On solving parametric polynomial systems

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Abstract. Border polynomial and discriminant variety are two important notions related to parametric polynomial system solving, in particular, for partitioning the parameter values into regions where the solutions of the system depend continuously on the parameters. In this paper, we study the relations between those notions in the case of parametric triangular systems. We also investigate the properties and computation of the non-properness locus of the canonical projection restricted at a parametric regular chain or at its saturated ideal.

1. Introduction

Many authors have contributed to the study of parametric polynomial systems, and there is a large collection of references, such as [7, 22, 13, 23, 18, 19, 27, 15, 5, 16], to name a few. Various notions have been formulated for investigating the properties of parametric polynomial systems from different aspects. Border polynomial [26, 27, 25, 4], discriminant variety [15], discriminant ideal [23], discriminant set [5] are some of those notions. For (parametric) semi-algebraic systems, methods based on *cylindrical algebraic decomposition* (CAD) and its variants [8, 9, 10] are applicable. However, these methods may compute much more than what is needed for the purpose of solving.

One central question in the study of parametric polynomial systems is the dependence of the solutions on the parameter values. There are different ways to express the fact that the zeros of a parametric system depends *continuously* on the parameters in a neighborhood of a given parameter value. The notions of a border polynomial and a discriminant variety aim at capturing the parameter values at which certain dependence is not continuous. This inspires us to unify various notions under a continuity framework, which will help us understand how different notions are related.

A first objective of the present work is to study the non-properness of the canonical projection restricted at T or $\mathrm{sat}(T)$ where T is a regular chain with free variables as parameters. Theorem 1 shows that, within the zero locus of the iterated resultant of the product of the initials of T, the number of solutions of T counted with multiplicities, is either infinite or less than the product of the main degrees of T. Theorem 2 strengthens this result and states that the iterated resultant of the product of the initials of T defines the non-properness locus (of the canonical projection restricted) at T; moreover, Theorem 2 states that any parameter value which is in the non-properness locus at T but not in that at $\mathrm{sat}(T)$ yields either no solutions or infinitely many solutions for T. In addition, Theorem 3 supplies a formula for computing the non-properness locus at $\mathrm{sat}(T)$.

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A second objective of the present work is to study the relations between the notions of a border polynomial and a discriminant variety. To this end, we gather key properties on these objects, including results from the PhD thesis in Chinese of the third author [25]. We stress the fact that most of our results assume that the input parametric system is triangular, since triangular decomposition methods [24, 14, 12, 21, 6] can help reducing the study of general parametric systems to the triangular case. Theorem 4 implies that the zero locus of the border polynomial of T is the minimal discriminant variety of the quasi-component of $V(T)$. Other results, more technical, such as Proposition 6 establish fine relations between the minimal discriminant varieties of $V(T)$ and $V(\mathrm{sat}(T))$. This leads us to answer the following question: among all regular chains that have the same saturated ideal as a given regular chain, the best choice to make the border polynomial set minimal is to choose a canonical regular chain, see Theorem 5.

With respect to our MACIS 2011 article, the present paper focuses on parametric algebraic system and enhances our work [17] in two directions. Firstly, we supply proofs for all the relevant results from [17]. Secondly, we expand the study of non-properness and devote one whole section (Section 3) to this subject. Extensions of our work on parametric semi-algebraic systems will appear elsewhere and are partially summarized in the concluding section.

This paper is organized as follows. In Section 2, we revisit the notions of a border polynomial and a discriminant variety in a unified framework. In the context of triangular parametric systems, we show that the two notions essentially coincide, see Theorem 4. In Section 3, we show several properties of the non-properness of polynomial maps for complex varieties. In Section 4, we compare the minimal discriminant variety of a regular chain and that of its saturated ideal. In Section 5, we discuss the possible extensions of the results of Sections 3 and 4 to a more general context.

2. Border polynomial and discriminant variety: two notions of discontinuity

In this paper, a parametric polynomial system S is a system of equations, inequations and/or inequalities given by polynomials in $\mathbb{Q}[U, X]$ where $U = u_1, u_2, \ldots, u_d$ are the parameters and $X = x_1, x_2, \ldots, x_s$ are the unknowns. All variables (parameters and unknowns) hold values from a fixed field $\mathbb{K}$, which is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. In the former case, we say that the system is algebraic[1] and in the latter case, we say that the system is semi-algebraic.

We denote by $Z(S)$ the solution set of S, which is a subset of $\mathbb{K}^{d+s}$. The canonical projection Π_{U} to the parameter space restricted at $Z(S)$ is defined as follows:

$$\Pi_{\mathrm{U}} : Z(S) \subset \mathbb{K}^{d+s} \mapsto \mathbb{K}^d$$
$$\Pi_{\mathrm{U}}(u_1, \ldots, u_d, x_1, \ldots, x_s) = (u_1, \ldots, u_d)$$

Let us denote by E (resp. I) the set of the polynomials of S defining its equations (resp. inequations and strict inequalities). The ideal $\langle E \rangle : (\prod_{h \in I} h)^\infty$ is called the *ideal associated with* S. For a given polynomial set $L \subset \mathbb{Q}[U, X]$, we denote by $V(L)$ the variety of L in $\mathbb{C}^{d+s}$.

Definition 1. *We say that S is* well-determinate *if the set U is an $\subseteq$-maximal algebraically independent variable set modulo the ideal associated with S, that is, the ideal $\mathbb{K}[U] \cap \langle E \rangle : (\prod_{h \in I} h)^\infty$ is $\langle 0 \rangle$ and U is $\subseteq$-maximal with that property.*

Note that the notion of " well-determinate" is more general than the notion of "well-behaved" used in [15], in the sense that it is less restrictive for E. Indeed, the polynomial set E is not required to have exactly s elements, nor to generate a radical ideal in $\mathbb{Q}(U)[X]$.

[1] No inequalities are present in an algebraic system.

Example 1. *Consider a semi-algebraic system*

$$S = \{x(x^2 + ay + b) = x(y^2 + bx + a) = 0, x > 0\}$$

with parameters a, b. The ideal generated by the polynomials defining the equations of S equals

$$\langle x\rangle \cap \langle x^2 + ay + b, y^2 + bx + a\rangle.$$

The polynomial system $\{x(x^2 + ay + b) = x(y^2 + bx + a) = 0\}$ with parameters a, b is not well-determinate, since $\{a, b\}$ is not a maximal algebraically independent set modulo $\langle x\rangle$. However, the ideal associated to S is $\mathcal{I} := \langle x^2 + ay + b, y^2 + bx + a\rangle$, and $\{a, b\}$ is a maximal algebraic independent variable set modulo $\mathcal{I}$. Therefore, S is a well-determinate parametric semi-algebraic system.

For a parametric polynomial system, we shall always assume that the parametric space[2] is positive dimensional. Throughout this paper, in order to keep the presentation concise, we assume that S is well-determinate. This assumption can be relaxed and the results discussed hereafter can be adapted to more general systems.

We rely on triangular decomposition techniques for studying parametric polynomial systems. We refer to [4] for the standard notions and notations on triangular decomposition, such as: regular chain, main variable (mvar), main degree (mdeg), initial (init), iterated resultant (ires).

An STAS is a pair $[T, H_{\neq}]$ where T is a squarefree regular chain of $\mathbb{Q}[U, X]$ and $H_{\neq}$ is a set of non-constant polynomials of $\mathbb{Q}[U, X]$ such that each of those is regular modulo $\mathrm{sat}(T)$, the saturated ideal of T, which is $\langle T\rangle : \mathrm{init}(T)^{\infty}$. A point of $\mathbb{K}^{d+s}$ is a zero of $[T, H_{\neq}]$ if it is a zero of T not canceling any of the polynomials of $H_{\neq}$.

An STSAS is a triple $[T, H_{\neq}, P_{>}]$ such that $[T, H_{\neq}]$ is an STAS and $P_{>}$ is a set of non-constant polynomials of $\mathbb{Q}[U, X]$ such that each of those is regular modulo $\mathrm{sat}(T)$. A point of $\mathbb{K}^{d+s}$ is a zero of $[T, H_{\neq}, P_{>}]$ if it is a zero of $[T, H_{\neq}]$ making each polynomial of $P_{>}$ strictly positive.

In the algebraic case, we shall decompose $Z(S)$ into zero sets of finitely many *squarefree triangular algebraic systems* (STAS); in the semi-algebraic case, we shall decompose $Z(S)$ into zero sets of finitely many *squarefree triangular semi-algebraic systems* (STSAS).

Let $\alpha \in \mathbb{K}^d$. As mentioned in the introduction, there are different ways to express the fact that the zeros of the parametric system S depends continuously on the parameters in a neighborhood of α in $\mathbb{K}^d$. In this paper, we focus on two of them.

