Polynomial Data-Types

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Plan

Polynomial system solvers in action

Algebraic structures

Polynomials in algebra

Univariate polynomial data-type

Multivariate polynomial data-type

Polynomial representations by values

Straigth-line programs

Big integers

Plan

MAPLE' `solve` command

```
> solve(x^2+x-2, {x});
> solve(x^3+x-2, {x});
> solve(x^5+x-2, {x});
```

$$\{x=1\}, \{x=-2\}$$

$$\{x=1\}, \left\{x=-\frac{1}{2}-\frac{1}{2}\,\mathrm{I}\sqrt{7}\right\}, \left\{x=-\frac{1}{2}+\frac{1}{2}\,\mathrm{I}\sqrt{7}\right\}$$

$$\{x=1\}, \{x=\mathit{RootOf}(\_Z^4+\_Z^3+\_Z^2+\_Z+2, \mathit{index}=1)\}, \{x=\mathit{RootOf}(\_Z^4+\_Z^3+\_Z^2+\_Z+2, \mathit{index}=2)\}, \{x=\mathit{RootOf}(\_Z^4+\_Z^3+\_Z^2+\_Z+2, \mathit{index}=3)\}, \{x=\mathit{RootOf}(\_Z^4+\_Z^3+\_Z^2+\_Z+2, \mathit{index}=4)\} \quad (2)$$

```
> solve({x^2 + y -1, x + y^2 -1}, {x,y});
```

$$\{x=1, y=0\}, \{x=0, y=1\}, \{x=\mathit{RootOf}(\_Z^2+\_Z-1), y=\mathit{RootOf}(\_Z^2+\_Z-1)\} \quad (3)$$

```
> solve(z^2 + z -1, {z});
```

$$\left\{z=\frac{1}{2}\sqrt{5}-\frac{1}{2}\right\}, \left\{z=-\frac{1}{2}-\frac{1}{2}\sqrt{5}\right\} \quad (4)$$

```
> fsolve({x^2 + y -1, x + y^2 -1}, {x,y});
```

$$\{x=1.000000000, y=0.\} \quad (5)$$

```
> fsolve(z^2 + z -1, {z});
```

$$\{z=-1.618033989\}, \{z=0.6180339887\} \quad (6)$$

Real root isolation for zero-dimensional systems

Applications Places System moreno

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$with(RegularChains) : with(SemiAlgebraicSetTools) : with(ParametricSystemTools) :$ with(ParametricSy

$R := PolynomialRing([x, y, z]);\ \ F := [x\verb|^|2 + y + z - 1, x + y\verb|^|2 + z - 1, x + y + z\verb|^|2 - 1];$

$$polynomial\_ring$$

$$[x^2 + y + z - 1, y^2 + x + z - 1, x + y + z^2 - 1]$$

$dec := Triangularize(F, R);\ \ map(Display, dec, R);$

$$[regular\_chain, regular\_chain, regular\_chain, regular\_chain]$$

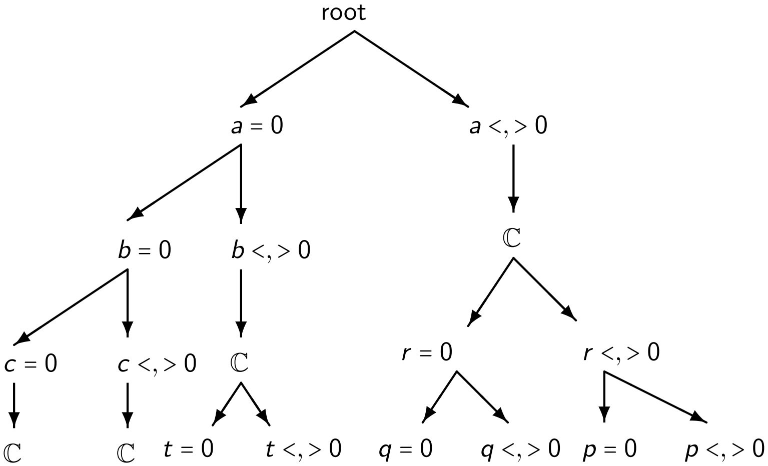
$$\left[\left\{\begin{array}{c} x - z = 0 \\ y - z = 0 \\ z^2 + 2\,z - 1 = 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z - 1 = 0 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y - 1 = 0 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x - 1 = 0 \\ y = 0 \\ z = 0 \end{array}\right.\right]$$

$boxes := \left[seq\left(op\left(RealRootIsolate\left(rc, R, 'rerr' = \frac{1}{2^9}\right)\right), rc = dec\right)\right];\ map\left(Display, boxes, R\right.$

$$[box, box, box, box, box]$$

$$\left[\left\{\begin{array}{c} x = \left[\frac{3393}{8192}, \frac{6791}{16384}\right] \\ y = \left[\frac{3393}{8192}, \frac{6791}{16384}\right] \\ z = \left[\frac{217167}{524288}, \frac{868669}{2097152}\right] \end{array}\right., \left\{\begin{array}{c} x = \left[-\frac{4947}{2048}, -\frac{2471}{1024}\right] \\ y = \left[-\frac{4947}{2048}, -\frac{2471}{1024}\right] \\ z = \left[-\frac{79109}{32768}, -\frac{316435}{131072}\right] \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y = 0 \\ z = 1 \end{array}\right., \left\{\begin{array}{c} x = 0 \\ y = 1 \\ z = 0 \end{array}\right., \left\{\begin{array}{c} x = \\ y = \\ z = \end{array}\right.\right.$$

Cylindrical algebraic decomposition of $\{ax^2 + bx + c\}$



The cylindrical algebraic decomposition of $\{ax^2 + bx + c\}$ is given by the tree above, where $t = bx + c$, $q = 2ax + b$, and $r = 4ac - b^2$. This is the best possible output for that method, leading to **27 cells**!

