On Fulton's Algorithm for Computing Intersection Multiplicities

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Overview

Let $f_1, \ldots, f_n \in k[x_1, \ldots, k_n]$ such that $V(f_1, \ldots, f_n) \subset \overline{k}[x_1, \ldots, k_n]$ is zero-dimensional. The intersection multiplicity $I(p; f_1, \ldots, f_n)$ at $p \in V(f_1, \ldots, f_n)$

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- We will combine Fulton's Algorithm approach and the theory of regular chains.
- Our algorithm is complete in the bivariate case.
- We propose algorithmic criteria for reducing the case of n variables to the bivariate one. Experimental results are also reported.

The case of two plane curves

Given an arbitrary field **k** and two bivariate polynomials $f, g \in k[x, y]$, consider the affine algebraic curves $C := V(f)$ and $D := V(g)$ in $\mathbb{A}^2 = \overline{\mathbf{k}}^2$, where $\overline{\mathbf{k}}$ is the algebraic closure of k. Let p be a point in the intersection.

Definition

The intersection multiplicity of p in $V(f, g)$ is defined to be

$$I(p; f, g) = \dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^2,p}/\langle f, g\rangle)$$

where $\mathcal{O}_{\mathbb{A}^2,p}$ and $\dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^2,p}/\langle f, g\rangle)$ are the local ring at p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^2,p}/\langle f, g\rangle$.

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Remark

As pointed out by Fulton in his book *Algebraic Curves*, the intersection multiplicities of the plane curves C and D satisfy a series of 7 properties which uniquely define $I(p; f, g)$ at each point $p \in V(f, g)$.
Moreover, the proof is constructive, which leads to an algorithm.

Fulton's Properties

The intersection multiplicity of two plane curves at a point satisfies and is uniquely determined by the following.

(2-1) $I(p; f, g)$ is a non-negative integer for any C, D, and p such that C and D have no common component at p. We set $I(p; f, g) = \infty$ if C and D have a common component at p.

(2-2) $I(p; f, g) = 0$ if and only if $p \notin C \cap D$.

(2-3) $I(p; f, g)$ is invariant under affine change of coordinates on $\mathbb{A}^2$.

(2-4) $I(p; f, g) = I(p; g, f)$

(2-5) $I(p; f, g)$ is greater or equal to the product of the multiplicity of p in f and g, with equality occurring if and only if C and D have no tangent lines in common at p.

(2-6) $I(p; f, gh) = I(p; f, g) + I(p; f, h)$ for all $h \in k[x, y]$.

(2-7) $I(p; f, g) = I(p; f, g + hf)$ for all $h \in k[x, y]$.

Fulton's Algorithm

Algorithm 1: $\mathrm{IM}_2(p; f, g)$

Input: $p = (\alpha, \beta) \in \mathbb{A}^2(\mathbf{k})$ and $f, g \in \mathbf{k}[y \succ x]$ such that $\mathbf{gcd}(f, g) \in \mathbf{k}$
Output: $I(p; f, g) \in \mathbb{N}$ satisfying (2-1)–(2-7)

if $f(p) \neq 0$ or $g(p) \neq 0$ **then**
- **return** 0;

$r, s = \mathbf{deg}\,(f(x, \beta))\,, \mathbf{deg}\,(g(x, \beta))\,;$ assume $s \geq r$.

if $r = 0$ **then**
- write $f = (y - \beta) \cdot h$ and $g(x, \beta) = (x - \alpha)^m\,(a_0 + a_1(x - \alpha) + \cdots)$;
- **return** $m + \mathrm{IM}_2(p; h, g)$;
- $\mathrm{IM}_2(p; (y - \beta) \cdot h, g) = \mathrm{IM}_2(p; (y - \beta), g) + \mathrm{IM}_2(p; h, g)$
- $\mathrm{IM}_2(p; (y - \beta), g) = \mathrm{IM}_2(p; (y - \beta), g(x, \beta)) = \mathrm{IM}_2(p; (y - \beta), (x - \alpha)^m) = m$

if $r > 0$ **then**
- $h \leftarrow \mathrm{monic}\,(g) - (x - \alpha)^{s-r}\mathrm{monic}\,(f)$;
- **return** $\mathrm{IM}_2(p; f, h)$;