Definition 2. *We say that S is Z-*continuous *at α if there exists a neighborhood $\mathcal{O}_\alpha$ of α such that for any two parameter values $\alpha_1, \alpha_2 \in \mathcal{O}_\alpha$, we have $\#(Z(S(\alpha_1)) = \#(Z(S(\alpha_2)) < \infty$. We say that S is Π_{U}-*continuous *at α if there exists a neighborhood $\mathcal{O}_\alpha$ of α such that there exists a finite partition $\{C_1, \ldots, C_k\}$ of $\Pi_{\mathrm{U}}^{-1}(\mathcal{O}_\alpha) \cap Z(S)$ such that the restriction $\Pi_{\mathrm{U}}|_{C_j} : C_j \xrightarrow{\Pi_{\mathrm{U}}} \mathcal{O}_\alpha$ is a diffeomorphism, for each $j \in \{1, \ldots, k\}$.*

Example 2. *Consider the semi-algebraic system*

$$S := \{x^2 + ay^2 - x = ax^2 + y^2 - y = 0, x \neq y\}$$

with parameter a. When the parameter takes value in the open interval $(-1, \frac{1}{3})$, there are two solutions, which are given by:

$$x = \frac{a + 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)}, y = \frac{-a - 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)},$$

and

$$x = \frac{-a - 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)}, y = \frac{a + 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)}.$$

Therefore, the system S is Z-continuous as well as Π_{U}-continuous at any point in $(-1, \frac{1}{3})$.

[2]Here parametric space refers to the set of all parameter values that does not specify the associate ideal of the system to $\langle 1\rangle$.

It is obvious that Π_{U}-continuity implies Z-continuity. Moreover, these two kinds of continuity are equivalent in many cases, e.g. for parametric STASes, as we shall show in Section 4. Another notion of continuity (or discontinuity) is *non-properness*. The canonical projection Π_{U} is said to be *non-proper* at the point α, if for any compact set $S \subseteq \mathbb{K}^d$ containing α, the set $\Pi_{\mathrm{U}}^{-1}(S)$ is not compact. In Definition 6, the notion of non-properness is stated for an arbitrary polynomial map. Non-properness is strongly linked to the following Z^**-continuity* (see Corollary 2).

Definition 3. *We say S is Z^*-continuous at α if there exists a neighborhood $\mathcal{O}_\alpha$ of α such that for any two parameter values $\alpha_1, \alpha_2 \in \mathcal{O}_\alpha$, the number of solutions, counted with multiplicities, of $S(\alpha_1)$ is finite and equals that of $S(\alpha_2)$.*

The notion of a *border polynomial* is based on the Z-continuity and was proposed in [26] for computing the real root classification of a parametric semi-algebraic system. We reformulate the definition here, for both parametric algebraic systems and parametric semi-algebraic systems.

Definition 4 (Border polynomial). *A non-zero squarefree polynomial b in $\mathbb{Q}[U]$ is called a* border polynomial *of the parametric polynomial system S if the zero set $V(b)$ of b in $\mathbb{K}^d$ contains all the points at which S is not Z-continuous.*

Example 3. *Consider the polynomial system $S := \{x^2 + bx - 1\}$ with parameter b. Regarding S as an algebraic system, it is easy to check that the system has two solutions for $b^2 + 4 \neq 0$ and has only one solution for $b^2 + 4 = 0$; therefore, $b^2 + 4$ is a border polynomial. In fact, it is a minimal border polynomial of S in the sense that it divides any other border polynomials of S.*

Viewing S as a semi-algebraic system, this system always has two real solutions; therefore, 1 is the minimal border polynomial. Indeed, recall that in the semi-algebraic case, the field $\mathbb{K}$ of Definition 4 *is $\mathbb{R}$.*

The notion of a *discriminant variety* is based on the Π_{U}-continuity and was proposed in [15] for general parametric algebraic systems. We reformulate the definition here, for both parametric algebraic systems and parametric semi-algebraic systems.

Definition 5 (Discriminant variety). *An algebraic set $\mathcal{W} \subsetneq \mathbb{K}^d$ is a* discriminant variety *of the parametric polynomial system S if $\mathcal{W}$ contains all the points at which S is not Π_{U}-continuous.*

Example 4 (Example 2 **Cont.).** *Consider again the semi-algebraic system*

$$S := \{x^2 + ay^2 - x = ax^2 + y^2 - y = 0, x \neq y\}$$

with parameter a. It is not hard to show that when either $a < -1$ or $a > \frac{1}{3}$ holds the system has no real solutions. So $\{-1, \frac{1}{3}\}$ is a (indeed, the minimal) discriminant variety of S and $(a+1)(a-\frac{1}{3})$ is a (again, the minimal) border polynomial of S.

If S is viewed as a parametric algebraic system, then the minimal discriminant variety would be $\{-1, \frac{1}{3}, 1\}$ and the minimal border polynomial of S would be $(a^2-1)(a-\frac{1}{3})$.

Remark 1. *The following facts can be easily deduced from the above definitions.*

(i) One can form a discriminant variety of S by taking the intersection of all discriminant varieties, which is the minimal discriminant variety *of S.*
(ii) If the hypersurface of a polynomial contains the minimal discriminant variety, then this polynomial is a border polynomial.
(iii) In general, there is no "minimal border polynomial". This will typically happen when the minimal discriminant variety of S is not the zero set of a single polynomial. However, we call a border polynomial quasi-minimal *if none of its proper factors is a border polynomial.*
(iv) In the algebraic case, the set of points where the Π_{U}-continuity of S does not hold is just the minimal discriminant variety of S; in the semi-algebraic case, the points at which the Π_{U}-continuity of S does not hold form a semi-algebraic set, which is not algebraic in general.

For both the Z-continuity and the Π_{U}-continuity, there are essentially two steps in solving the parametric system S:

(1) describe the parameter values where the continuity does not hold,
(2) describe the (groups of) regions where the continuity is maintained.

Step (1) is achieved by computing a border polynomial or a discriminant variety, depending on the continuity notion. It is not hard to show that the computation of a border polynomial or a discriminant variety of S in the semi-algebraic case can be reduced to the computation of a discriminant variety in the algebraic case. Based on this observation, we devote Sections 3 and 4 to the algebraic case.

For simplicity with Step (2), let us assume that a border polynomial of S is a polynomial whose hypersurface is also a discriminant variety of S. In the algebraic case, the complement of an algebraic set in $\mathbb{C}^d$ has only one connected component, thus Step (2) is rather simple. However, in the semi-algebraic case, there are usually more than one connected components in the complement of an algebraic set in $\mathbb{R}^d$ and the description of those connected components is more challenging. The notion of a finger polynomial set (see [3]) and an effective boundary (see [4]) are dedicated to this. Here we use a simple example to illustrate the difficulty.

Example 5. *Consider the polynomial system $S := \{x^4+bx^2+c=0\}$ with parameters b, c. Viewing S as an algebraic system: $c(b^2-4c)$ is a border polynomial, which defines the minimal discriminant variety as well; when $c(b^2-4c) \neq 0$, the system is Z-continuous and has 4 simple solutions.*

Regarding S as a semi-algebraic system, $c(b^2-4c)$ is a border polynomial and defining the minimal discriminant variety. However, there are 4 regions (see Figure 1) where S is Z-continuous, namely:

(I) $b^2-4c<0$, *above which S has no real solutions;*
(II) $b^2-4c>0 \wedge c>0 \wedge b>0$, *above which S has no real solutions;*
(III) $b^2-4c>0 \wedge c>0 \wedge b<0$, *above which S has 4 real solutions;*
(IV) $c<0$, *above which S has 2 real solutions.*

One can see that, in addition to the factors of the border polynomial, at least one polynomial (here b) is required to describe all the regions.

3. Non properness locus of a polynomial map

In this section, we shall discuss the properties and computation of the non-properness of a polynomial map, in particular, the canonical projection restricted at T or $\mathrm{sat}(T)$ where T is a regular chain with its free variables regarded as parameters. To this end, we recall a more intuitive concept, the finiteness (see Definition 7) of a continuous map, which, in the case of polynomial maps between complex varieties, is equivalent to that of non-properness (see [20]).

3.1. General properties

Throughout this section, let V and W be two complex varieties; let $f : V \to W$ be a polynomial map such that we have $\overline{f(V)} = W$. Note that for a complex variety V, the Zariski closure of $f(V)$ coincides with the closure of $f(V)$ in the usual topology.[3] The following definition of properness of a polynomial map is more general than others, see for instance [11]. Indeed, following [15], we do not require that the target variety W is irreducible.