Can a computer program be as good as a high-school student?

For the equation $ax^2 + bx + c = 0$, can a computer program produce?

$$\begin{cases} ax^2 + bx + c = 0 \\ a \neq 0 \wedge 4ac - b^2 > 0 \end{cases} \quad \begin{cases} 2ax + b = 0 \\ 4ac - b^2 = 0 \\ a \neq 0 \end{cases}$$

$$\begin{cases} bx + c = 0 \\ a = 0 \\ b \neq 0 \end{cases} \quad \begin{cases} c = 0 \\ b = 0 \\ a = 0 \end{cases}$$

Yes, RealTriangularize in MAPLE can do that!

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with(RegularChains) : with(SemiAlgebraicSetTools) : with(ParametricSystemTools) : with(ParametricSy

$R := PolynomialRing([x, c, b, a]);\ F := [a \cdot x^2 + b \cdot x + c];$

$$polynomial\_ring$$

$$[a x^2 + b x + c]$$

Solving for the real solutions:

$RealTriangularize(F, R, output = record);$

$$\left\{\begin{array}{c} a x^2 + b x + c = 0 \\ -4 c a + b^2 > 0 \\ a \neq 0 \end{array}\right., \left\{\begin{array}{c} b x + c = 0 \\ b \neq 0 \\ a = 0 \end{array}\right., \left\{\begin{array}{c} c = 0 \\ b = 0 \\ a = 0 \end{array}\right., \left\{\begin{array}{c} 2 a x + b = 0 \\ 4 a c - b^2 = 0 \\ a \neq 0 \end{array}\right.$$

Solving **for** *the complex solutions*

$dec := Triangularize(F, R, output = lazard);\ map(Display, dec, R);$

$$[regular\_chain, regular\_chain, regular\_chain]$$

$$\left[\left\{\begin{array}{c} a x^2 + b x + c = 0 \\ a \neq 0 \end{array}\right., \left\{\begin{array}{c} b x + c = 0 \\ a = 0 \\ b \neq 0 \end{array}\right., \left\{\begin{array}{c} c = 0 \\ b = 0 \\ a = 0 \end{array}\right.\right]$$

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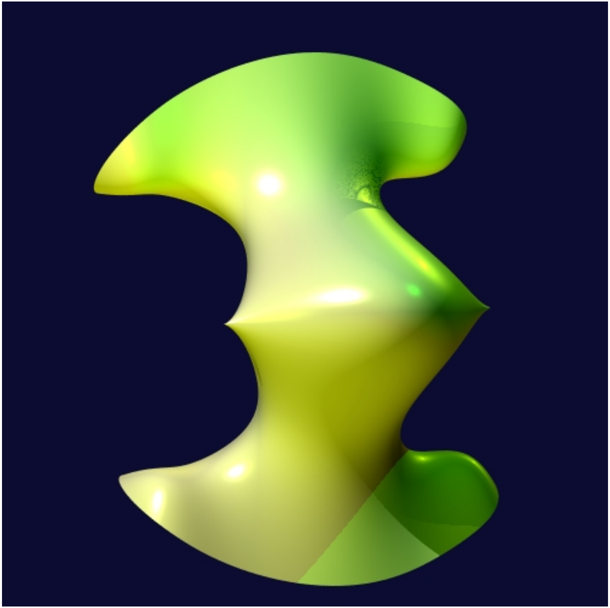
RealTriangularize applied to the *Eve* surface (1/2)



RealTriangularize applied to the *Eve* surface (2/2)

Applications Places System moreno

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```
R := PolynomialRing([x, y, z]); F := [5*x^2 + 2*x*z^2 + 5*y^6 + 15*y^4 + 5*z^2 - 15*y^5 - 5
```

$$polynomial\_ring$$

$$[5\,x^2+2\,x\,z^2+5\,y^6+15\,y^4+5\,z^2-15\,y^5-5\,y^3]$$

