) \in \mathbb{K}[\alpha][x]$ is called a *univariate, parametric polynomial* in x and takes the form

$$p(x; \alpha) = a_n(\alpha)x^n + \cdots + a_1(\alpha)x + a_0(\alpha) \tag{8}$$

where the coefficients $a_i(\alpha)$ are polynomials in $\mathbb{K}[\alpha]$.

Let $p(x; \alpha) = \prod_{i=1}^{\ell} b_i(x; \alpha)^i$ be a square-free factorization of $p(x; \alpha)$, regarded as a univariate polynomial in $\mathbb{K}[\alpha][x]$. Then, the following properties must hold:

1. each polynomial $b_i(x; \alpha)$ is square-free as a polynomial in $\mathbb{K}[\alpha][x]$, and
2. the GCD of $b_i(x; \alpha)$ and $b_j(x; \alpha)$, as polynomials in $\mathbb{K}[\alpha][x]$, has degree zero in x, for all $1 \leq i < j \leq \ell$.

We note that each of the square-free factors $b_1, \ldots, b_\ell$ of $p(x; \alpha)$ is uniquely defined up to a multiplicative element of $\mathbb{K}[\alpha]$.

Definition 1 *We say that the sequence of polynomials* $b_1, \ldots, b_\ell$ specializes well *at a point* $\alpha^* = (\alpha_1^*, \ldots, \alpha_s^*) \in \overline{\mathbb{K}}^s$ *whenever*

1. *the degree in x of the specialized polynomial $b_i(x;\alpha^*)$ is the same as the degree in x of b_i as a polynomial in $\mathbb{K}[\alpha][x]$, for all $1 \le i \le \ell$;*
2. *each specialized polynomial $b_i(x;\alpha^*)$ is square-free, as a polynomial in $\mathbb{K}[x]$, for all $1 \le i \le \ell$; and*
3. *the GCD of $b_i(x;\alpha^*)$ and $b_j(x;\alpha^*)$, as polynomials in $\mathbb{K}[x]$, has degree zero in x, for all $1 \le i < j \le \ell$.*

From the theory of *border polynomials* [25, 27, 36] the following result holds.

Proposition 1 *The set of points $\alpha \in \overline{\mathbb{K}}^s$ at which the sequence of polynomials $b_1, \ldots, b_\ell$ specializes well is the complement of the algebraic set given by*

$$\left\{ \bigcup_{i=1}^{i=e} V(\Delta_i) \right\} \quad \cup \quad \left\{ \bigcup_{1 \le i < j \le e} V(R_{i,j}) \right\}, \tag{9}$$

where $\Delta_i := \mathrm{discr}(b_i(x;\alpha), x)$ denotes the discriminant of $b_i(x;\alpha)$ w.r.t. x and $R_{i,j} := \mathrm{res}(b_i(x;\alpha), b_j(x;\alpha), x)$ denotes the resultant of $b_i(x;\alpha)$ and $b_j(x;\alpha)$ w.r.t. x.

Definition 2 *We call the* proviso *of the sequence of polynomials $b_1, \ldots, b_\ell$ the algebraic set (actually hypersurface) given by Equation* (9) *and denote it by* $\mathsf{Proviso}(b_1, \ldots, b_\ell)$. *We call the* square-free factorization with proviso *of $p(x;\alpha)$ the pair $(\prod_{i=1}^{\ell} b_i(x;\alpha)^i, \mathsf{Proviso}(b_1, \ldots, b_\ell))$.*

We note that the zero set of the border polynomial of $p(x;\alpha)$ (in the sense [27, 36]) is usually defined whenever $p(x;\alpha)$ is square-free w.r.t. x, in which case it coincides with $\mathsf{Proviso}(b_1, \ldots, b_\ell)$.

We are now interested in obtaining a complete case discussion for the square-free factorization of $p(x;\alpha)$, that is, including the cases where $\alpha^* \in \mathsf{Proviso}(p(x;\alpha), x)$ holds. This can be achieved by using the function $\mathsf{Squarefree\_RC}(p, T, H)$ specified in Section 3.1.