Definition 6 (Properness of a map). *We say that f is* proper *at a point $\alpha \in W$, if there exists a compact neighborhood C_α (which has the same local dimension at α as W does) of α such that $f^{-1}(C_\alpha)$ is compact in V. We denote by $\mathcal{O}_\infty(f)$ the set of all points where f is not proper. We call $\mathcal{O}_\infty(f)$ the* non-properness locus *of f.*

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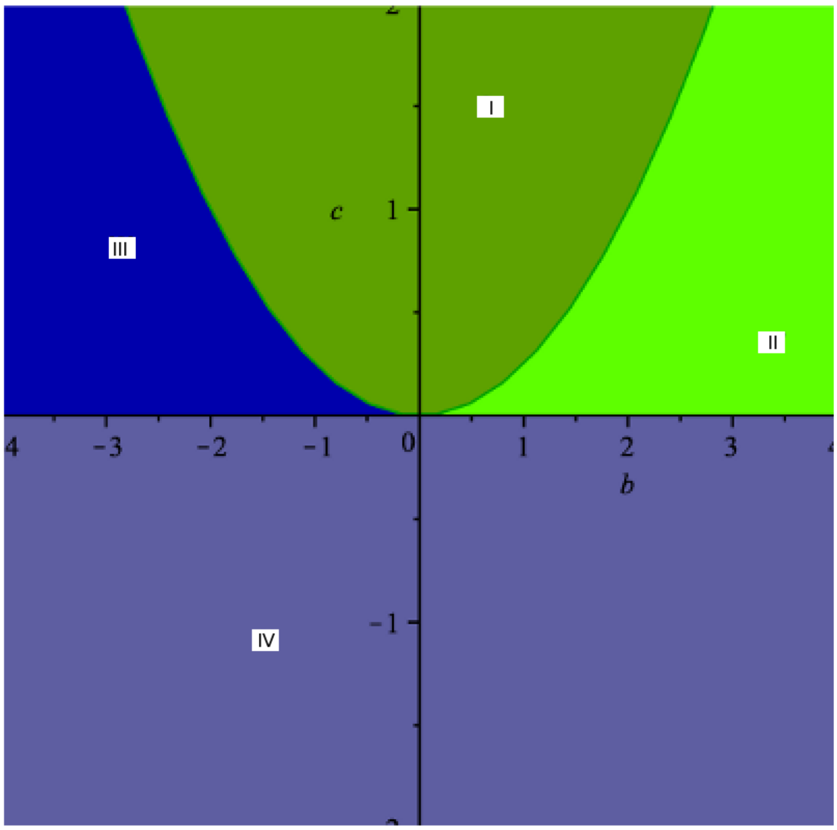
[3]By usual topology of $\mathbb{C}^n$, we mean the topology induced by the identification between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, this latter being equipped with the Euclidean topology.

FIGURE 1. The 4 regions



Notation 1. *Consider a parametric polynomial system S in $\mathbb{Q}[U, X]$ (as defined in Section 2) and I its associated ideal. We denote by $\mathcal{O}_\infty(S)$ or $\mathcal{O}_\infty(V(I))$ the set $\mathcal{O}_\infty(\Pi_{\mathrm{U}}|_{V(I)})$, where $\Pi_{\mathrm{U}}|_{V(I)}$ is the restriction of Π_{U} at $V(I)$.*

One key geometric property of non-properness is stated in the following lemma, from [11].

Lemma 1 ([11]). *Given two irreducible varieties V and W, let $f : V \to W$ be a polynomial map. If $\overline{f(V)} = W$ and $\dim(V) = \dim(W)$ both hold, then $\mathcal{O}_\infty(f)$ is either empty or is a hypersurface in W.*

Lemma 1 implies that, for a well-determinate parametric polynomial system, the non-properness locus of its standard projection on the parameter space is always a variety, which is either empty or has a dimension strictly less than that of the parameter space. Next, we recall from [15] an algorithmic description of the non-properness locus of a general parametric polynomial system by means of Gröbner basis computations.

Lemma 2 ([15]). *Given a parametric ideal I with parameters $U = u_1, u_2, \ldots, u_d$ and variables $X = x_1, x_2, \ldots, x_s$, let $\mathcal{G}$ be a reduced Gröbner basis of I w.r.t a block ordering $\prec_{U,X}$ where $\prec_X$ is a degree reverse lexicographic ordering. For $i = 1, \ldots, s$, define*

$$\Xi_i^\infty = \{\mathrm{lc}_{\prec_X}(g) \mid g \in \mathcal{G},\ \mathrm{lm}_{\prec_X}(g) = x_i^m \text{ for some } m \geq 0\}.$$

Then we have $\mathcal{O}_\infty(I) = \cup_{i=1}^s V(\Xi_i^\infty)$.

For polynomial maps between complex varieties, the definition of non-properness coincides with the following notion of non-finiteness. In Definition 7, the norm is the Euclidean norm.

Definition 7 (Finiteness of a map). *We say that* f *is* not finite *at a point* $\alpha \in W$*, if there exists a sequence of points*

$$\{\mathbf{y}_1, \mathbf{y}_2, \ldots \mathbf{y}_n, \ldots\} \in V,$$

such that we have

$$\lim_{n\to\infty} \|\mathbf{y}_n\| = \infty \text{ and } \lim_{n\to\infty} f(\mathbf{y}_n) = \alpha.$$

From the above characterization of non-properness, we easily deduce the following facts.

1. If $f : V \to W$ is a polynomial map and V is the union of two varieties V_1 and V_2, then $\mathcal{O}_\infty(f)$ is the union of $\mathcal{O}_\infty(f|_{V_1})$ and $\mathcal{O}_\infty(f|_{V_2})$.
2. If $\alpha \in W$ satisfies $\dim(\overline{f^{-1}(\alpha)}) > 0$, then we have $\alpha \in \mathcal{O}_\infty(f)$.

We shall use Definition 7 as a definition of non-properness in the proof of the results in this section. Lemma 3, taken from [25], could be known in a more general setting but we have not found a reference for it. And Corollary is a direct application of Lemma 3 to parametric algebraic systems.

Lemma 3. *Let* V, W *be complex varieties and* $f : V \to W$ *be a polynomial map. Assume that* W *is of positive dimension and that* $\overline{f(V)} = W$ *holds. Then* $W \setminus f(V)$ *is contained in* $\mathcal{O}_\infty(f)$*.*

Proof. Let $\alpha \in W \setminus f(V)$. Since $\dim(W) > 0$ and $W = \overline{f(V)}$ hold, there exists a sequence of points $\{\beta_i\}_{i=1,2,\ldots,\infty}$ in $f(V)$ such that $\lim_{i\to\infty} \beta_i = \alpha$, since $W = \overline{f(V)}$ holds. For each β_i, let $\mathbf{y}_i$ be a point in $f^{-1}(\beta_i)$. Then we obtain a sequence of points $\mathbf{y}_i \in V$, for $i = 1, 2, \ldots, \infty$. We claim that $\lim_{i\to\infty} \|\mathbf{y}_i\| = \infty$ holds. Otherwise, the sequence $(\mathbf{y}_i \in V, i \in \mathbb{N})$ admits a bounded sub-sequence, from which we can extract a convergent sub-sequence; since V is closed, the limit$\mathbf{y}^*$ of this latter sub-sequence belongs to V. Therefore, we have $\mathbf{y}^* \in V$ and $f(\mathbf{y}^*) = \alpha$ holds, which contradicts the assumption $\alpha \in W \setminus f(V)$. Therefore, the claim $\lim_{i\to\infty} \|\mathbf{y}_i\| = \infty$ holds, which implies that $\alpha \in \mathcal{O}_\infty(f)$ holds. □

Corollary 1. *Consider a parametric algebraic system* S *in* $\mathbb{Q}[U, X]$ *with no inequations. Then the non-properness locus of* Π_U *contains the parameter values at which* S *is inconsistent.*

The following Lemma from [25] shows that the set of non-properness is well adapted to the composition of polynomial maps.

Proposition 1. *Consider* 3 *varieties* V_1, V_2, V_3 *and* 3 *polynomial maps* $f_{1,3} : V_1 \to V_3$, $f_{1,2} : V_1 \to V_2$, $f_{2,3} : V_2 \to V_3$ *satisfying* $f_{1,3} = f_{2,3} \circ f_{1,2}$, $V_2 = \overline{f_{1,2}(V_1)}$ *and* $V_3 = \overline{f_{2,3}(V_2)}$*. Then we have*

$$\mathcal{O}_\infty(f_{1,3}) = \mathcal{O}_\infty(f_{2,3}) \cup f_{2,3}(\mathcal{O}_\infty(f_{1,2})).$$

Proof. Let us first show that $\mathcal{O}_\infty(f_{1,3}) \subseteq \mathcal{O}_\infty(f_{2,3}) \cup f_{2,3}(\mathcal{O}_\infty(f_{1,2})$ holds. Let $\alpha \in \mathcal{O}_\infty(f_{1,3})$. Then there exists an unbounded point sequence $\{\mathbf{x}_i\}_{i=1,2,3,\ldots,\infty}$ in V_1 such that $\lim_{i\to\infty} f_{1,3}(\mathbf{x}_i) = \alpha$ holds. If there exists a sub-sequence of $f_{1,2}(\mathbf{x}_i)$ $(i = 1, 2, \ldots, \infty)$ which converges to a point $\mathbf{y}$ in V_2, then $\mathbf{y} \in \mathcal{O}_\infty(f_{1,2})$ holds by definition and we have $\alpha = f_{2,3}(\mathbf{y}) \in f_{2,3}(\mathcal{O}_\infty(f_{1,2}))$. Otherwise, the sequence $f_{1,2}(\mathbf{x}_i)$ $(i = 1, 2, \ldots, \infty)$ is contained in V_2 and is unbounded, which implies that $\alpha \in \mathcal{O}_\infty(f_{2,3})$ holds.