```
RealTriangularize(F, R, output = record);
```

$$\left\{\begin{array}{c} 5\,x^2+2\,z^2\,x+5\,y^6+15\,y^4-5\,y^3-15\,y^5+5\,z^2=0 \\ 25\,y^6-75\,y^5+75\,y^4-z^4-25\,y^3+25\,z^2<0 \end{array}\right.,$$

$$\left\{\begin{array}{c} 5\,x+z^2=0 \\ 25\,y^6-75\,y^5+75\,y^4-25\,y^3-z^4+25\,z^2=0 \\ 64\,z^4-1600\,z^2+25>0 \\ z\neq 0 \\ z-5\neq 0 \\ z+5\neq 0 \end{array}\right., \left\{\begin{array}{c} x=0 \\ y-1=0 \\ z=0 \end{array}\right., \left\{\begin{array}{c} x=0 \\ y=0 \\ z=0 \end{array}\right., \left\{\begin{array}{c} x+5=0 \\ y-1=0 \\ z-5=0 \end{array}\right.,$$

$$\left\{\begin{array}{c} x+5=0 \\ y=0 \\ z-5=0 \end{array}\right., \left\{\begin{array}{c} x+5=0 \\ y-1=0 \\ z+5=0 \end{array}\right., \left\{\begin{array}{c} x+5=0 \\ y=0 \\ z+5=0 \end{array}\right., \left\{\begin{array}{c} 5\,x+z^2=0 \\ 2\,y-1=0 \\ 64\,z^4-1600\,z^2+25=0 \end{array}\right.$$

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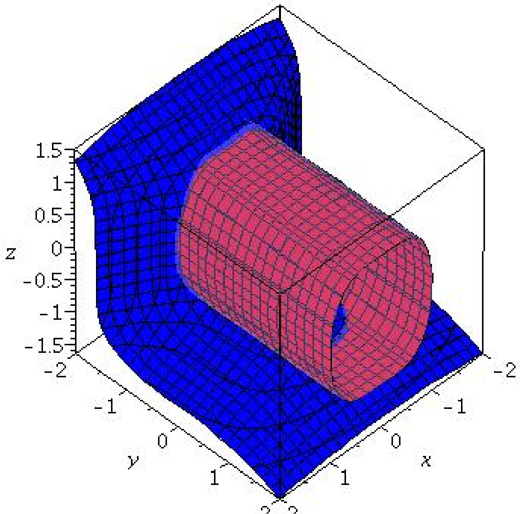
Triangularize (not RealTriangularize) applied to *sofa* and *cylinder* (1/2)

$$x^2 + y^3 + z^5 = x^4 + z^2 - 1 = 0$$



Triangularize applied to *sofa* and *cylinder* (2/2)

```
> R := PolynomialRing([z, y, x]): F := [x^2+y^3+z^5, x^4+z^2-1]: dec :=
  Triangularize(F, R): map(Display, dec, R);
```

$$\left[\left\{\begin{array}{c} \left(-2x^4+x^8+1\right)z+x^2+y^3=0 \\ y^6+2x^2y^3+10x^{12}-10x^8+x^{20}-5x^{16}+6x^4-1=0 \\ -2x^4+x^8+1\neq 0 \end{array}\right.\right]$$

