Polynomials over Power Series and their Applications to Symbolic Analysis

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Plan

Plan

Does the parametrization reach all points of the surface? (1/8)

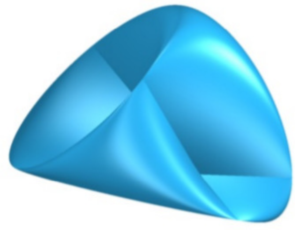


Figure: Steiner's Roman surface

https://upload.wikimedia.org/wikipedia/commons/e/ea/Steiner%27s_Roman_Surface.gif

An implicit formula of Steiner's Roman surface S is $f = 0$, where:

$$\begin{array}{l} f := 4\,x^4 - 8\,yx^3 + 9\,x^2y^2 - 8\,yzx^2 - 5\,y^3x + 8\,y^2zx + y^4 \\ -2\,y^3z + 3\,y^2z^2 - 2\,yz^3 + z^4 - 8\,yx^2 + 8\,zx^2 + 8\,y^2x \\ -8\,xyz - 2\,y^3 + 2\,y^2z - 2\,yz^2 + 4\,x^2 - 4\,yx + y^2. \end{array} \quad (1)$$

Does the parametrization reach all points of the surface? (2/8)

- With $q(s,t) := s^2 + t^2 + s - t + 1$, consider also the following map

$$\vec{r} : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (s,t) & \mapsto & \left(\frac{s^2}{q(s,t)} \, , \, \frac{s^2+t^2}{q(s,t)} \, , \, \frac{s^2+s\,t+s+t}{q(s,t)} \right) , \end{array} \quad (2)$$

- Do we have $\mathrm{Image}(\vec{r}) = S$?
- A preliminary question is whether $q(s,t)$ vanishes or not.

```
> R := PolynomialRing([s, t, x, y, z]) : q := s^2 + t^2 + s - t + 1 :
  RealTriangularize([q], R);

                                  [ ]
```

Figure: `RegularChains:-RealTriangularize` proves $q(s,t)$ has no real points.

Does the parametrization reach all points of the surface? (3/8)

Let us verify that the image of the map $\vec{r}$ is contained in the surface S.

$$> f := 4\cdot x^4 - 8\cdot y\cdot x^3 + 9\cdot x^2\cdot y^2 - 8\cdot y\cdot z\cdot x^2 - 5\cdot y^3\cdot x + 8\cdot y^2\cdot z\cdot x + y^4 - 2\cdot y^3\cdot z + 3\cdot y^2\cdot z^2 - 2\cdot y\cdot z^3 + z^4 - 8\cdot y\cdot x^2 + 8\cdot z\cdot x^2 + 8\cdot y^2\cdot x - 8\cdot x\cdot y\cdot z - 2\cdot y^3 + 2\cdot y^2\cdot z - 2\cdot y\cdot z^2 + 4\cdot x^2 - 4\cdot y\cdot x + y^2 :$$

```
R := PolynomialRing([s, t, x, y, z]) :
dec1 := Triangularize([f], R); S := GeneralConstruct(dec1[1], map(Initial,
    Equations(dec1[1], R), R), R);
```

dec1 := [regular_chain]

S := constructible_set

$$> q := s^2 + t^2 + s - t + 1 :$$

$$F := [q\cdot x - s^2,\ q\cdot y - (s^2 + t^2),\ q\cdot z - (s^2 + s\cdot t + s + t)] :$$

```
dec2 := Triangularize(F, R); ImageR := GeneralConstruct(dec2[1], map(Initial, F, R), R);
```

dec2 := [regular_chain]

ImageR := constructible_set

```
> LM1 := Difference(ImageR, S, R); IsEmpty(LM1, R);
```

LM1 := constructible_set

true

Figure: The command `Difference` computes the points in the image of $\vec{r}$ that do not belong to surface S, which is empty.

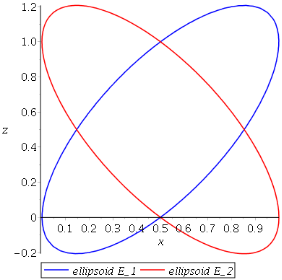
Does the parametrization reach all points of the surface? (4/8)