Now we show that $\mathcal{O}_\infty(f_{1,3}) \supseteq \mathcal{O}_\infty(f_{2,3}) \cup f_{2,3}(\mathcal{O}_\infty(f_{1,2}))$ holds. Let $\alpha \in \mathcal{O}_\infty(f_{2,3}) \cup f_{2,3}(\mathcal{O}_\infty(f_{1,2}))$. If $f_{2,3}^{-1}(\alpha) \cap V_2 = \emptyset$ holds, then we have $\alpha \in (V_3 \setminus f_{1,3}(V_1))$, which implies that $\alpha \in \mathcal{O}_\infty(f_{1,3})$ holds. Now we consider only the case $f_{2,3}^{-1}(\alpha) \cap V_2 \neq \emptyset$.

We first assume $\alpha = f_{2,3}(\mathbf{z})$ for some $\mathbf{z}$ in $\mathcal{O}_\infty(f_{1,2})$. Then there exists an unbounded point sequence $\{\mathbf{x}_i\}_{i=1,2,3,\ldots,\infty}$ such that $\lim_{i\to\infty} f_{1,2}(\mathbf{x}_i) = \mathbf{z}$ holds. Therefore, $\lim_{i\to\infty} f_{1,3}(\mathbf{x}_i) = \alpha$ holds, which implies that $\alpha \in \mathcal{O}_\infty(f_{1,3})$ holds.

Next, we assume that $\alpha \in \mathcal{O}_\infty(f_{2,3}) \setminus f_{2,3}(\mathcal{O}_\infty(f_{1,2}))$ holds. Then there exists an unbounded point sequence $\mathbf{z}_{j=1,2,\ldots,\infty}$ in $f_{1,2}(V_1)$, since $f_{1,2}(V_1)$ is dense in V_2, such that $\lim_{j\to\infty} f_{2,3}(\mathbf{z}_j) = \alpha$ holds. For each $\mathbf{z}_j$, we can choose one $\mathbf{y}_j$ in $f_{1,2}^{-1}(\mathbf{z}_j)$ and obtain an unbounded point sequence $\mathbf{y}_j$ in V_1 such that we have $\lim_{j\to\infty} f_{1,3}(\mathbf{y}_j) = \lim_{j\to\infty} f_{2,3}(\mathbf{z}_j) = \alpha$, which implies that $\alpha \in \mathcal{O}_\infty(f_{1,3})$ holds as well. □

Consider a parametric polynomial $p \in \mathbb{Q}[U, x]$, with $U = u_1, \ldots, u_d$ as parameters. One can expect that the zero locus of its leading coefficient equals $\mathcal{O}_\infty(V(p))$, see Lemma 2. In fact, Proposition 2 is a more general result, taken from [25]. Two notations are needed: for a polynomial set $F \subset \mathbb{Q}[U]$ we write $V_U(F)$ the zero set of F in $\mathbb{C}^d$; in the lemma below, the norm is the modulus (or absolute value) of a complex number.

Lemma 4. *Let $p := a_0x^m + a_1x^{m-1} + \cdots + a_m$ be a univariate polynomial in $\mathbb{C}[x]$ and $\mathbf{x}$ be the root of p in $\mathbb{C}$ with the maximum norm. Then, for each $j = 1, \ldots, m$, we have $\|\mathbf{x}\| \geq \left(\frac{\|a_j\|}{\|a_0\|\binom{m}{j}}\right)^{\frac{1}{j}}$.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be the m roots of p. Then we have

$$(-1)^{j-1}\frac{a_j}{a_0} = \sum_{\{i_1,i_2,\ldots,i_j\}\subseteq\{1,2,\ldots,m\}} \mathbf{x}_{i_1}\mathbf{x}_{i_2}\cdots\mathbf{x}_{i_j}.$$

Hence, we have $\|(-1)^j\frac{a_j}{a_0}\| \leq \binom{m}{j}\|\mathbf{x}\|^j$, which implies the conclusion. □

Proposition 2. *Let P be a prime ideal in $\mathbb{Q}[U]$ and $p \in \mathbb{Q}[U, x]$ be a polynomial with positive degree in x. We regard U as parameters. Let $I := \langle p, P\rangle$ be the polynomial ideal in $\mathbb{Q}[U, x]$ generated by P and p. Assume that the leading coefficient $\mathrm{lc}(p, x)$ of p w.r.t. x is regular modulo P and assume that $\dim(P) > 0$ holds. Then, we have $\mathcal{O}_\infty(V(I)) = V_U(\mathrm{lc}(p, x) + P)$.*

Proof. According to the continuity of the roots of univariate polynomials [2], we have

$$\mathcal{O}_\infty(V(p)) \subseteq V_U(\mathrm{lc}(p, x)).$$

Therefore, we deduce that $\mathcal{O}_\infty(V(I)) \subseteq V_U(\mathrm{lc}(p, x))$ holds, since we have $V(I) \subseteq V(p)$. Hence, with $\mathcal{O}_\infty(V(I)) \subseteq V_U(P)$, we obtain,

$$\mathcal{O}_\infty(V(I)) \subseteq V_U(\mathrm{lc}(p, x) + P).$$

Next, we show that $\mathcal{O}_\infty(V(I)) \supseteq \mathcal{O}_\infty(V(p))$ holds. This is trivially true if $V_U(\mathrm{lc}(p, x) + P) = \emptyset$ holds, thus we assume that we have $V_U(\mathrm{lc}(p, x)+P) \neq \emptyset$. Let $\alpha \in V_U(\mathrm{lc}(p, x)+P)$. If α specializes p to the zero polynomial, then $\dim(\Pi_{\mathrm{U}}^{-1}(\alpha)) = 1$, which implies that α is in $\mathcal{O}_\infty(V(I))$. Let us assume that α does not specialize p to the zero polynomial. Thus α cancels $(\mathrm{lc}(p, x)$ but does not cancel all the other coefficients of p w.r.t. x. Since $V_U(P)$ is a variety of positive dimension, we can choose a point sequence $\{\mathbf{z}_i\}$ ($i = 1, 2, \ldots, \infty$) in $V_U(P) \setminus V_U(\mathrm{lc}(p, x))$ such that $\lim_{i\to\infty} \mathbf{z}_i = \alpha$. For each $\mathbf{z}_i$, we have $\Pi_{\mathrm{U}}^{-1}(\mathbf{z}_i) \neq \emptyset$. Let $\mathbf{x}_i$ be a point of $\Pi_{\mathrm{U}}^{-1}(\mathbf{z}_i)$ such that its last coordinate (that is, its x-coordinate) has maximum norm. Then, according to Lemma 4, the sequence $\{\mathbf{x}_i\}$ ($i = 1, 2, \ldots, \infty$) must be unbounded, which implies that $\alpha \in \mathcal{O}_\infty(V(I))$ holds. This completes the proof. □

3.2. Non-properness locus of parametric algebraic systems in triangular shape

In this section, we focus on the non-properness locus of Π_{U} restricted at the variety of a parametric regular chain or the variety of the saturated ideal of a regular chain. Let $R := [T, H]$ be an STAS as defined in Section 2. We view R as a parametric algebraic system with the free variables of T as parameters, that is, $u_1, \ldots, u_d$. The following notations are related to the triangular structure of R.

Notation 2. *We denote by* $B_{sep}(T)$, $B_{ini}(T)$, $B_{ie}([T,H])$ *the set of the irreducible factors of*

$$\prod_{t\in T} \mathrm{ires}(\mathrm{discrim}(t, \mathrm{mvar}(t)), T), \prod_{t\in T} \mathrm{ires}(\mathrm{init}(t), T), \quad \text{and} \quad \prod_{f\in H} \mathrm{ires}(f, T),$$

respectively. The set $B_{sep}(T) \cup B_{ini}(T) \cup B_{ie}([T,H])$ *is called the* border polynomial set *of* R*, denoted by* $\mathbf{BPS}(R)$.

With Proposition 3 we will establish that $\mathbf{BPS}(R)$ is the set of factors of the minimal border polynomial of R: this will justify the above terminology. The following lemma is a basic property of the iterated resultant of a polynomial w.r.t. a regular chain, see for instance [6].

Lemma 5. *Let* $G \in \mathbb{Q}[X]$ *be a zero-dimensional regular chain and let* $f \in \mathbb{Q}[X]$ *be a polynomial. Then* $\mathrm{ires}(f, G) = 0$ *holds if and only if there exists a point* $\alpha \in V(G)$ *satisfying* $f(\alpha) = 0$.

Recall that the regular chain T is viewed as a parametric algebraic system with its free variables $U = u_1, \ldots, u_d$ as parameters and with its main variables $X = x_1, \ldots, x_s$ as unknowns.

Notation 3. *Denote by* N *the number* $\prod_{f\in T} \mathrm{mdeg}(f)$. *For each parameter value* $\alpha \in \mathbb{C}^d$*, denote by* N_α *the number of solutions of* $T(\alpha)$*, counted with multiplicities.*

Theorem 1. *Let* $b = \prod_{f\in B_{ini}(T)} f$ *and let* $\alpha \in \mathbb{C}^d$. *Then the following statements hold:*
- (i) *if* $b(\alpha) \neq 0$*, then* $N_\alpha = N$ *holds;*
- (ii) *if* $b(\alpha) = 0$*, then* N_α *is either infinite or less than* N.