Our goal: extending Fulton's Algorithm

Limitations of Fulton's Algorithm

Fulton's Algorithm

- does not generalize to $n > 2$, that is, to n polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ since $k[x_1, \ldots, x_{n-1}]$ is no longer a PID.
- is limited to computing the IM at a single point with rational coordinates, that is, with coordinates in the base field k. (Approaches based on standard or Gröbner bases suffer from the same limitation)

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Our contributions

- We adapt Fulton's Algorithm such that it can work at any point of $V(f_1, f_2)$, rational or not.
- For $n > 2$, we propose an algorithmic criterion to reduce the n-variate case to that of $n - 1$ variables.

A first algorithmic tool: *regular chains* (1/2)

Definition

$T \subset k[x_n > \cdots > x_1]$ is a triangular set if $T \cap k = \emptyset$ and $\mathrm{mvar}(p) \neq \mathrm{mvar}(q)$ for all $p, q \in T$ with $p \neq q$.

For all $t \in T$ write $\mathrm{init}(t) := \mathrm{lc}(t, \mathrm{mvar}(t))$ and $h_T := \prod_{t \in T} \mathrm{init}(t)$. The saturated ideal of T is:

$$\mathrm{sat}(T) = \langle T \rangle : h_T^{\infty}.$$

Theorem (J.F. Ritt, 1932)

Let $V \subset \overline{k}^n$ be an irreducible variety and $F \subset k[x_1, \ldots, x_n]$ s.t. $V = V(F)$. Then, one can compute a (reduced) triangular set $T \subset \langle F \rangle$ s.t.

$$(\forall\ g \in \langle \mathbf{F} \rangle)\ \ \mathrm{prem}(g, T) = 0.$$

Therefore, we have

$$V = V(\mathrm{sat}(T)).$$

A first algorithmic tool: *regular chains* (2/2)

Definition (M. Kalkbrner, 1991 - L. Yang, J. Zhang 1991)

T is a regular chain if $T = \emptyset$ or $T := T' \cup \{t\}$ with $\mathrm{mvar}(t)$ maximum s.t.

- T' is a regular chain,
- $\mathrm{init}(t)$ is regular modulo $\mathrm{sat}(T')$

Kalkbrener triangular decomposition

For all $F \subset k[x_1, \ldots, x_n]$, one can compute a family of regular chains $T_1, \ldots, T_e$ of $k[x_1, \ldots, x_n]$, called a Kalkbrener triangular decomposition of $V(F)$, such that we have

$$V(F) = \cup_{i=1}^{e} V(\mathrm{sat}(T_i)).$$

A second algorithmic tool: *the D5 Principle*

Original version (Della Dora, Discrescenzo & Duval)

Let $f, g \in k[x_1]$ such that f is squarefree. Without using irreducible factorization, one can compute $f_1, \ldots, f_e \in k[x_1]$ such that

- $f = f_1 \ldots f_e$ holds and,
- for each $i = 1 \cdots e$, either $g \equiv 0 \mod f_i$ or g is invertible modulo f_i.

Multivariate version

Let $T \subset k[x_1, \ldots, x_n]$ be a regular chain such that $\mathrm{sat}(T)$ is zero-dimensional, thus $\mathrm{sat}(T) = \langle T \rangle$ holds. Let $f \in k[x_1, \ldots, x_n]$.

The operation Regularize (f, T) computes regular chains $T_1, \ldots, T_e \subset k[x_1, \ldots, x_n]$ such that

- $V(T) = V(T_1) \cup \cdots \cup V(T_e)$ holds and,
- for each $i = 1 \cdots e$, either $V(T_i) \subseteq V(f)$ or $V(T_i) \cap V(f) = \emptyset$ holds.

Moreover, only polynomial GCDs and resultants need to be computed, that is, irreducible factorization is not required.