```
> dec := Triangularize(F, R, output=lazard): map(Display, dec, R);
```

$$\left[\left\{\begin{array}{c} \left(-2x^4+x^8+1\right)z+x^2+y^3=0 \\ y^6+2x^2y^3+10x^{12}-10x^8+x^{20}-5x^{16}+6x^4-1=0 \\ -2x^4+x^8+1\neq 0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y^2+y+1=0 \\ x^2+1=0 \end{array}\right.,\right.$$

$$\left\{\begin{array}{c} z=0 \\ y-1=0 \\ x^2+1=0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y^2-y+1=0 \\ x+1=0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y^2-y+1=0 \\ x-1=0 \end{array}\right., \left\{\begin{array}{c} z=0 \\ y+1=0 \\ x+1=0 \end{array}\right.,$$

$$\left.\left\{\begin{array}{c} z=0 \\ y+1=0 \\ x-1=0 \end{array}\right.\right]$$

Solving for the integer solutions of a linear system (1/3)

Solve integer programming:

$$\begin{gathered}\min_{\text{lex}}(x_1, \ldots, x_d) \\ \mathbf{Ax} \le \mathbf{b}, \\ \mathbf{x} \in \mathbb{Z}^d\end{gathered}$$

Example

Problem:

$$\begin{aligned}\min_{\text{lex}}(x_3, x_2, x_1) \\ 3x_1 - 2x_2 + x_3 &\le 7 \\ -2x_1 + 2x_2 - x_3 &\le 12 \\ -4x_1 + x_2 + 3x_3 &\le 15 \\ -x_2 &\le -25 \\ x_1, x_2, x_3 &\in \mathbb{Z}\end{aligned}$$

Solving for the integer solutions of a linear system (2/3)

Example

Input: $K_1 : \begin{cases} 3x_1 - 2x_2 + x_3 \le 7 \\ -2x_1 + 2x_2 - x_3 \le 12 \\ -4x_1 + x_2 + 3x_3 \le 15 \\ -x_2 \le -25 \end{cases}$, assume $x_1 > x_2 > x_3$.

Output: $K_1^1, K_1^2, K_1^3, K_1^4, K_1^5$ given by:

$$\begin{cases} 3x_1 - 2x_2 + x_3 \le 7 \\ -2x_1 + 2x_2 - x_3 \le 12 \\ -4x_1 + x_2 + 3x_3 \le 15 \\ 2x_2 - x_3 \le 48 \\ -5x_2 + 13x_3 \le 67 \\ -x_2 \le -25 \\ 2 \le x_3 \le 17 \end{cases}, \begin{cases} x_1 = 15 \\ x_2 = 27 \\ x_3 = 16 \end{cases}, \begin{cases} x_1 = 18 \\ x_2 = 33 \\ x_3 = 18 \end{cases}, \begin{cases} x_1 = 14 \\ x_2 = 25 \\ x_3 = 15 \end{cases}, \begin{cases} x_1 = 19 \\ x_2 = 50 + t \\ x_3 = 50 + 2t \\ -25 \le t \le -16. \end{cases}$$

Solving for the integer solutions of a linear system (2/3)

$$
\begin{tikzcd}[column sep=small]
 & & \begin{array}{c}\min(x_3,x_2,x_1)\\ K_1\cap\mathbb{Z}^3\end{array} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\begin{array}{c}\min(x_3,x_2,x_1)\\ K_1^1\cap\mathbb{Z}^3\end{array} \arrow[d] & \begin{array}{c}\min(x_3,x_2,x_1)\\ K_1^2\cap\mathbb{Z}^3\end{array} \arrow[d] & \begin{array}{c}\min(x_3,x_2,x_1)\\ K_1^3\cap\mathbb{Z}^3\end{array} \arrow[d] & \begin{array}{c}\min(x_3,x_2,x_1)\\ K_1^4\cap\mathbb{Z}^3\end{array} \arrow[d] & \begin{array}{c}\min(x_3,x_2,x_1)\\ K_1^5\cap\mathbb{Z}^3\end{array} \arrow[d] \\
(2,-8,-4) & (16,27,15) & (18,33,18) \arrow[d, Rightarrow] & (15,25,14) & (0,25,19) \\
 & & (0,25,19) & &
\end{tikzcd}
$$

Hence $(0,25,19)$ is a solution to the posed ILP problem.

Plan

Operation (1/2)

Definition

Given a non-empty set $\mathbb{M}$, an *internal operation* (or simply *operation*) over $\mathbb{M}$ is a function f that maps any couple (x, y) of elements from $\mathbb{M}$ with an element $f(x, y)$ of $\mathbb{M}$. The operation f

- is *associative* if the following holds

$$(\forall x, y, z \in \mathbb{M}) \quad f(x, f(y, z)) = f(f(x, y), z),$$

- is *commutative* if the following holds

$$(\forall x, y \in \mathbb{M}) \quad f(x, y) = f(y, x).$$

The set $\mathbb{M}$ possesses an *identity element* if there exists $e \in \mathbb{M}$ such that

$$(\forall x \in \mathbb{M}) \quad f(e, x) = x = f(x, e)$$

Moreover, in this case, an element $x \in \mathbb{M}$ possesses a *symmetric element* (or *reciprocal element*) if the following holds

$$(\exists x' \in \mathbb{M}) \quad f(x, x') = f(x', x) = e$$

Operation (2/2)

Proposition

Let $\mathbb{M}$ be a non-empty set with an operation f.

(*i*) *If $\mathbb{M}$ possesses an identity element, then it is unique.*

(*ii*) *Moreover, in this case, if an element $x \in \mathbb{M}$ possesses a symmetric element $x' \in \mathbb{M}$, then it is unique.*

Remark

For a non-empty set $\mathbb{M}$ with an associative operation f it is natural to define $f(x_1, x_2, \ldots, x_n)$ for $x_1, x_2, \ldots, x_n \in \mathbb{M}$ with $n \geq 3$ by

$$f(x_1, x_2, \ldots, x_n) = f(x_1, f(x_2, \ldots, x_n))$$

Semi-group, group

A *semi-group* is a set $\mathbb{M}$ endowed with an operation such that this operation is associative.

- If for this operation, the set $\mathbb{M}$ admits an identity element, then it is said to be a *monoid*. Furthermore, if for this operation every element possesses a symmetric element, then the monoid is said to be a *group*.
- If this operation is commutative, then it is usually denoted additively (provided that this does lead to confusion with another operation) and the semi-group is said *abelian* or *commutative*. Otherwise this operation is usually denoted multiplicatively.
- If $\mathbb{M}$ is an abelian semi-group and a monoid, then its identity element is denoted 0 and $\mathbb{M}$ is said to be an *abelian monoid*.
- If $\mathbb{M}$ is a monoid which is not known to be commutative then its identity element is denoted 1.