Proof. We prove this by induction on s, the number of polynomials in T. When $s = 1$, the result is trivially true according to the Fundamental Theorem of Algebra. Assume that, for $i = 1, 2, \ldots, s-1$, the conclusion holds. We write $T = T_{<s} \cup \{t_s\}$, where t_s has the largest main variable, namely x_s. Define $b_1 := \prod_{f\in B_{ini}(T_{<s})} f$.

Assume first that $b(\alpha) \neq 0$ holds. Then, we have $b_1(\alpha) \neq 0$ and $\mathrm{ires}(\mathrm{init}(t_s)(\alpha), T_{<s}(\alpha)) \neq 0$. Thus, by induction hypothesis, the number of solutions of the specialized regular chain $T_{<s}(\alpha)$, counted with multiplicities, is $\prod_{f\in T_{<s}} \mathrm{mdeg}(f)$. And for each solution β of $T_{<s}(\alpha)$, we have $\mathrm{init}(t_s)(\beta) \neq 0$, by Lemma 5. Therefore, β can be extended to $\mathrm{mdeg}(t_s)$ solutions of t_s counting multiplicities. Therefore, $N(\alpha) = N$ holds whenever $b(\alpha) \neq 0$ holds.

Assume from now on that $b(\alpha) = 0$ holds. Then, there are two scenarios:

1. either $b_1(\alpha) = 0$ holds,
2. or $\mathrm{ires}(\mathrm{init}(t_s)(\alpha), T_{<s}(\alpha)) = 0$ and $b_1(\alpha) \neq 0$ both hold.

Consider first the case where $b_1(\alpha) = 0$ holds. By induction hypothesis, the number of solutions of $T_{<s}(\alpha)$ is either infinite, or less than $\prod_{f\in T_{<s}} \mathrm{mdeg}(f)$. If the number of solutions of $T_{<s}(\alpha)$ is infinite, then $T(\alpha)$ is either inconsistent or has infinitely many solutions and the claim holds. (Consider whether or not finitely many of those solutions cancel $\mathrm{init}(t_s)$.) Assume now that the number of solutions of $T_{<s}(\alpha)$ is less than $\prod_{f\in T_{<s}} \mathrm{mdeg}(f)$. If one solution of $T_{<s}(\alpha)$ can be extended to infinitely many solution of t_s, then the claim is clearly true. Otherwise, each solution of $T_{<s}(\alpha)$ can be extended to at most $\mathrm{mdeg}(t_s)$ solutions of t_s, we have $N_\alpha < N$ holds and the claim is true again.

Consider now the second scenario, that is, $\mathrm{ires}(\mathrm{init}(t_s)(\alpha), T_{<s}(\alpha)) = 0$ and $b_1(\alpha) \neq 0$ both hold. There must exist one solution β of $T_{<s}(\alpha)$ such that $\mathrm{init}(t_s)(\beta) = 0$. For each of such solution β, if β specializes t_s to a zero polynomial, then the conclusion is clearly true; otherwise, β specializes $\mathrm{init}(t_s)$ to zero and β can only be extended to at most $\mathrm{mdeg}(t_s) - 1$ solutions of t_s, which implies that $N_\alpha < N$ holds. □

Consider again a regular chain T that we regard as a polynomial system, parametric in its free variables. Theorem 2 shows that the non-properness locus of T depends essentially on the initials of T. The first claim of this appeared in [25], with a different proof.

Observe that, from Lemma 1, the non-properness locus $\mathcal{O}_\infty(\mathrm{sat}(T))$ is the zero set of a polynomial (which is 1 in case this set is empty) in $\mathbb{C}^d$.

Theorem 2. *Let $b := \prod_{f \in B_{ini}(T)} f$. Then we have $\mathcal{O}_\infty(T) = V_U(b)$. Let b_{sat} be a polynomial such that $\mathcal{O}_\infty(\mathrm{sat}(T)) = V_U(b_{sat})$. Observe that b_{sat} divides b since we have $\mathcal{O}_\infty(T) \supseteq \mathcal{O}_\infty(\mathrm{sat}(T))$. Then for each α satisfying $\frac{b}{b_{sat}}(\alpha) = 0$, the number of solutions N_α is either 0 or infinite.*

Proof. We prove all the claims by induction on s, the number of polynomials in T. When $s = 1$, let $T := \{t\}$. The first part of the claim is true according to Proposition 2. If $\frac{b}{b_{sat}}$ is a constant, the second part is clear. If $\frac{b}{b_{sat}}$ is not constant, then $\frac{b}{b_{sat}}$ must be the content of t; in this case, for each α satisfying $\frac{b}{b_{sat}}(\alpha) = 0$, the parameter value α specializes t to the zero polynomial, and again the conclusion is true.

From now on we assume $s > 1$. We also assume that for $k = 1, 2, \ldots, s-1$, the conclusion holds. When $k > 1$, write T as $T := T_{<k} \cup \{t_k\}$, where t_k has the largest main variable, namely x_k.

Denote by $b_{<s}$ the polynomial $\prod_{f \in B_{ini}(T_{<s})} f$. We claim that

$$V_U(b_{<s}) \subseteq \mathcal{O}_\infty(T) \tag{1}$$

holds. Let $b_{<s,sat}$ be a polynomial such that $\mathcal{O}_\infty(\mathrm{sat}(T_{<s})) = V_U(b_{<s,sat})$ holds. By induction hypothesis, for each point $\alpha \in V_U(\frac{b_{<s}}{b_{<s,sat}})$, either $T_{<s}(\alpha)$ is inconsistent or it has infinitely many solutions; therefore, $T(\alpha)$ is either inconsistent or has infinitely many solutions. Therefore, we have

$$V_U(\frac{b_{<s}}{b_{<s,sat}}) \subseteq \mathcal{O}_\infty(T). \tag{2}$$

Define

$$\Pi_{1\cdots s-1} : \begin{array}{ccc} \mathbb{C}^{d+s} & \to & \mathbb{C}^{d+s-1} \\ (u_1, \ldots, u_d, x_1, \ldots, x_s)) & \mapsto & (u_1, \ldots, u_d, x_1, x_2, \ldots, x_{s-1}) \end{array}$$

and

$$\Pi_{U,s-1} : \begin{array}{ccc} \mathbb{C}^{d+s-1} & \to & \mathbb{C}^{d} \\ (u_1, \ldots, u_d, x_1, x_2, \ldots, x_{s-1})) & \mapsto & (u_1, \ldots, u_d) \end{array}$$

Then we have $\Pi_{\mathrm{U}} = \Pi_{U,s-1} \circ \Pi_{1\cdots s-1}$. It follows from the results of [1] that we have

$$\mathrm{sat}(T) \cap \mathbb{Q}[U, x_1, \ldots, x_{s-1}] = \mathrm{sat}(T_{<s}).$$

Thus, we also have

$$\overline{\Pi_{1\cdots s-1}(V(\mathrm{sat}(T)))} = V_{(U,x_1,x_2,\ldots,x_{s-1})}(\mathrm{sat}(T_{<s})), \tag{3}$$

which implies

$$V_U(b_{<s,sat}) = \mathcal{O}_\infty(\mathrm{sat}(T_{<s})) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T)) \tag{4}$$

by applying Proposition 1 to the composition $\Pi_{\mathrm{U}} = \Pi_{U,s-1} \circ \Pi_{1\cdots s-1}$ restricted at $V(\mathrm{sat}(T))$. Therefore, combining Relations (2) and (4) we obtain

$$V_U(b_{<s}) = V_U(b_{<s,sat}) \cup V_U(\frac{b_{<s}}{b_{<s,sat}}) \subseteq \mathcal{O}_\infty(T),$$

which completes the proof of Relation (1).

Consider now the composition $\Pi_{\mathrm{U}} = \Pi_{U,s-1} \circ \Pi_{1\cdots s-1}$ restricted at $V(t_s + \mathrm{sat}(T_{<s}))$. Denote by O the set

$$\Pi_{U,s-1}\left(\mathcal{O}_\infty\left(\Pi_{1\cdots s-1}|_{V(t_s+\mathrm{sat}(T_{<s}))}\right)\right) \setminus V_U(b_{<s}).$$

Next we show that both

$$V_U(b) \setminus V_U(b_{<s}) \subseteq O \subseteq V_U(b) \tag{5}$$

and

$$O \cup V_U(b_{<s}) = \mathcal{O}_\infty(T) \tag{6}$$

hold. From there, Relations (5) and (6) combined with the fact that $V_U(b_{<s}) \subseteq V_U(b)$ holds by definition of b and $b_{<s}$, we can conclude that $V_U(b) = \mathcal{O}_\infty(T)$ holds.