- Disproving $\mathrm{Image}(\vec{r}) = S$ can be done by specialization
- Computing $\mathrm{Image}(\vec{r}) \cap \{y = 1\}$ yields

$$2\,x^2 + 2\,x\,z + z^2 - 3\,x - 2\,z + 1 = 0$$

- While computing $S \cap \{y = 1\}$ brings more:

$$(2\,x^2 - 2\,x\,z + z^2 - x)\,(2\,x^2 + 2\,x\,z + z^2 - 3\,x - 2\,z + 1) = 0$$



```
> R := PolynomialRing([s, t, x, y, z]):
  q := s^2 + t^2 + s - t + 1:
  F := [x*q - s^2, y*q - (s^2 + t^2), z*q-(s^2 + s*t + s + t)]:
  dec2 := Projection([op(F), y - 1], [ ], [ ], [ ], 3, R): Display(%, R)
```

$$\left[\left\{\begin{array}{c} 4x+2z-3=0 \\ y-1=0 \\ 4z^2-4z-1=0 \end{array}\right., \left\{\begin{array}{c} x=0 \\ y-1=0 \\ z-1=0 \end{array}\right., \left\{\begin{array}{c} 2x^2+(2z-3)x+z^2-2z+1=0 \\ y-1=0 \\ 4z^2-4z<1 \text{ and } z-1\neq 0 \end{array}\right.\right]$$

```
> f := 4 x^4 - 8 y x^3 + (9 y^2 + (-8 z - 8) y + 8 z + 4) x^2 + (-5 y^3 + (8 z + 8) y^2 + (-8 z - 4) y) x
      + y^4 + y^3 (-2 z - 2) + (3 z^2 + 2 z + 1) y^2 + (-2 z^3 - 2 z^2) y + z^4:
  R := PolynomialRing([s, t, x, y, z]):
  dec1 := RealTriangularize([f, y - 1], R): Display(dec1, R);
```

$$\left[\left\{\begin{array}{c} 2x^2+(2z-3)x+z^2-2z+1=0 \\ y-1=0 \\ 4z^2-4z<1 \end{array}\right., \left\{\begin{array}{c} 4x+2z-3=0 \\ y-1=0 \\ 4z^2-4z-1=0 \end{array}\right., \left\{\begin{array}{c} 2x^2+(-2z-1)x+z^2=0 \\ y-1=0 \\ 4z^2-4z<1 \end{array}\right., \left\{\begin{array}{c} 4x-2z-1=0 \\ y-1=0 \\ 4z^2-4z-1=0 \end{array}\right.\right]$$

Does the parametrization reach all points of the surface? (6/8)

```
> Difference(dec1, dec2, R) : Display(%, R);
```

$$\left[\left\{\begin{array}{l} x-1=0 \\ y-1=0 \\ z-1=0 \end{array}\right., \left\{\begin{array}{l} x=0 \\ y-1=0 \\ z=0 \end{array}\right., \left\{\begin{array}{l} 2x-1=0 \\ y-1=0 \\ z-1=0 \end{array}\right., \left\{\begin{array}{l} 4x-2z-1=0 \\ y-1=0 \\ 4z^2-4z-1=0 \end{array}\right.,\right.$$

$$\left.\left\{\begin{array}{l} 2x^2+(-1-2z)x+z^2=0 \\ y-1=0 \\ 4z^2-4z<1 \textbf{ and } z\neq 0 \textbf{ and } z-1\neq 0 \textbf{ and } 2z-1\neq 0 \end{array}\right.\right]$$

Figure: The points on Steiner surface S and the plane $y=1$ which do not belong to the intersection of the image of the parametrization $\vec{r}$ and the plane $y=1$.

Observe that these calculations are done over the `reals`!

Does the parametrization reach all points of the surface? (7/8)

The next question

1. Therefore, $\mathrm{Image}(\vec{r}) = S$ does not hold!
2. Next question: can we recover from S what $\mathrm{Image}(\vec{r})$ is missing?
3. if the missing point are $\overline{\mathrm{Image}(\vec{r})} \setminus \mathrm{Image}(\vec{r})$, then the answer is yes.

The closure of a constructible set

1. Denote by $\overline{\mathrm{Image}(\vec{r})}$ the closure of $\mathrm{Image}(\vec{r})$ in the Euclidean topology (over $\mathbb{C}$).
2. Thanks to a theorem of David Mumford, $\overline{\mathrm{Image}(\vec{r})}$ is also the closure of $\mathrm{Image}(\vec{r})$ in Zariski topology.
3. Thus $\overline{\mathrm{Image}(\vec{r})}$ is the intersection of all algebraic sets containing $\mathrm{Image}(\vec{r})$.
4. By the way, Gröbner basis techniques can capture Zariski closures over algebraically closed fields.