- If $\mathbb{M}$ is an abelian monoid and a group, then the symmetric element of an element $x \in \mathbb{M}$ is denoted $-x$ and called the *opposite* of x. Moreover, in this case, $\mathbb{M}$ is said to be an *abelian group*,
- If $\mathbb{M}$ is a group which is not known to be commutative then the symmetric element of an element $x \in \mathbb{M}$ is denoted x^{-1} and called the *multiplicative inverse* of x (or simply the *inverse* of x).

Semi-ring

A *semi-ring* is a set $\mathbb{A}$ endowed with two operations one being denoted additively and the other being denoted multiplicatively, called respectively the *addition* of $\mathbb{A}$ and the *multiplication* of $\mathbb{A}$ such that

(*i*) $\mathbb{A}$ is an abelian monoid for its addition,

(*ii*) $\mathbb{A}^*$ is a semi-group for its multiplication,

(*iii*) the multiplication of $\mathbb{A}$ is *distributive* w.r.t. its addition, which means that the following two conditions hold:

- $(\forall x, y, z \in \mathbb{A}) \quad x(y+z) = xy + xz$ *(left-distributivity)*,
- $(\forall x, y, z \in \mathbb{A}) \quad (y+z)x = yx + zx$ *(right-distributivity)*.

where $\mathbb{A}^* = \mathbb{A} \setminus \{0\}$.

Ring

If $\mathbb{A}$ is an abelian group for its addition, then $\mathbb{A}$ is said to be a *ring*. From now on, we assume that $\mathbb{A}$ is a ring.

- If $\mathbb{A}^*$ is a monoid for its multiplication, then $\mathbb{A}$ is said to be a *ring with identity element*.
- If $\mathbb{A}^*$ is an abelian semi-group for its multiplication, then $\mathbb{A}$ is said to be a *commutative ring*.
- If $\mathbb{A}^*$ is an abelian monoid for its multiplication, then $\mathbb{A}$ is said to be a *commutative ring with identity element*.
- If $\mathbb{A}^*$ is a group for its multiplication, then $\mathbb{A}$ is said to be a *division ring* (or a *skew field*).
- If $\mathbb{A}^*$ is an abelian group for its multiplication, then $\mathbb{A}$ is said to be a *field*.

Some properties of rings (1/2)

Let $\mathbb{A}$ be a ring. For $x, y, z \in \mathbb{A}$ we have

$$x(y-z) + xz = x((y-z)+z) = xy \quad \text{and} \quad (y-z)x + zx = ((y-z)+z)x = yx$$

We deduce:

$$x(y-z) = xy - xz \quad \text{and} \quad (y-z)x = yx - zx. \tag{1}$$

By setting $y = z$ we obtain

$$x \times 0 = 0 = 0 \times x. \tag{2}$$

By setting $y = 0$ in Equation (1) we obtain

$$x \times (-z) = -(xz) \quad \text{and} \quad (-z)x = -(zx) \tag{3}$$

which implies

$$(-x)(-z) = xz. \tag{4}$$

Then, for every positive integer $n \in \mathbb{N}$ we deduce from Equation (4)

$$(-x)^n = (-1)^n x^n \tag{5}$$

Some properties of rings (2/2)

Let $\mathbb{A}$ be a commutative ring with identity element. Let $x \in \mathbb{A}$. Because of the rule $x^{n+m} = x^n x^m$ with n, m positive integers, it is natural to define

$$x^0 = 1 \tag{6}$$

Then, one obtains the *Newton binomial formula* for every $x, y \in \mathbb{A}$

$$(x+y)^n = \Sigma_{k=0}^{k=n} \begin{pmatrix} n \\ k \end{pmatrix} x^k y^{n-k} \tag{7}$$

Examples

We illustrate the above definitions.

- The set of the natural integer numbers $\mathbb{N}$ (endowed with its natural addition and multiplication) is a semi-ring but not a ring.
- The set of the integer numbers $\mathbb{Z}$ is a commutative ring with identity element, but not a field.
- For $p \in \mathbb{Z}$ with $p \geq 2$, the subset $p\mathbb{Z}$ of $\mathbb{Z}$ consisting of the multiples of p is a commutative ring, but not a commutative ring with identity element.
- For $n \geq 2$, the set $\mathcal{M}_{n,n}(\mathbb{Z})$ of the square matrices of order n with integer coefficients, is a ring with identity element, but not a commutative ring.

Complex numbers

- Let $\mathbb{F}$ be a field such that for every element $x \in \mathbb{F}$ we have $x^2 \neq -1$.
- Then, the subset $\mathrm{Complex}(\mathbb{F})$ of $\mathcal{M}_{2,2}(\mathbb{F})$ (the ring of square matrices with order 2 and coefficients in $\mathbb{F}$) consisting of the matrices of the form

$$C(a,b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

  is a field (for the addition and the multiplication of $\mathcal{M}_{2,2}(\mathbb{F})$), called the *complex field* of $\mathbb{F}$.
- It is also a vector subspace of $\mathcal{M}_{2,2}(\mathbb{F})$ with dimension 2.

Quaternions

- Let $\mathbb{F}$ be a field such that for all $x, y, z \in \mathbb{F}$ we have $x^2 + y^2 + z^2 \neq -1$.
- Then, the subset $\mathrm{Quaternion}(\mathbb{F})$ of $\mathcal{M}_{4,4}(\mathbb{F})$ (the ring of square matrices with order 4 and coefficients in $\mathbb{F}$) consisting of the matrices of the form

$$H(a,b,c,d) = \begin{pmatrix} d & a & b & c \\ -a & d & -c & b \\ -b & c & d & -a \\ -c & -b & a & d \end{pmatrix}$$

  is a division ring, which is not a field, called the *quaternion ring* of $\mathbb{F}$.
- It is also a vector subspace of $\mathcal{M}_{4,4}(\mathbb{F})$ with dimension 4.

Plan

Monoid Rings (1/6)

Notation