We will first show that Relation (5) holds. By Proposition 2, we have

$$O = \Pi_{U,s-1}\left(V_{(U,x_1,\ldots,x_{s-1})}(\mathrm{init}(t_s) + \mathrm{sat}(T_{<s}))\right) \setminus V_U(b_{<s}).$$

On one hand, clearly, we have $b \in \mathrm{init}(t_s) + \mathrm{sat}(T_{<s})$; therefore, we have $O \subset V_U(b)$. On the other hand, for each point $\alpha \in V_U(b) \setminus V_U(b_{<s})$, we have

$$V_{(x_1,x_2,\ldots,x_{s-1})}(\mathrm{sat}(T_{<s})(\alpha)) = V_{(x_1,x_2,\ldots,x_{s-1})}(T_{<s}(\alpha)) \neq \emptyset.$$

Indeed, α specializes $T_{<s}$ well to a regular chain, and $\mathrm{ires}(\mathrm{init}(t_s), T_{<s})(\alpha) = 0$ holds if and only if $b(\alpha) = 0$ holds. Therefore, each α can be lifted to a solution of $V_{(U,x_1,\ldots,x_{s-1})}(\mathrm{init}(t_s) + \mathrm{sat}(T_{<s}))$, which implies $V_U(b) \setminus V_U(b_{<s}) \subseteq O$. The above two arguments complete the proof of Relation (5).

Now the only thing remaining to show is Equation (6). It follows from (3) that we have

$$\overline{\Pi_{1\cdots s-1}(V(t_s + \mathrm{sat}(T_{<s})))} = V_{(U,x_1,x_2,\ldots,x_{s-1})}(\mathrm{sat}(T_{<s})), \tag{7}$$

since $\overline{\Pi_{1\cdots s-1}(V(t_s + \mathrm{sat}(T_{<s})))} \supseteq \overline{\Pi_{1\cdots s-1}(V(\mathrm{sat}(T)))}$ holds. We apply Proposition 1 to the composition $\Pi_{\mathrm{U}} = \Pi_{U,s-1} \circ \Pi_{1\cdots s-1}$ restricted at $V(t_s + \mathrm{sat}(T_{<s}))$, thanks to Relation (7), we deduce

$$\mathcal{O}_\infty(V(t_s + \mathrm{sat}(T_{<s}))) = \mathcal{O}_\infty(\Pi_{U,s-1}|_{V(\mathrm{sat}(T_{<s}))}) \cup \Pi_{U,s-1}\left(\mathcal{O}_\infty\left(\Pi_{1\cdots s-1}|_{V(t_s+\mathrm{sat}(T_{<s}))}\right)\right). \tag{8}$$

On one hand, from Equation (8) and the inclusion

$$\mathcal{O}_\infty(\Pi_{U,s-1}|_{V(\mathrm{sat}(T_{<s}))}) \subseteq \mathcal{O}_\infty(\Pi_{U,s-1}|_{V(T_{<s})}) = V_U(b_{<s}),$$

we deduce that the inclusion

$$O = \mathcal{O}_\infty(V(t_s + \mathrm{sat}(T_{<s}))) \setminus V_U(b_{<s}). \tag{9}$$

On the other hand, we observe that

$$V(T) \setminus V(b_{<s}) = V(t_s + \mathrm{sat}(T_{<s})) \setminus V(b_{<s})$$

holds, thus we have

$$\mathcal{O}_\infty(T) \setminus V_U(b_{<s}) = \mathcal{O}_\infty(V(t_s + \mathrm{sat}(T_{<s}))) \setminus V_U(b_{<s}). \tag{10}$$

Combining Equations (9) and (10), with Relation (1), we deduce that Equation (6) holds. This completes the proof of the first claim of the conclusion.

Next, let us prove the second claim of the theorem, To this end, we observe that it is sufficient to establish the following statement: if for $\alpha \in V_U(b)$ the polynomial system $T(\alpha)$ has at least one but finitely many solutions, then $\alpha \in \mathcal{O}_\infty(\mathrm{sat}(T))$ holds.

If $\alpha \in V_U(b_{<s})$, the claim is clearly true by induction. Now we assume that $b_{<s}(\alpha) \neq 0$ holds. Then there must exists one solution β of $T_{<s}(\alpha)$ such that β specializes $\mathrm{init}(t_s)$ to 0 and specializes t_s to be a polynomial of degree greater or equal than 1. We can find a sequence of points $\alpha_1, \ldots, \alpha_n, \ldots$ in $\mathbb{C}^d \setminus V_U(b)$, such that $\lim_{i\to\infty} \alpha_i = \alpha$ holds. Then, by the continuity of the roots of a univariate polynomial (see [2]), for each α_i, we can find one solution of $T_{<s}(\alpha_i)$, say β_i, such that, $\lim_{i\to\infty} \beta_i = \beta$. For each β_i, let $a_{s,i}$ be the root of $t_s(\beta_i)$ with the maximum norm. We observe, for each i, $(\beta_i, a_{s,i})$ is in $V(\mathrm{sat}(T))$ since each α_i is chosen to satisfy $b(\alpha_i) \neq 0$. Also, we deduce that $\lim_{i\to\infty} \|(\beta_i, a_{s,i})\| = \infty$, according to Lemma 4. Therefore, α is in $\mathcal{O}_\infty(\mathrm{sat}(T))$. This proves the above statement and thus completes the proof of the theorem. □

The above two results, Theorem 1 and Theorem 2, show that for a regular chain regarded as a parametric algebraic system, Z^*-continuity is equivalent to the properness of the Π_{U} map. We state this equivalence formally in the following corollary.

Corollary 2. *Consider a regular chain T, regarded as a parametric system with its free variables as parameters. Let α be any parameter value. Then T is Z^*-continuous at α if and only if Π_{U} restricted at $V(T)$ is proper at α.*

An algorithm for computing the non-properness locus of a general polynomial map can be found in [20]. With the next proposition, we show a nicer construction of $\mathcal{O}_\infty(\mathrm{sat}(T))$, which can be exploited to design new algorithms to compute the $\mathcal{O}_\infty$ set of a parametric polynomial system.

Recall that T is a squarefree regular chain with $U = u_1, \ldots, u_d$ and $X = x_1, x_2, \ldots, x_s$ as free variables and algebraic variables respectively; let $I = \mathrm{sat}(T)$.

Lemma 6. *For each $i = 1 \cdots s$, the ideal $I \cap \mathbb{Q}[U, x_i]$ is a principal ideal generated by a polynomial $g_i \in \mathbb{Q}[U, x_i]$ whose content w.r.t. x_i belongs to $\mathbb{Q}$.*

Proof. Let $\{P_j \mid j = 1, 2, \ldots, e\}$ be the set of the associated primes of I. Then for each j, the set U is a variable set which is algebraically independent modulo P_j and $\subseteq$-maximal with that property. For each $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, e$, we denote by $Q_{j,i}$ the ideal $P_j \cap \mathbb{Q}[U, x_i]$. Clearly, the ideal $Q_{j,i}$ is prime and U is a $\subseteq$-maximal algebraically independent modulo $Q_{j,i}$.

Consider two distinct polynomials $f, g \in Q_{j,i}$. Since their resultant lies in $Q_{j,i}$ and has degree zero in x_i, this latter polynomial must be null. Thus $h := \gcd(f, g)$ has a positive degree w.r.t. x_i. Since $Q_{j,i}$ is prime, either h or f/h must belong to $Q_{j,i}$. From there, it is routine (proceeding by contradiction) to show that $Q_{j,i}$ is a principal ideal. Moreover, the fact that $Q_{j,i}$ is prime implies that $Q_{j,i}$ is generated by an irreducible polynomial, say $g_{j,i}$.

Denote by g_i the polynomial $\prod_{j=1}^{e} g_{j,i}$. Note that $I \cap \mathbb{Q}[U, x_i] = \bigcap_{j=1}^{e} Q_{j,i}$ holds. Therefore, $I \cap \mathbb{Q}[U, x_i] = \langle g_i \rangle$. And it is obvious that g_i is content free. □

Theorem 3. *For each $i = 1, \ldots, s$, let g_i be a polynomial generating the principal ideal $\mathrm{sat}(T) \cap \mathbb{Q}[U, x_i]$. Then, we have*

$$\mathcal{O}_\infty(\mathrm{sat}(T)) = \cup_{i=1}^{s} V_U(\mathrm{init}(g_i)).$$

Proof. Let $\mathcal{I} = \langle g_1, g_2, \ldots, g_s \rangle$. We observe that $\mathcal{I}$ is a regular chain and Theorem 2 applies. Therefore, we have

$$\mathcal{O}_\infty(\mathcal{I}) = \cup_{i=1}^{s} V_U(\mathrm{init}(g_i)).$$

We define

$$\Pi_i : \mathbb{C}^{d+s} \to \mathbb{C}^{d+1}, \Pi_i((U, x_1, \ldots, x_s)) = (U, x_i)$$

and

$$\Pi_{i+} : \mathbb{C}^{d+1} \to \mathbb{C}^{d}, \Pi_i((U, x_i)) = (U).$$

We have $\Pi_{\mathrm{U}} = \Pi_{i+} \circ \Pi_i$. For each $i = 1, \ldots, s$, we have

$$\overline{\Pi_i(V(\mathrm{sat}(T)))} = V_{(U,x_i)}(g_i)$$

and

$$\mathcal{O}_\infty(\Pi_{i+}|_{V_{(U,x_i)}(g_i)} = V_U(\mathrm{init}(g_i).$$

Therefore, by Proposition 1, for each $i = 1, \ldots, s$, we have

$$V_U(\mathrm{init}(g_i)) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T)),$$

which implies

$$\cup_{i=1}^{s} V_U(\mathrm{init}(g_i)) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T)).$$

Since $V(\mathrm{sat}(T)) \subseteq V(\mathcal{I})$ holds, we have $\mathcal{O}_\infty(\mathrm{sat}(T)) \subseteq \mathcal{O}_\infty(\mathcal{I})$. Finally, we have

$$\cup_{i=1}^{s} V_U(\mathrm{init}(g_i)) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T)) \subseteq \mathcal{O}_\infty(\mathcal{I}) = \cup_{i=1}^{s} V_U(\mathrm{init}(g_i)),$$

which yields the conclusion. □

We conclude this section with a simple example illustrating Theorem 2 and Theorem 3.