Dealing with non-rational points

Working with regular chains

To deal with non-rational points, we extend Fulton's Algorithm to compute $\mathrm{IM}_2(T; f_1, f_2)$, where $T \subset k[x_1, x_2]$ is a regular chain such that we have $V(T) \subseteq V(f_1, f_2)$.

- This makes sense thanks to the theorem below, which is non-trivial since intersection multiplicity is really a local property.
- For an arbitray zero-dimensional regular chain T, we apply the D5 Principle to Fulton's Algorithm in order to reduce to the case of the theorem.

Theorem

Recall that $V(f_1, f_2)$ is zero-dimensional. Let $T \subset k[x_1, x_2]$ be a regular chain such that we have $V(T) \subset V(f_1, f_2)$ and the ideal $\langle T \rangle$ is maximal. Then $\mathrm{IM}_2(p; f_1, f_2)$ is the same at any point $p \in V(T)$.

A third tool toward our algorithm: Taylor series

Fulton's Algorithm works with Taylor series

We observe that this algorithm works the Taylor series of f_1, f_2 at a rational point p. To extend this idea when working with $V(T)$, instead of a point p, we introduce two new variables y_1 and y_2 representing $x_1 - \alpha$ and $x_2 - \beta$ respectively, for an arbitrary point $(\alpha, \beta) \in V(T)$. These variables are simply used as place holders in the following definition, where $f \in \{f_1, f_2\}$.

Definition

Let $F \in k[x_1, x_2][y_1, y_2]$ and $T \subset k[x_1, x_2]$ be a regular chain such that we have $V(T) \subset V(f_1, f_2)$. We say that F is an expansion of f about $V(T)$ if at every point $(\alpha, \beta) \in V(T)$ we have

$$F(\alpha, \beta)(x_1 - \alpha, x_2 - \beta) = f(x_1, x_2).$$

The fundamental example is

$$F = \sum_j \left(\sum_i f_{i,j} \, y_1^i\right) y_2^j \quad \text{where} \quad f_{i,j} = \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}.$$

Our Algorithm in the bivariate case

Input: F^1 and F^2 as described in the previous slide.

Output: Finitely many pairs (T_i, m_i) where $T_i \subset k[x_1, x_2]$ are regular chains and $m_i \in \mathbb{Z}^+$ such that $\forall p \in V(T^i)$ $I(p; f_1, f_2) = m_i$.

```
for (F_1^1, T) ∈ Regularize(F_1^1, T) do
    if F_1^1 ∉ ⟨T⟩ then
        output(T, 0);
    else
        for (T, F_1^2) ∈ Regularize(F_1^2, T) do
            if F_1^2 ∉ ⟨T⟩ then
                output(T, 0);
            else
                for (T, a_{F^1}) ∈ LT(F^1_{<y_2}, T) do
                    for (T, a_{F^2}) ∈ LT(F^2_{<y_2}, T) do
                        /* Wlog deg(F^1_{<y_2}) ≤ deg(F^2_{<y_2})                */
                        if a_{F^1} ∈ ⟨T⟩ then
                            for (T, d) ∈ TDeg(F^2_{<y_2}, T) do
                                for (T, i) ∈ IM_2(T, (F^1 − F^1_{<y_2})/y_2, F^2) do
                                    output(T, (d + i));
                        else
                            H ← F^2 − a_{F^2} · Inverse(a^1_F, T) · F^1;
                            output(IM_2(T, F^1, H));
```

Reducing the n-dimensional case to the $n-1$ case (1/5)

Definition

The intersection multiplicity of p in $V(f_1, \ldots, f_n)$ is given by

$$I(p; f_1, \ldots, f_n) := \dim_{\overline{k}} \left(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle \right).$$

where $\mathcal{O}_{\mathbb{A}^n,p}$ and $\dim_{\overline{k}}(\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle)$ are respectively the local ring at the point p and the dimension of the vector space $\mathcal{O}_{\mathbb{A}^n,p} / \langle f_1, \ldots, f_n \rangle$.