Does the parametrization reach all points of the surface? (8/8)

```
> q := s^2 + t^2 + s - t + 1 :
  R := [x*q - s^2, y*q - (s^2 + t^2), z*q-(s^2 + s*t + s + t)] :
  with(PolynomialIdeals) :
  sat := Saturate(⟨op(R)⟩, q) :
  closure_of_Image_of_r := EliminationIdeal(sat, {x, y, z});
```

$$closure\_of\_Image\_of\_r := \langle 4x^4 - 8x^3y + 9x^2y^2 - 8x^2yz - 5xy^3 + 8xy^2z + y^4 - 2y^3z + 3y^2z^2 - 2yz^3 + z^4 - 8x^2y + 8x^2z + 8xy^2 - 8xyz - 2y^3 + 2y^2z - 2yz^2 + 4x^2 - 4xy + y^2\rangle$$

Figure: Closure of $\mathrm{Image}(\vec{r})$.

- We retrieve the polynomial defining the implicit representation of S
- According to the so-called *Elimination Theorem* (see the book *Ideals, varieties and Algorithms*) the algebraic set of the elimination ideal $\mathcal{I} \subset \mathbb{K}[x_1 < \cdots < x_n]$ w.r.t. $x_1, \ldots, x_k$ (for some $1 \leq k < n$) is equal to the Zariski closure of the projection of $V(\mathcal{I})$ onto $x_1, \ldots, x_k$.

Summary 1

- Computing Zariski closures of constructible sets (that is, systems of polynomial equations and inequation) and semi-algebraic sets (that is, systems of polynomial equations and inequalities) appear naturally in practice: reachable sets, projection of constructible sets and semi-algebraic sets.
- Gröbner basis techniques can deal with the case of constructible sets.
- We are mainly interested here with the real case, that is, semi-algebraic sets .

Topological closure and limit points

Let (X, τ) be a topological space and $S \subseteq X$ be a subset.

Topological closure

The *closure* of S, denoted $\overline{S}$, is the intersection of all closed sets containing S.

Limit point

- A point $p \in X$ is a *limit point* of S if every neighbourhood of p contains at least one point of S different from p itself.
- The limit points of S which do not belong to S are called non-trivial, denoted by $\lim(S)$.

Properties

- If X is a metric space, the point p is a limit point of S if and only if there exists a sequence $(x_n, n \in \mathbb{N})$ of points of $S \setminus \{p\}$ such that $\lim_{n\to\infty} x_n = p$.
- We have $\overline{S} = S \cup \lim(S)$.

Zariski topology and the Euclidean topology

The relation between the two topologies

- With $\mathbb{K} = \mathbb{C}$, the affine space $\mathbb{A}^s$ is endowed with both topologies.
- The basic open sets of the Euclidean topology are the open balls.
- The basic open sets of Zariski topology are the complements of hypersurfaces.
- Thus, a Zariski closed (resp. open) set is closed (resp. open) in the Euclidean topology on $\mathbb{A}^s$.
- That is, Zariski topology is coarser than the Euclidean topology.

The relation between the two closures (D. Mumford)

- Let $V \subseteq \mathbb{A}^s$ be an irreducible affine variety.
- Let $U \subseteq V$ be nonempty and open in Zariski topology induced on V.

Then, U has the same closure in both topologies. In fact, we have

$$V = \overline{U}^Z = \overline{U}^E.$$

Limit points: a first example

- Let S be the zero-set of a polynomial system and $\overline{S}$ be the topological closure S in the Euclidean topology.
- It can be proved that the set-theoretic difference $\overline{S} \setminus S$ can be obtained via a *limit computation process* illustrated below

Consider S below together with a Puiseux series expansion around $z = 0$:

$$S := \begin{cases} z\,x - y^2 = 0 \\ y^5 - z^4 = 0 \\ z \neq 0 \end{cases} \quad \text{and} \quad \begin{cases} x = \frac{t^{8/5}}{t} \\ y = t^{4/5} \\ z = t \\ t \neq 0 \end{cases}$$

Then we have:

$$\lim_{t \to 0} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \overline{S} \setminus S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Limit points: a second example

Consider S below together with a Puiseux series expansion around $z = 0$:

$$S := \left\{ \begin{array}{l} z\,x - y^2 = 0 \\ y^5 - z^2 = 0 \\ z \neq 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x = t^{-1/5} \\ y = t^{2/5} \\ z = t \\ t \neq 0 \end{array} \right.$$

Then we have:

$$\lim_{t \to 0} \left(\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array} \right) = \left(\begin{array}{c} \pm\infty \\ 0 \\ 0 \end{array} \right) \quad \text{and} \quad \overline{S} \setminus S = \emptyset$$

The Puiseux series solutions of a regular chain (1/2)

Regular chains in a nutshell

- Regular chains generalize the concept of *triangular system* from linear algebra to polynomial algebra.
- Thus, they are polynomial systems with a triangular shape and additional algebraic properties which support a back substitution process.
- Every (non-constant) bivariate polynomial forms a regular chain.