Example 6. *Let $T := \{ax+b, by+a\}$ be a regular chain with variable ordering $a < b < x < y$. We regard T as a parametric polynomial system with a, b as parameters. Observe that $\mathrm{sat}(T) \cap \mathbb{Q}[a,b,x] = \langle ax+b \rangle$ holds. Indeed the irreducible polynomial $ax+b$ lies in $\mathrm{sat}(T) \cap \mathbb{Q}[a,b,x]$, which is a principal ideal, thanks to Lemma 6. Similarly $\mathrm{sat}(T) \cap \mathbb{Q}[a,b,y] = \langle by+a \rangle$ holds. Now applying Theorem 3, we deduce $\mathcal{O}_\infty(\mathrm{sat}(T)) = V_{(a,b)}(ab)$.*

This latter fact can also be justified by combining the two following observations.

(1) *We have $\mathcal{O}_\infty(T) = V_{(a,b)}(ab)$ by applying Theorem 2. Indeed, we have $B_{ini}(T) = \{a, b\}$. Thus, since $\mathcal{O}_\infty(\mathrm{sat}(T))$ is a hypersurface contained in $\mathcal{O}_\infty(T)$, we have $\mathcal{O}_\infty(\mathrm{sat}(T)) \subseteq V_{(a,b)}(ab)$.*

(2) *On one hand, for all parameter value α satisfying either $a = 0, b \neq 0$ or $b = 0, a \neq 0$, we have $V_{(x,y)}(\mathrm{sat}(T)(\alpha)) \subseteq V_{(x,y)}(T(\alpha)) = \emptyset$ which implies $\alpha \in \mathcal{O}_\infty(\mathrm{sat}(T))$ by Lemma 3. One the other hand, the Zariski closure of $V_{(a,b)}(a) \setminus V_{(a,b)}(b) \cup V_{(a,b)}(b) \setminus V_{(a,b)}(a)$ is $V_{(a,b)}(ab)$. Thus, we deduce $V_{(a,b)}(ab) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T))$.*

4. Z-continuity and Π_{U}-continuity of Parametric Algebraic Triangular Systems

In this section, we study the minimal discriminant variety of an STAS, regarded as a parametric system in the free variables of its regular chain. We show that for this type of parametric systems the notions of Z-continuity and Π_{U}-continuity coincide. Then, we compare the minimal discriminant variety of a regular chain T and that of its saturated ideal, both regarded as a parametric system in the free variables of T. Finally, we show that among all regular chains having the same saturated ideal as T, the canonical regular chain associated with T has a $\subseteq$-minimal border polynomial set.

4.1. The minimal discriminant variety of a parametric STAS

In this subsection, we focus on the characterization of the minimal discriminant variety of an STAS $R := [T, H]$, as defined in Section 2. We view an STAS as a parametric algebraic system with the free variables of T as parameters.

Proposition 3 and Theorem 4 imply that the notions of Z-continuity and Π_{U}-continuity coincide for STASes. In particular, Theorem 4 shows that the minimal discriminant variety of R can be characterized by $\mathbf{BPS}(R)$ (see Notation 2).

Lemma 7. *Let $b := \prod_{f \in B_{ini}(T)} f$ and α be any parameter value satisfying $b(\alpha) \neq 0$. Then, we have $\mathrm{ires}(f,T)(\alpha) = 0$ if and only if $\mathrm{ires}(f(\alpha), T(\alpha)) = 0$ holds.*

Proof. This follows from Theorem 8 and Proposition 11 in [6]. □

Proposition 3. *Let $b = \prod_{f \in \mathbf{BPS}(R)} f$; let $N := \prod_{f \in T} \mathrm{mdeg}(f)$. Then for each parameter value $\alpha \in \mathbb{C}^d$:*

(a) *if $b(\alpha) \neq 0$, then $\# Z(R(\alpha)) = N$ holds;*
(b) *if $b(\alpha) = 0$, then $\# Z(R(\alpha))$ is either infinite or less than N.*

Proof. Let $b_i := \prod_{f \in B_{ini}(T)} f$; $b_s := \prod_{f \in B_{sep}(T)} f$; $b_f := \prod_{f \in B_{ie}(R)} f$. Let $\alpha \in \mathbb{C}^d$ be a parameter value. Assume first that $b(\alpha) \neq 0$ holds. Then, by Lemma 7, the following facts hold:

(i) $b_i(\alpha) \neq 0$,
(ii) for each $i \in \{1, 2, \ldots, s\}$, we have $\mathrm{ires}(\mathrm{discrim}(t_i(\alpha), x_i), T) \neq 0$,
(iii) $\mathrm{ires}(h(\alpha), T(\alpha)) \neq 0$.

From Fact (i), we deduce by Theorem 1 that $T(\alpha)$ has N zeros, counted with multiplicities. Thanks to Lemma 5, Fact (ii) implies that T does not have multiple zeros. Fact (iii) means that, for each polynomial $h \in H$ and each zero $\mathbf{x}$ of $T(\alpha)$, we have $h(\mathbf{x}) \neq 0$. Therefore, Claim (a) holds.

From now on, we assume that $b(\alpha) = 0$ holds. Three cases can occur:

(i) $b_i(\alpha) = 0$ holds, then $T(\alpha)$ either has infinitely many solutions or has less than N solutions, counted with multiplicities;
(ii) $b_i(\alpha) \neq 0$ and $b_s(\alpha) = 0$ hold, then $T(\alpha)$ is a regular chain with multiple zeros; and it has N zeros, counted with multiplicities;
(iii) $b_i(\alpha) \neq 0$, $b_s(\alpha) \neq 0$ and $b_f(\alpha) = 0$ hold, then $T(\alpha)$ has N simple zeros, and at least one of them vanishes some polynomials in H.

In any case, Claim (b) holds. This completes the proof. □

The following Theorem appeared in [25]. Here we supply a new proof, which relies directly on the concept of Π_{U}-continuity.

Theorem 4. *Let $b = \prod_{f\in \mathbf{BPS}(R)} f$. Then, the hypersurface $V_U(b)$ of $\mathbb{C}^d$ is the minimal discriminant variety of R.*

Proof. By Proposition 3, we know that $Z(b = 0)$ contains the minimal discriminant variety. Next, we shall show that, R is Π_{U}-continuous at each α where $b(\alpha) \neq 0$ holds.

Let $(\alpha, \mathbf{y}_1), (\alpha, \mathbf{y}_2), \ldots, (\alpha, \mathbf{y}_N)$ be the N simple solutions of $R(\alpha)$. Then by the Implicit Function Theorem, there exists a neighborhood C_α of α in $\mathbf{C}^d$ such that for each point $(\alpha, \mathbf{y}_i)$, there exist a diffeomorphic function ϕ_i such that

$$S_i := \{(U, \phi_i(U)) | U \in C_\alpha\} \subset Z(R)$$

and $\phi_i(\alpha) = \mathbf{y}_i$ hold. Moreover, we can choose C_α such that $S_i \cap S_j = \emptyset$ when $i \neq j$. It is obvious that each S_i is diffeomorphic to C_α. By Proposition 3, it is easy to deduce that $\Pi_{\mathrm{U}}^{-1}(C_\alpha) = \cup S_i$ holds. This shows that R is Π_{U}-continuous at α. □

Corollary 3. *Let $R := [T, H]$ be an STAS, regarded as a parametric system with the free variables of T as parameters. Let α be any parameter value. Then R is Z-continuous at α if and only if R is Π_{U}-continuous at α.*

4.2. The minimal discriminant variety of a saturated ideal

As before, let us denote by $U = u_1, u_2, \ldots, u_d$ and $X = x_1, x_2, \ldots, x_s$ the set of free and algebraic variables of our regular chain T. Since $\mathrm{sat}(T)$ is a strongly equidimensional ideal, [4] it is natural to view it as a parametric system with U as parameters and compare its minimal discriminant variety with that of T, also regarded as a parametric system in U.

In this section, we perform this comparison, see the results of Proposition 4 and 5. We shall also show, with Theorem 4 and Theorem 5, that among all regular chains having $\mathrm{sat}(T)$ as saturated ideal, the *canonical regular chain* associated with T has a discriminant variety of is $\subseteq$-minimal.

We denote by DV_T (resp. $DV_{\mathrm{sat}(T)}$) the minimal discriminant variety of T (resp. $\mathrm{sat}(T)$).