The next theorem reduces the n-dimensional case to $n-1$, under assumptions which state that f_n does not contribute to $I(p; f_1, \ldots, f_n)$.

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Theorem

Assume that $h_n = V(f_n)$ is non-singular at p. Let v_n be its tangent hyperplane at p. Assume that h_n meets each component (through p) of the curve $\mathcal{C} = V(f_1, \ldots, f_{n-1})$ transversely (that is, the tangent cone $TC_p(\mathcal{C})$ intersects v_n only at the point p). Let $h \in k[x_1, \ldots, x_n]$ be the degree 1 polynomial defining v_n. Then, we have

$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

Reducing the n-dimensional case to the $n-1$ case (2/5)

The theorem again:

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$$I(p; f_1, \ldots, f_n) = I(p; f_1, \ldots, f_{n-1}, h).$$

How to use this theorem in practise?

Assume that the coefficient of x_n in h is non-zero, thus $h = x_n - h'$, where $h' \in k[x_1, \ldots, x_{n-1}]$. Hence, we can rewrite the ideal $\langle f_1, \ldots, f_{n-1}, h \rangle$ as $\langle g_1, \ldots, g_{n-1}, h \rangle$ where g_i is obtained from f_i by substituting x_n with h'. Then, we have

$$I_n(p; f_1, \ldots, f_n) = I_{n-1}(p|_{x_1, \ldots, x_{n-1}}; g_1, \ldots, g_{n-1}).$$

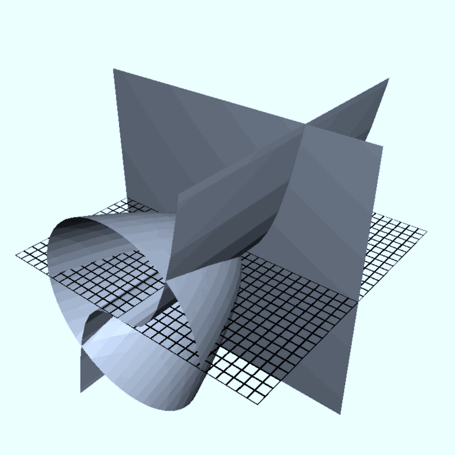
Reducing the n-dimensional case to the $n-1$ case (3/5)

Example

Consider the system

$$f_1 = x, \quad f_2 = x + y^2 - z^2, \quad f_3 := y - z^3$$

near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$



Reducing the n-dimensional case to the $n-1$ case (4/5)

Example

Recall the system

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near the origin $o := (0,0,0) \in V(f_1, f_2, f_3)$.

Computing the IM using the definition

Let us compute a basis for $\mathcal{O}_{\mathbb{A}^3,o} / \langle f_1, f_2, f_3 \rangle$ as a vector space over $\overline{k}$.

Setting $x = 0$ and $y = z^3$, we must have $z^2(z^4+1) = 0$ in $\mathcal{O}_{\mathbb{A}^3,o} = \overline{k}[x,y,z]_{(z,y,z)}$.

Since $z^4 + 1$ is a unit in this local ring, we see that

$$\mathcal{O}_{\mathbb{A}^3,o} / \langle f_1, f_2, f_3 \rangle = \langle 1, z \rangle$$

as a vector space, so $I(o; f_1, f_2, f_3) = 2$.

Example

Recall the system again

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Computing the IM using the reduction

We have

$$\mathcal{C} := V(x, x + y^2 - z^2) = V(x, (y-z)(y+z)) = TC_o(\mathcal{C})$$

and we have

$$h = y.$$

Thus $\mathcal{C}$ and $V(f_3)$ intersect transversally at the origin. Therefore, we have

$$I_3(p; f_1, f_2, f_3) = I_2((0,0); x, x - z^2) = 2.$$

Reducing the n-dimensional case to the $n-1$ case (5/5)

In practise, this reduction from n to $n-1$ variables does not always apply. For instance, this is the case for *Ojika 2*:

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0.$$

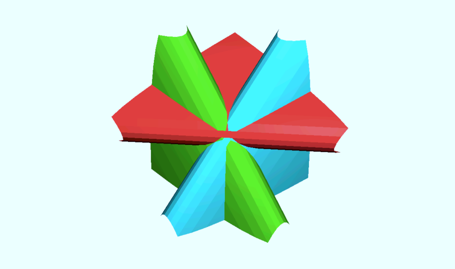


Figure: The real points of $V(x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$.