The solutions of a regular chain

- Like Gröbner bases, regular chains can be used to compute and describe the solutions of polynomial systems over algebraically closed fields, say $\mathbb{C}$.
- Regular chains can also be used to solve over real closed fields, say $\mathbb{R}$ but also Puiseux series.

The Puiseux series solutions of a regular chain (2/2)

```
> with(AlgebraicGeometryTools):
> R := PolynomialRing([x, y, z]):
> rc := Chain([-z^2+y, x*z-y^2], Empty(R), R):
> br := RegularChainBranches(rc, R, [z]);
                                                      2       3
                              br := [[z = T, y = T , x = T ]]
> rc := Chain([y^2*z+y+1, (z+2)*z*x^2+(y+1)*(x+1)], Empty(R),R):
> RegularChainBranches(rc, R, [z]);
                                    2        2
                         (T - 2) (T  + 4) (T  - 9 T - 54)
[[z = T, y = -T - 1, x = --------------------------------],
                                       432

                                     5   11   4          3          2
    [z = T, y = -T - 1, x = -1/432 T  + --- T  + 5/432 T  - 5/216 T  + 1/12 T - 1/2]]
                                         432
```

Limit points: yet another example

$R := PolynomialRing([x, y, z]) : rc := Chain([-y^\wedge 3 + y^\wedge 2 + z^\wedge 5, z^\wedge 4 * x + y^\wedge 3 - y^\wedge 2], Empty(R), R) :$
$Display(rc, R);$

$$\begin{cases} z^4 x + y^3 - y^2 = 0 \\ -y^3 + y^2 + z^5 = 0 \\ z^4 \neq 0 \end{cases}$$

$RegularChainBranches(rc, R, [z]);$

$$\Big[\Big[z = T^2, y = \frac{1}{2} T^5 \left(-T^5 + 2\, RootOf(\_Z^2 + 1)\right), x = -\frac{1}{8} T^2 \left(-T^{20} + 6\, T^{15}\, RootOf(\_Z^2 + 1) + 10\, T^{10} + 8\right)\Big], \Big[z = T^2, y = -\frac{1}{2} T^5 \left(T^5 + 2\, RootOf(\_Z^2 + 1)\right), x = \frac{1}{8} T^2 \left(T^{20} + 6\, T^{15}\, RootOf(\_Z^2 + 1) - 10\, T^{10} - 8\right)\Big], \left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\Big]$$

$lp := LimitPoints(rc, R) : Display(lp, R);$

$$\left[\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}, \begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0 \end{cases} \right]$$

Figure: Computation of (non-trivial) limit points with the `RegularChains` library

Limit points: statement of our quest

- Let $R := \{t_2(x_1, x_2), \ldots, t_n(x_1, \ldots, x_n)\}$
- We regard t_i as a univariate polynomial w.r.t. x_i, for $i = 2, \ldots, n$:
- We denote by h_i the leading coefficient (also called initial) of t_i w.r.t. x_i, and assume that h_i depends on x_1 only; hence we have
  $t_i \; = \; h_i(x_1)x_i^{d_i} + c_{d_i-1}(x_1, \ldots, x_{i-1})x_i^{d_i-1} + \cdots + c_0(x_1, \ldots, x_{i-1})$
- Consider the system

$$W(R) := \begin{cases} t_n(x_1, \ldots, x_n) = 0 \\ \vdots \\ t_2(x_1, x_2) = 0 \\ (h_2 \cdots h_n)(x_1) \neq 0 \end{cases}$$

Main Goal

- Where do the points of $W(R)$ go when x_1 approaches a root of $h_2 \cdots h_n$?
- In other words, we want to compute the points which belong to the topological closure of $W(R)$ but to $W(R)$ itself.