Proposition 4. *Let $R := [\mathrm{sat}(T), B_{ini}(T)_{\neq}]$ [5] and denote by DV_R the minimal discriminant of R. Then $DV_R = DV_T$ holds. In particular, we have*

$$DV_T = V_U(\prod_{f\in B_{ini}(T)\cup B_{sep}(T)} f) = DV_{\mathrm{sat}(T)} \cup V_U(\prod_{f\in B_{ini}(T)} f).$$

Proof. It is obvious that $V_U(\prod_{f\in B_{ini}(T)} f)$ is contained in DV_R, since they are not in the image of $\Pi_{\mathrm{U}}(Z(R))$. By Theorem 4, we know that $V_U(\prod_{f\in B_{ini}(T)} f) \subseteq DV_T$ holds.

Now we consider any point $\alpha \notin V_U(\prod_{f\in B_{ini}(T)} f)$. It is easy to deduce that $Z(\mathrm{sat}(T)(\alpha) = Z(T(\alpha))$ holds, which implies that: R is Π_{U}-continuous at α if and only if T is. That is, $DV_R \setminus$

[4] More precisely, $\mathrm{sat}(T)$ is an equidimensional ideal of dimensional d such that U a $\subseteq$-maximal algebraically independent set modulo each associated prime of $\mathrm{sat}(T)$.
[5] Here, $[\mathrm{sat}(T), B_{ini}(T)_{\neq}]$ is regarded as the parametric algebraic system with equations defined by any basis of $\mathrm{sat}(T)$ and with inequations defined by $B_{ini}(T)_{\neq}$.

$V_U(\prod_{f\in B_{ini}(T)} f) = DV_T \setminus V_U(\prod_{f\in B_{ini}(T)} f)$ holds. This completes the proof of the fact $DV_R = DV_T$. The latter statement holds since $DV_{\mathrm{sat}(T)} \subseteq DV_R$ holds, which can be checked by the definition of Π_{U}-continuity. □

The following proposition gives an upper bound on the set theoretic difference $DV_T \setminus DV_{\mathrm{sat}(T)}$.

Proposition 5. *We have*

$$DV_T \setminus DV_{\mathrm{sat}(T)} \subseteq V_U(\prod_{f\in B_{ini}(T)} f) \setminus \mathcal{O}_\infty(\mathrm{sat}(T)).$$

Proof. Since $DV_T = DV_{\mathrm{sat}(T)} \cup V_U(\prod_{f\in B_{ini}(T)} f)$ holds (see Proposition 4), we have

$$DV_T \setminus DV_{\mathrm{sat}(T)} \subseteq V_U(\prod_{f\in B_{ini}(T)} f) \setminus DV_{\mathrm{sat}(T)},$$

hence,

$$DV_T \setminus DV_{\mathrm{sat}(T)} \subseteq V_U(\prod_{f\in B_{ini}(T)} f) \setminus \mathcal{O}_\infty(\mathrm{sat}(T))$$

holds, since we have $\mathcal{O}_\infty(\mathrm{sat}(T)) \subseteq DV_{\mathrm{sat}(T)}$ holds. □

The following proposition shows that the difference of $DV_T \setminus DV_{\mathrm{sat}(T)}$ is actually dominated by the difference of the non-properness locus of T and that of $\mathrm{sat}(T)$, respectively denoted by $\mathcal{O}_\infty(T)$ and $\mathcal{O}_\infty(\mathrm{sat}(T))$.

Since different regular chains may have the same saturated ideal, a natural question to ask is: which regular chain(s) will be the best choice in the sense that the set theoretic difference of DV_T and $DV_{\mathrm{sat}(T)}$ is minimal. This question is answered by Proposition 5 and Theorem 5.

Let us recall the notion of a canonical regular chain [21, 25, 4], which is used in Theorem 5.

Definition 8 (canonical regular chain). *Let T be a regular chain of $\mathbb{Q}[U, X]$. If each polynomial t of T satisfies:*

1. *the initial of t involves only the free variables of T,*
2. *for any polynomial $f \in T$ with $\mathrm{mvar}(f) < \mathrm{mvar}(t)$, we have $\deg(t, \mathrm{mvar}(f)) < \mathrm{mdeg}(f)$,*
3. *t is primitive over $\mathbb{Q}$, w.r.t. its main variable,*

then we say that T is canonical.

Remark 2. *Let $T = \{t_1, \ldots, t_m\}$ be a regular chain; let $d_k = \mathrm{mdeg}(t_k)$, for $k = 1 \ldots m$. One constructs a canonical regular chain $T^* = \{t_1^*, t_2^*, \ldots, t_m^*\}$ such that $\mathrm{sat}(T) = \mathrm{sat}(T^*)$ in the following way:*

1. *set t_1^* to be the primitive part of t_1 w.r.t. y_1;*
2. *for $k = 2, \ldots, m$, let r_k be the iterated resultant $\mathrm{ires}(\mathrm{init}(t_k), \{t_1, \ldots, t_{k-1}\})$. Write $r_k = a_k \,\mathrm{init}(t_k) + \sum_{i=1}^{k-1} c_i t_i$, for some appropriate polynomials $a_k, c_1, \ldots, c_{k-1}$. Compute t as the pseudo-reminder of $a_k t_k + (\sum_{i=1}^{k-1} c_i t_i) y_k^{d_k}$ by $\{t_1^*, \ldots, t_{k-1}^*\}$. Set t_k^* to be the primitive part of t w.r.t. y_k.*

Proposition 6. *Let T_1 and T_2 be two regular chains satisfying $\mathrm{sat}(T_1) = \mathrm{sat}(T_2)$. If $B_{ini}(T_1) \subseteq B_{ini}(T_2)$ holds, then $\mathbf{BPS}(T_1) \subseteq \mathbf{BPS}(T_2)$ holds.*

Proof. The conclusion is a consequence of Proposition 4 and Theorem 4. □

Theorem 5. *Given a squarefree regular chains T of $\mathbb{Q}[U, X]$, there exists a unique canonical regular chain T^* such that $\mathrm{sat}(T) = \mathrm{sat}(T^*)$ holds. Moreover, we have $\mathbf{BPS}(T^*) \subseteq \mathbf{BPS}(T)$.*

Proof. By Remark 2, we can always construct a canonical regular chain T^* such that $\mathrm{sat}(T) = \mathrm{sat}(T^*)$. Moreover, for each $t \in T$, $\mathrm{init}(t^*)$ divides $\mathrm{ires}(\mathrm{init}(t), T)$. Therefore, $B_{ini}(T^*) \subseteq B_{ini}(T)$ holds, which implies $\mathbf{BPS}(T^*) \subseteq \mathbf{BPS}(T)$ by Proposition 6.

Suppose $T^\diamond$ is any given canonical regular chain such that $\mathrm{sat}(T^\diamond) = \mathrm{sat}(T)$ holds. It is sufficient to show that $T^* = T^\diamond$ holds to complete the proof.

Note that $T^\diamond$, T^* and T have the same set of free and algebraic variables, denoted respectively by U and X. Given $\mathbf{I}$ an ideal in $\mathbb{Q}[U, X]$, denote by $\mathbf{I}^{ext}$ the extension of $\mathbf{I}$ in $\mathbb{Q}(U)[X]$. Since $\mathfrak{p}^{ext} = \langle 1 \rangle$ holds for any prime ideal $\mathfrak{p}$ in $\mathbb{Q}[U, X]$ with U algebraically dependent, we have $\langle T^* \rangle^{ext} = \langle T^\diamond \rangle^{ext} = \mathrm{sat}(T)^{ext}$ holds. Therefore, the polynomials in T^* (or $T^\diamond$) form a Gröbner basis of $\mathrm{sat}(T)^{ext}$ (w.r.t. the lexicographical ordering on X) since their leading power products are pairwise coprime. Dividing each polynomial in T^* (or $T^\diamond$) by its initial, we obtain the unique reduced Gröbner basis of $\mathrm{sat}(T)^{ext}$. This implies $T^* = T^\diamond$. □

5. Conclusion and discussion

As we mentioned in Section 2, there are essentially two steps in solving a parametric system S:

(1) describe the parameter values where the continuity does not hold,
(2) describe the (groups of) regions where the continuity is maintained.

The present paper was dedicated to Step (1). As mentioned also previously, in the semi-algebraic case, Step (1) can be reduced to the algebraic case. Thus, this paper has mainly focused on Step (1) for parametric algebraic systems.

We have shown that for parametric polynomial triangular systems the notion of the Z-continuity is equivalent to that of Π_{U}-continuity. We are currently working on a generalization of this equivalence to a broader context, e.g. when the ideal associated to the parametric system is equidimensional. Thus, we would like to prove the following statement. Given a well-determinate parametric algebraic system S such that its associated ideal is equidimensional, for any parametric value α, the system S is Z-continuous at α if and only if S is Π_{U}-continuous at α.

We are also working on generalizing the results of Theorem 1 and Theorem 2. In this case, we would like to establish the following result. Given a well-determinate parametric algebraic system S, such that its associated ideal I is equidimensional, for any parametric value α, the system S is Z^*-continuous at α if and only if Π_{U} (restricted at $V(I)$) is proper at α.

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