Reducing the n-dimensional case to the $n-1$ case (6/5)

Recall the system

$$x^2 + y + z - 1 = x + y^2 + z - 1 = x + y + z^2 - 1 = 0.$$

If one uses the first equation, that is $x^2 + y + z - 1 = 0$, to eliminate z from the other two, we obtain two bivariate polynomials $f, g \in k[x, y]$.

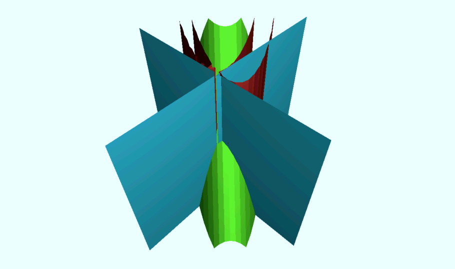
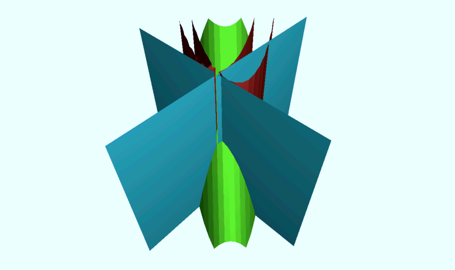


Figure: The real points of $V(x^2 + y + z - 1, x + y^2 - x^2 - y, x - y + x^4 + 2\,x^2 y - 2\,x^2 + y^2)$ near the origin.

Reducing the n-dimensional case to the $n-1$ case (7/5)

At any point of $p \in V(h, f, g)$ the tangent cone of the curve $V(f, g)$ is independent of z; in some sense it is "vertical". On the other hand, at any point of $p \in V(h, f, g)$ the tangent space of $V(h)$ is not vertical.

Thus, the previous theorem applies without computing any tangent cones.



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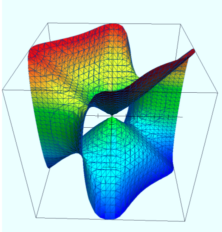
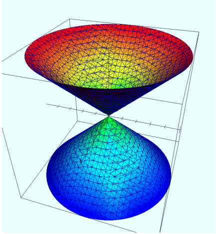
Tangent cone computation without standard bases





Assume $\overline{k} = \mathbb{C}$ and none of the $V(f_i)$ is singular at p. For each component $\mathcal{G}$ through p of $\mathcal{C} = V(f_1, \ldots, f_{n-1})$,

- There exists a neighbourhood B of p such that $V(f_i)$ is not singular at all $q \in (B \cap \mathcal{G}) \setminus \{p\}$, for $i = 1, \ldots, n-1$.
- Let $v_i(q)$ be the tangent hyperplane of $V(f_i)$ at q. Regard $v_1(q) \cap \cdots \cap v_{n-1}(q)$ as a parametric variety with q as parameter.
- Then, $TC_p(\mathcal{G}) = v_1(q) \cap \cdots \cap v_{n-1}(q)$ when q approaches p, which we compute by a variable elimination process.

Finally, $TC_p(\mathcal{C})$ is the union of all the $TC_p(\mathcal{G})$. This approach avoids standard basis computation and extends easily for working with $V(T)$ instead of p.

TriangularizeWithMultiplicity

We specify TriangularizeWithMultiplicity for the bivariate case.

Input $f, g \in \mathbf{k}[x, y]$ such that $V(f, g)$ is zero-dimensional.