From now on, we consider a semi-group $\mathbb{M}$ whose operation is denoted multiplicatively and a ring $\mathbb{A}$ which may not be commutative and which may not have an identity element.

- Let $\underline{a} = (a_m)_{m \in \mathbb{M}}$ be a sequence of elements of $\mathbb{A}$ indexed by $\mathbb{M}$, that is a map from $\mathbb{M}$ to $\mathbb{A}$.
- For every $m \in \mathbb{M}$ the element a_m of the sequence $\underline{a}$ is called the *coefficient* at m of $\underline{a}$. The *support* of $\underline{a}$ is the subset of $\mathbb{M}$ defined by

$$\mathrm{supp}(\underline{a}) = \{ m \in \mathbb{M} \mid a_m \neq 0 \} \tag{8}$$

  The elements of $\mathrm{supp}(\underline{a})$ are called the *monomials* of $\underline{a}$.
- The sequence $\underline{a}$ is a *linear combination of elements from* $\mathbb{M}$ *with coefficients from* $\mathbb{A}$ if its support is finite.
- The set of the linear combinations of elements from $\mathbb{M}$ with coefficients from $\mathbb{A}$ is denoted by $\mathbb{A}[\mathbb{M}]$ and called the *monoid ring* of $\mathbb{M}$ over $\mathbb{A}$.
- An element of $\mathbb{A}[\mathbb{M}]$ which has only one monomial is called a *term*. In the case where $\mathbb{M}$ is a group, then $\mathbb{A}[\mathbb{M}]$ is said to be a *group ring*.

Monoid Rings (2/6)

- Let $\underline{a}$, $\underline{b}$ be in $\mathbb{A}[\mathbb{M}]$.
- The *sum* of $\underline{a}$ and $\underline{b}$ is the map $\underline{s}$ from $\mathbb{M}$ to $\mathbb{A}$ defined for every $m \in \mathbb{M}$ by
$$s_m = a_m + b_m$$
and denoted by $\underline{a} + \underline{b}$.
- The *product* of $\underline{a}$ and $\underline{b}$ is the map $\underline{p}$ from $\mathbb{M}$ to $\mathbb{A}$ defined for every $m \in \mathbb{M}$ by
$$p_m = \sum_{\substack{(m', m'') \in \mathbb{M} \times \mathbb{M} \\ m'm'' = m}} a_{m'} b_{m''}$$
and denoted by $\underline{ab}$.

Monoid Rings (3/6)

Proposition

For $\underline{a}, \underline{b}$ *in* $\mathbb{A}[\mathbb{M}]$ *the sum* $\underline{a} + \underline{b}$ *and the product* $\underline{ab}$ *belong to* $\mathbb{A}[\mathbb{M}]$*.*

Proposition

The set $\mathbb{A}[\mathbb{M}]$ *endowed with the addition*

$$(\underline{a}, \underline{b}) \longmapsto \underline{a} + \underline{b}$$

and the multiplication

$$(\underline{a}, \underline{b}) \longmapsto \underline{ab}$$

is a ring.

Monoid Rings (4/6)

Proposition

- *Assume that* $\mathbb{M}$ *is a monoid with identity element* $1_{\mathbb{M}}$ *and that* $\mathbb{A}$ *is a ring with (multiplicative) identity element* $1_{\mathbb{A}}$.
- *Let* $\underline{1}$ *be the element of* $\mathbb{A}[\mathbb{M}]$ *with support* $\{1_{\mathbb{M}}\}$ *and with coefficient* $1_{\mathbb{A}}$ *at* $1_{\mathbb{M}}$.
- *Then, the ring* $\mathbb{A}[\mathbb{M}]$ *has* $\underline{1}$ *as (multiplicative) identity element.*

Monoid Rings (5/6)

Definition

Assume again that $\mathbb{M}$ is a monoid with identity element $1_{\mathbb{M}}$ and that $\mathbb{A}$ is a ring with identity element $1_{\mathbb{A}}$. Then, we define a map from $\mathbb{A}$ to $\mathbb{A}[\mathbb{M}]$ by

$$a \longmapsto a1_{\mathbb{M}} \tag{9}$$

where $a1_{\mathbb{M}}$ is the element of $\mathbb{A}[\mathbb{M}]$ whose support is $\{1_{\mathbb{M}}\}$ and whose coefficient at $1_{\mathbb{M}}$ is a. This map allows us to view $\mathbb{A}$ as a subset of $\mathbb{A}[\mathbb{M}]$.

Proposition

With the hypothesis of the above definition, let us assume that $\mathbb{A}$ is a commutative ring. Then, for every $a \in \mathbb{A}$ and every $\underline{b} \in \mathbb{A}[\mathbb{M}]$ we have

$$(a1_{\mathbb{M}})\ \underline{b} = \underline{b}\ (a1_{\mathbb{M}}).$$

Monoid Rings (6/6)

Remark

It follows from Proposition 5 that every element of $\mathbb{A}$ commutes *with every element of* $\mathbb{A}[\mathbb{M}]$*. However, commutativity of the multiplication in* $\mathbb{A}[\mathbb{M}]$ *requires also commutativity for* $\mathbb{M}$*.*

Proposition

Assume that $\mathbb{M}$ *is an abelian monoid and that* $\mathbb{A}$ *is a commutative ring. Then, the ring* $\mathbb{A}[\mathbb{M}]$ *is commutative too.*

The free abelian monoid

- Let X be a set. The *free monoid generated by* X is the set denoted by X^* of all words (or finite sequences) over X endowed with the concatenation as multiplication and with the empty word ε as identity element.
- For later user, we define $X^+ = X^* \setminus \{\varepsilon\}$.
- We consider in X^* the following equivalence relation: two words w, w' over X are equivalent if for every $x \in X$ the number of occurrences of x is the same in both w and w'.
- The set of the residue classes of this relation is an abelian monoid (for the multiplication induced by that of X^*) called the *free abelian monoid generated by* X. Let us denote it by $\mathbb{X}$.

Multivariate polynomials (1/4)

- Let m be any element of $\mathbb{X}$. For any $x \in X$, the number of occurrences of x in a representative of m is called the *degree* of m w.r.t. x and is denoted by $\deg(m, x)$.