5.2 JCF of a companion matrix with parameters

From now on, we assume that the field $\overline{\mathbb{K}}$ is $\mathbb{C}$. Let $C \in \mathbb{K}[\alpha]^{n \times n}$ be a companion matrix with characteristic polynomial $\psi(x;\alpha) \in \mathbb{K}[\alpha][x]$. Let $\prod_{i=1}^{\ell} b_i(x;\alpha)^i$ be a square-free factorization of $\psi(x;\alpha)$. We observe that in the complement of $\mathsf{Proviso}(b_1, \ldots, b_\ell)$, the roots (in x) of $b_1, \ldots, b_\ell$, as functions of α, define continuous, disjoint graphs. Let us denote those functions by $\lambda_{i,1}, \ldots, \lambda_{i,n_i}$ corresponding to the polynomial b_i, for $1 \le i \le \ell$. Therefore, one can construct the JCF of C uniformly over the complement of $\mathsf{Proviso}(b_1, \ldots, b_\ell)$ as follows

$$\bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{n_i} \mathrm{JBM}_i(\lambda_{i,j}) \,. \tag{10}$$

More generally, for a regular system $[T, H]$ let $((b_{i,1}, \ldots, b_{i,\ell_i}), T_i, H_i)$, with $1 \le i \le e$, be the output of $\mathsf{Squarefree\_RC}(\psi(x;\alpha), T, H)$. Then, for every $1 \le i \le e$, one can construct the JCF of C uniformly over $Z(T_i, H_i)$ as the regular systems $[T_1, H_1], \ldots, [T_e, H_e]$ form a triangular decomposition of $Z(T, H)$.

5.3 Frobenius form to JCF with parameters

Let $F \in \mathbb{K}[\alpha]^{n\times n}$ be a matrix in Frobenius form with invariant factors $\psi_i(x;\alpha) \in \mathbb{K}[\alpha][x]$ for $1 \leq i \leq m$. Let $\prod_{i=1}^{\ell} b_i(x;\alpha)^i$ be a square-free factorization of the minimal polynomial, $\psi_1(x;\alpha)$. The JCF over the complement of $\mathsf{Proviso}(b_1,\ldots,b_\ell)$ is defined continuously for each companion matrix $C(\psi_i(x;\alpha))$, $1 \leq i \leq m$. This is a consequence of the property that each subsequent $\psi_i(x;\alpha)$ divides $\psi_1(x;\alpha)$.

The construction of the JCF of $C(\psi_1(x;\alpha))$ defines a decomposition of the complement of $\mathsf{Proviso}(b_1,\ldots,b_\ell)$ into the zero sets of finitely many square-free regular systems $[T_1,H_1],\ldots,[T_e,H_e]$. Over each regular system, the JCF of each companion matrix $C(\psi_i(x;\alpha))$ for $1 \leq i \leq m$ is defined continuously.

6 Experimentation

We are actively developing a package called `ParametricMatrixTools` in Maple that implements algorithms for computations on matrices with parameters. The source for this package, including numerous examples, is available at github.com/StevenThornton/ParametricMatrixTools and is compatible the version of the `RegularChains` library included in Maple 2016 and later. The `ComprehensiveJordanForm` method implements the algorithm discussed in section 5. Further details can be found at regularchains.org.

For each of the examples that follow, we have first computed a full case discussion for the Frobenius form using the `ComprehensiveFrobeniusForm` routine in our package. The details of the Frobenius form implementation have been omitted and we are actively working to improve our current implementation.

6.1 Kac-Murdock-Szegö matrices

The inverse matrix $K_n^{-1}(\rho)$ from [34] is

$$\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}.$$

The cost to compute a full case discussion of the JCF of $(1-\rho^2)K_n^{-1}(\rho)$ grows exponentially with n. See figure 2.

6.2 The Belousov-Zhabotinskii reaction

The report [16] contains a very readable account of the famous B-Z reaction and its history. This is a chemical oscillator. In non-dimensional form with $\varepsilon = \delta = 1$

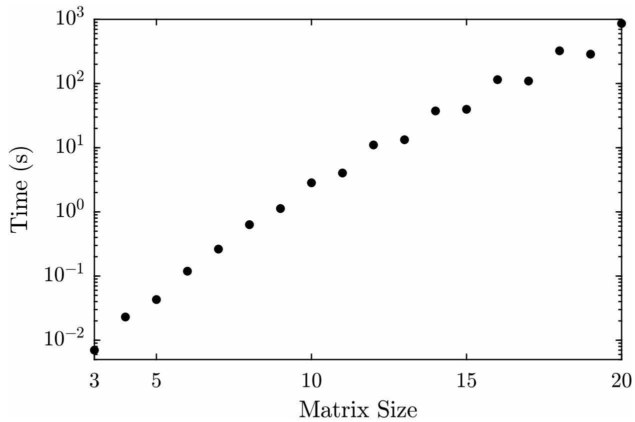

Fig. 2. Time to compute the JCF of each Frobenius form in the full case discussion of the Frobenius form of the matrix in section 6.1. For all n, the Frobenius form splits into two cases: $\rho = 0$ and $\rho \neq 0$. The JCF is computed over each of these branches. Note the exponential growth. Timing was done on a 2016, 3.3GHz quad-core Intel Core i7 iMac with 16GB of RAM using Maple 2016.2.