Output Finitely many pairs $[(T_1, m_1), \ldots, (T_\ell, m_\ell)]$ of the form $(T_i :: \text{RegularChain}, m_i :: \text{nonnegint})$ such that for all $p \in V(T_i)$

$$\mathrm{I}(p; f, g) = m_i \quad \text{and} \quad V(f, g) = V(T_1) \uplus \cdots \uplus V(T_\ell).$$

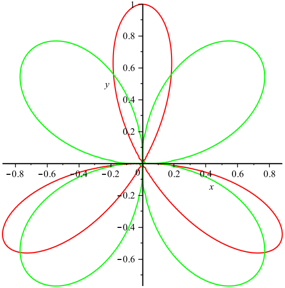
Implementating TriangularizeWithMultiplicity is done by

- first calling Triangularize (which encode the points of $V(f, g)$ with regular chains, and
- secondly calling $\mathrm{IM}_2(T; f, g)$ for all $T \in \mathtt{Triangularize}(f, g)$.

This approach allows optimizations such that using the Jacobian criterion to quickly discover points of IM equal to 1.

$> Fs := \left[\left(x^2+y^2\right)^2+3x^2y-y^3, \left(x^2+y^2\right)^3-4x^2y^2\right]:$

> plots[implicitplot](Fs,x=-2..2,y=-2..2);



$> R := PolynomialRing\left([x, y], 101\right):$

$> rcs := Triangularzie\left(Fs, R, normalized = `yes`\right):$

$> \mathrm{seq}\left(\mathrm{TriangularizeWithMultiplicity}\left(Fs, T, R\right), T \ \mathbf{in} \ rcs\right):$

$$\left[\left[1, \left\{\begin{array}{c} x-1=0 \\ y+14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x+1=0 \\ y+14=0 \end{array}\right.\right]\right], \left[\left[1, \left\{\begin{array}{c} x-47=0 \\ y-14=0 \end{array}\right.\right]\right],$$

$$\left[\left[1, \left\{\begin{array}{c} x+47=0 \\ y-14=0 \end{array}\right.\right]\right], \left[\left[14, \left\{\begin{array}{c} x=0 \\ y=0 \end{array}\right.\right]\right]$$

$> Fs := \left[x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\right]:$

$> R := PolynomialRing\left([x, y, z], 101\right):$

$>$ TriangularizeWithMultiplicity (Fs, T, R):

$$\left[\left[1, \left\{\begin{array}{c} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0 \end{array}\right.\right]\right], \left[\left[2, \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z - 1 = 0 \end{array}\right.\right]\right],$$

$$\left[\left[1, \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right.\right]\right], \left[\left[2, \left\{\begin{array}{c} x - 1 = 0 \\ y = 0 \\ z = 0 \end{array}\right.\right]\right]$$

Experiments

System	Dim	Time(△ize)	#rc's	Time(rc_im)
$\langle 1, 3\rangle$	888	9.7	20	19.2
$\langle 1, 4\rangle$	1456	226.0	8	9.023
$\langle 1, 5\rangle$	1595	169.4	8	25.4
$\langle 3, 5\rangle$	1413	22.5	27	28.6
$\langle 4, 5\rangle$	1781	218.4	9	13.9
$\langle 5, 1\rangle$	1759	113.0	10	15.8
$\langle 6, 8\rangle$	1680	99.7	12	37.6
$\langle 6, 9\rangle$	2560	299.3	10	22.9
$\langle 6, 10\rangle$	1320	131.9	7	8.4
$\langle 6, 11\rangle$	1440	59.8	17	27.5
$\langle 7, 8\rangle$	1152	32.8	12	16.2
$\langle 7, 9\rangle$	756	18.5	16	11.2
$\langle 7, 10\rangle$	595	8.1	17	13.0
$\langle 7, 11\rangle$	648	9.2	25	11.1
$\langle 8, 9\rangle$	1984	374.5	10	11.3
$\langle 8, 10\rangle$	1362	232.5	7	9.3
$\langle 8, 11\rangle$	1256	49.6	17	45.7
$\langle 9, 10\rangle$	2080	504.9	12	34.812
$\langle 9, 11\rangle$	1792	115.1	16	17.2
$\langle 10, 11\rangle$	1180	40.9	17	21.3