- The *total degree* of m is the sum of the numbers $\deg(m, x)$ where x runs over the elements of X occurring in m.
- The ring $\mathbb{A}[\mathbb{X}]$ is also denoted by $\mathbb{A}[X]$ and its elements are called *multivariate polynomials in* X with coefficients in $\mathbb{A}$. If X is a finite set $\{x_1, \ldots, x_p\}$ then
- $\mathbb{A}[X]$ is also denoted by $\mathbb{A}[x_1, \ldots, x_p]$. Let $p \in \mathbb{A}[\mathbb{X}]$ be non-zero.
- For any $x \in X$, the maximum value of $\deg(m, x)$ for $m \in \mathrm{supp}(p)$ is the *degree* of p w.r.t. x and is denoted by $\deg(p, x)$.
- The maximum total degree of a monomial of p is called the *total degree* of p.

Univariate polynomials

- Assume from now on that X is a singleton $\{x\}$.
- Observe that the free monoid generated by X is clearly identical to the free abelian monoid generated by X.
- Moreover, every element of $\mathbb{A}[x]$ is called a *univariate polynomial in x with coefficients in* $\mathbb{A}$.
- In addition, the total degree of a non-zero element p of $\mathbb{A}[x]$ is simply called its *degree* and is denoted by $\deg(p)$.

Multivariate polynomials (2/4)

Because the monoid ring $\mathbb{A}[\mathbb{M}]$ is a generalization of the polynomial ring $\mathbb{A}[x]$, it is natural and convenient to use the following notation. An element $\underline{a} = (a_m)_{m \in \mathbb{M}}$ of $\mathbb{A}[\mathbb{M}]$ can be written

$$\underline{a} = \sum_{m \in \mathbb{M}} a_m$$

Multivariate polynomials (3/4)

Without any additional assumption on $\mathbb{M}$, computing in $\mathbb{A}[\mathbb{M}]$ is not easy. First, one would like to have a *canonical way* to represent the elements of $\mathbb{A}[\mathbb{M}]$. That would make the comparison or the addition of two elements from $\mathbb{A}[\mathbb{M}]$ simpler. Second, computing the product of two elements $\underline{a}$ and $\underline{b}$ of $\mathbb{A}[\mathbb{M}]$ implies to compute all the couples $(m', m'') \in \mathbb{M} \times \mathbb{M}$ such that $m'm''$ is equal to a given $m \in \mathbb{M}$. If $\mathbb{M}$ is a group, then the equation $m'm'' = m$ is simpler since we must have $m'' = m'^{-1}m$.

Definition

A total order $\leq$ on an abelian monoid $\mathbb{M}$ is a *term order* if the following two conditions hold

(*i*) for every $m \in \mathbb{M}$ we have $1_{\mathbb{M}} \leq m$

(*ii*) for every $m, m', m'' \in \mathbb{M}$ we have $m \leq m' \Rightarrow mm'' \leq m'm''$

Multivariate polynomials (4/4)

Assume that $\mathbb{M}$ is an abelian monoid endowed with a term order $\leq$ and let $\underline{a} \in \mathbb{A}[\mathbb{M}]$ be a non-zero element.

- The maximum (w.r.t. the total order of $\mathbb{M}$) element of $\mathrm{supp}(\underline{a})$ is called the *leading monomial of* $\underline{a}$ and is denoted by $\mathrm{lm}(\underline{a})$.
- The coefficient of $\underline{a}$ at $\mathrm{lm}(\underline{a})$ is called the *leading coefficient of* $\underline{a}$ and is denoted by $\mathrm{lc}(\underline{a})$.
- The term of $\mathbb{A}[\mathbb{M}]$ whose leading monomial is $\mathrm{lm}(\underline{a})$ and whose leading coefficient is $\mathrm{lc}(\underline{a})$ is called the *leading term* of $\underline{a}$ and is denoted by $\mathrm{lt}(\underline{a})$.
- The element $\underline{a} - \mathrm{lt}(\underline{a})$ is called the *reductum* of $\underline{a}$.
- Finally, the leading coefficient, the leading term and the reductum of 0 are defined to be 0.
- It is sometimes convenient to set $\mathrm{lm}(\underline{0}) = 0$ as well.

Example (1/4)

- This example is taken from *Automata Theory* and assume that the reader is familiar with the notion of a finite automaton.
- Let us consider an alphabet Σ, a finite automaton (not necessarily deterministic) $\mathcal{A}$ recognizing a language $\mathcal{L}$ over Σ and a positive integer n.
- We are interested in computing the words of $\mathcal{L}$ with length n.

Example (2/4)

- Let $Q = \{1, \ldots, q\}$ be the set of states of $\mathcal{A}$ and let Σ^* be the set of words over Σ.
- Recall that Σ^* is a monoid whose identity element is the empty word.
- Let $\mathbb{A}$ be the ring $\mathbb{Z}[\Sigma^*]$ of linear combinations of words from Σ^* with coefficients from the ring of integer numbers $\mathbb{Z}$.
- Let $\delta : (Q, \Sigma \cup \{\varepsilon\}) \longmapsto 2^Q$ be the transition function of $\mathcal{A}$. To every couple $(i, j) \in Q \times Q$ of states we associate the element $T_{i,j}$ of $\mathbb{Z}[\Sigma^*]$ defined by

$$T_{i,j} = \sum_{\begin{array}{ccc} x & \in & \Sigma \cup \{\varepsilon\} \\ j & \in & \delta(i, x) \end{array}} x.$$

- In broad words, the element $T_{i,j}$ is the sum of the $x \in \Sigma \cup \{\varepsilon\}$ such that one transits from state i to state j by reading x.

Example (3/4)

- Let T be the square matrix of order q with coefficients in $\mathbb{Z}[\Sigma^*]$ such that $T_{i,j}$ is the element of T at the intersection of row i and column j.