we have

$$\begin{aligned}\dot{x} &= qy - xy + x(1-x)\\ \dot{y} &= -qy - xy + fz\\ \dot{z} &= x - z\end{aligned}$$

The equilibria include $x = z$ being a positive root of the quadratic

$$x(x - 1 + f) + q(x - 1 - f) = 0\,. \tag{11}$$

The Jacobian at the equilibrium is

$$A = \begin{bmatrix} 1 - x - y & q - x & 0 \\ -y & -(q + x) & f \\ 1 & 0 & -1 \end{bmatrix} \tag{12}$$

and the Jordan form of A splits into many cases. One non-trivial example is

$$J = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{bmatrix} \tag{13}$$

where

$$\begin{aligned}\alpha &= \frac{1}{9994}(-81q^5 + 804q^4 - 3882q^3 + 12209q^2 - 6288q - 59636)\\ \beta &= \frac{1}{2}(-\alpha + 3q - 10)\end{aligned}$$

under the following constraints on the indeterminates of A:

$$
\begin{gathered}
x = z = -2y \\
f = -1 \\
(q^5 - 13q^4 + 86q^3 - 359q^2 + 911q - 742)z - 4q^2 - 8 = 0 \\
q^6 - 15q^5 + 112q^4 - 531q^3 + 1633q^2 - 2564q + 1492 = 0\,.
\end{gathered}
$$

There are real values of q satisfying this equation, and hence this case is real.

6.3 Nuclear magnetic resonance

In [18], section 2.2, we find a concise description of an application of the matrix exponential to solve the so-called Solomon equations

$$
\dot{M} = -RM, \quad M(0) = I \quad \text{by} \quad M(t) = e^{-Rt}\,. \tag{14}
$$

Here R is a symmetric, diagonally dominant matrix called the relaxation matrix, and M is the matrix of intensities. Suppose R is in fact tridiagonal, with ones on the sub- and super-diagonals, and diagonal parameters $|r_i| > 1$. Using MAPLE's built-in `MatrixExponential` gets answers (e.g. when the dimension n is 3) but we are not convinced that the generic answer returned is correct, always. So we try computing the JCF. Doing so, we find that indeed there are special cases that the generic code missed. For example, when R is of dimension 3, the JCF of R is

$$
\begin{bmatrix} (r_1+r_2+r_3)/3 & 1 & \\ 0 & (r_1+r_2+r_3)/3 & 1 \\ 0 & 0 & (r_1+r_2+r_3)/3 \end{bmatrix} \tag{15}
$$

when

$$
r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_1r_3 - r_2r_3 + 6 = 0 \tag{16}
$$

$$
((r_1 - r_3)^2 - 1)((r_1 - r_3)^2 + 8) = 0\,. \tag{17}
$$

When $\mathrm{discr}(\texttt{CharPoly}(A)) \neq 0$ the JCF is simply $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ for the distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$. And for the remaining parameter values, the JCF consists of a Jordan block of dimension 2 with eigenvalue λ_1, and a Jordan block of dimension 1 with eigenvalue λ_2 for $\lambda_1 \neq \lambda_2$. The only case corresponding to real values of r_1, r_2, r_3 is the trivial diagonal case. In the cases where the JCF is not a diagonal matrix, the result computed by the `MatrixExponential` function in MAPLE contains discontinuities.

6.4 Bifurcation studies

The mathematical methods used in bifurcation studies are highly sophisticated, both symbolically and numerically. Tools used include normal forms and the action of symmetry groups. Consider the matrix

$$
J = \begin{bmatrix} 0 & 2\rho & 0 \\ a & 2\beta & 2v \\ b & -2v & 2\beta \end{bmatrix} \tag{18}
$$

which is the Jacobian matrix of a dynamical system at equilibrium. The analysis of this system in [35] is quite complete, yet the evolution of trajectories near the equilibria, governed by

$$\xi' = J\xi, \quad \xi(0) = I \tag{19}$$

or $\xi = \exp(tJ)$, is of interest. When the JCF of J is nontrivial, one can anticipate phenomena such as greater sensitivity to modelling error, for instance. Our implementation is able to find a complete case discussion of the JCF, starting from the complete case discussion of the Frobenius form, in approximately 2 seconds. We find cases corresponding to each of the 5 possible Jordan structures for a 3×3 matrix with a total of 46 cases. Of the 46 cases, 14 are defined by polynomials of total degree greater than 4. The worst case contains a polynomial of degree 12 in the parameters with 19 terms.

One non-trivial case we were able to automatically identify is where the JCF of J is given by

$$\begin{bmatrix} 2\beta & 0 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{bmatrix} \tag{20}$$

when $2\rho a + \beta^2 = 0$, $v = 0$, and a, ρ and β are non-zero.

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