Limit points: yet again another example

```
> R := PolynomialRing([x, y, z]):
  rc := Chain([y^(3)-2*y^(3)+y^(2)+z^(5), z^(4)*x+y^(3)-y^(2)], Empty(R), R): Display(rc, R);
  br := RegularChainBranches(rc, R, [z], coefficient = complex);
```

$$\begin{cases} z^4 x + y^3 - y^2 = 0 \\ -y^3 + y^2 + z^5 = 0 \\ z^4 \neq 0 \end{cases}$$

$$br := \left[\left[z = T^2, y = \frac{1}{2} T^5 \left(-T^5 + 2\, RootOf(\_Z^2+1)\right), x = -\frac{1}{8} T^2 \left(-T^{20} + 6\, T^{15}\, RootOf(\_Z^2+1) + 10\, T^{10} + 8\right)\right], \left[z = T^2, y = -\frac{1}{2} T^5 \left(T^5 + 2\, RootOf(\_Z^2+1)\right), x = \frac{1}{8} T^2 \left(T^{20} + 6\, T^{15}\, RootOf(\_Z^2+1) - 10\, T^{10} - 8\right)\right], \left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

```
> br := RegularChainBranches(rc, R, [z], coefficient = real);
```

$$br := \left[\left[z = T, y = T^5 + 1, x = -T\left(T^{10} + 2\, T^5 + 1\right)\right]\right]$$

Figure: The command `RegularChainBranches` computes a parametrization for the complex and real paths of the quasi-component defined by rc. When coefficient argument is set as real, then the command `RegularChainBranches` computes the real branches.

Application 1: limit of multivariate rational functions

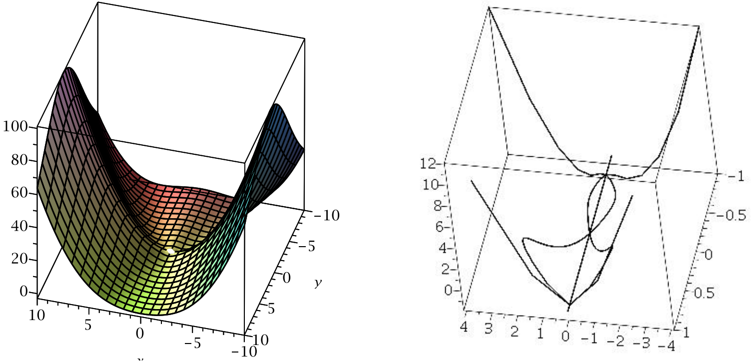


Figure: On the left: the surface defined by $q := \frac{x^4+3\,x^2\,y-x^2-y^2}{x^2+y^2} = z$ around the origin. On the right: the three paths of discriminant variety of q going through the point (0,0,-1).

Application 2: tangent cone computations

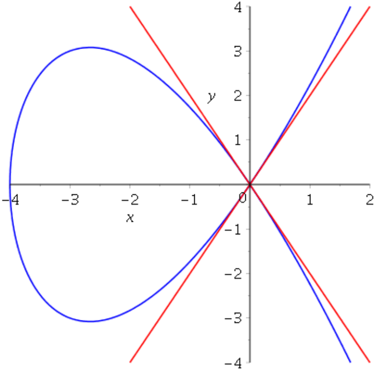
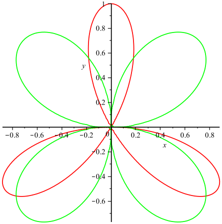


Figure: The tangent cone of the "fish" given by $f := y^2 - x^2\,(x+4) = 0$ at the origin consists of two tangent lines: $y = 2\,x$ and $y = -2\,x$.

Application 3: computing intersection multiplicities

$> F := \left[(x^2+y^2)^2 + 3x^2y - y^3, (x^2+y^2)^3 - 4x^2y^2\right]:$

$> \text{plots[implicitplot]}(Fs, x = -2..2, y = -2..2):$



(3)

$> R := \text{PolynomialRing}([x, y], 101):$

$> \text{TriangularizeWithMultiplicity}(F, R);$

$$\left[\left[1, \begin{cases} x-1=0 \\ y+14=0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x+1=0 \\ y+14=0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x-47=0 \\ y-14=0 \end{cases}\right]\right], \left[\left[1, \begin{cases} x+47=0 \\ y-14=0 \end{cases}\right]\right], \left[\left[14, \begin{cases} x=0 \\ y=0 \end{cases}\right]\right]$$

The command RegularChains:-TriangularizeWithMultiplicity computes the intersection multiplicities for each point of $V(F)$. In the above Maple session, computations are performed modulo a prime number for the only reason of keeping output expressions small. The same calculations can be performed with the TriangularizeWithMultiplicity command over the reals.

Summary 2

- The theory of regular chains allows us to reduce the question of computing limit points of constructible sets and semi-algebraic sets to that of computing limit points of zero sets of regular chains.
- We will restrict ourselves here to regular chains in dimension 1, that is, where only one variable is free.
- Then, the above question can be solved by computing the Puiseux series solutions of regular chains.