- Let S be the *horizontal* vector of length q with coefficients in $\mathbb{Z}$ such that $S_i = 1$ if i is an initial state and $S_i = 0$ otherwise.
- Let F be the *vertical* vector of length q with coefficients in $\mathbb{Z}$ such that $F_i = 1$ if i is a final state and $F_i = 0$ otherwise.
- Then we define the following element of $\mathbb{Z}[\Sigma^*]$

$$p_n(\mathcal{A}) = ST^nF.$$

- Let us compute this quantity for $n = 2$, $\Sigma = \{a, b\}$, $Q = \{1, 2\}$, the initial state 1, the final state 2 and the following transition function

	a	b	ε
1	1	1, 2	$\varnothing$
2	2	2	$\varnothing$

Example (4/4)

Then, the matrix T is

$$T = \begin{pmatrix} a+b & b \\ 0 & a+b \end{pmatrix}$$

and its square is

$$\begin{aligned} T^2 &= \begin{pmatrix} (a+b)(a+b) & (a+b)b + b(a+b) \\ 0 & (a+b)(a+b) \end{pmatrix} \\ &= \begin{pmatrix} a^2 + ab + ba + b^2 & ab + ba + 2b^2 \\ 0 & a^2 + ab + ba + b^2 \end{pmatrix} \end{aligned}$$

Then we have

$$\begin{aligned} p_2(\mathcal{A}) &= ST^2F \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} T^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= ab + ba + 2b^2 \end{aligned}$$

Observe that ab, ba and b^2 are the three words of length 2 that the automaton $\mathcal{A}$ recognizes. The coefficient 2 of b^2 comes from the fact that there are two ways to recognize this word.

Plan

Univariate polynomial data-type

Let $\mathbb{A}$ be ring. The univariate polynomial of $\mathbb{A}[x]$ can be implemented using different data-types in a computer program:

- dense univariate polynomial (DUP)
- sparse univariate polynomial (SUP)
- Straigth-line program (SLP)
- ...

Dense univariate polynomial (DUP)

The polynomial

$$p(x) \quad = \quad a_n x^n + \cdots + a_1 x + a_0 \tag{10}$$

is coded by a record consisting of

- a single integer s,
- a single integer $d \leq s + 1$,
- an array of size s such that $a_0 + \ldots + a_n x^n$ is represented by $[a_0, \ldots, a_n, \ldots]$ and $d = n$.
- This representation is said *dense* becasue all a_i are coded, even those which are null.
- This representation is said *canonical* (when $d = s + 1$ holds) because two different polynomials have different such representations.
- Hence operations like DEGREE, LEADING COEFFICIENT, REDUCTUM are in $\mathcal{O}(1)$.
- Addition and equality-test are in $\mathcal{O}(n)$ and multiplication is in $\mathcal{O}(n^2)$.
- This representation is especially good when the ring of coefficients is a small prime field, i.e. $\mathbb{Z}/p\mathbb{Z}$ with p prime and in the range $[2, 2^N - 1]$, for a fixed N.

Sparse univariate polynomial (SUP)

The polynomial

$$p(x) \quad = \quad a_n x^n + \cdots + a_1 x + a_0 \tag{11}$$

is coded by the list L of records $[a_i, i]$ where a_i is a nonzero coefficient and such that L is sorted decreasingly w.r.t. i.

- This representation is said *sparse*, since only the nonzero a_i are coded.
- This representation is also canonical.
- Hence operations like DEGREE, LEADING COEFFICIENT, REDUCTUM are in $\mathcal{O}(1)$.
- Moreover the operation REDUCTUM does not require coefficient duplication (on the contrary of the previous representation).
- Addition and equality-test are in $\mathcal{O}(n)$ and multiplication is in $\mathcal{O}(n^2)$.
- This representation is especially good when the ring of coefficients is itself a ring of sparse polynomials.

Division with remainder

Input: univariate polynomials $f = \Sigma_0^n \, a_i x^i$ and $g = \Sigma_0^m \, b_i x^i$ in $\mathbb{A}[x]$ with respective degrees n and m such that b_m is a unit.

Output: the quotient q and the remainder r of f w.r.t. g.

$n < m \Rightarrow$ **return** $(0, f)$
$r := f$
for $i = n - m, n - m - 1, \ldots, 0$ **repeat**
  if $\deg r = m + i$ **then**
    $q_i := \mathrm{lc}(r) \, / \, b_m$
    $r := r - q_i x^i g$
  else $q_i := 0$
$q := \Sigma_0^{n-m} \, q_i x^i$
return (q, r)

Exercise

Assuming that each element of $\mathbb{A}$ can fit a machine word, what is the minimum space requirement for implementing the above algorithm in the case of DUP? SUP?

Plan

Multivariate polynomial data-type

Let again $\mathbb{A}$ be ring and let $X = \{x_1, \ldots, x_n\}$ be a finite set of variables. The multivariate polynomials of $\mathbb{A}[X] = \mathbb{A}[\mathbb{X}]$ (hence in commutative variables) can be implemented using different data-types in a computer program:

- recursively based on SUP
- recursively based on DUP
- Expanded (or distributed) multivariate data-type
- Straight-line program (SLP)
- ...

Recursively

Recall $X = \{x_1, \ldots, x_n\}$ and we want to implement $\mathbb{A}[X]$.

- If $n = 1$ we can use a univariate representation
- otherwise we can view $\mathbb{A}[X]$ as a univariate polynomial ring with a multivariate polynomial ring as coefficient ring, say for instance $\mathbb{A}[x_1, \ldots, x_{n-1}][x_n]$.

This representation

- implies to choose an ordering on the variables and a representation for univariate polynomials.
- is well adapted for certain operations, in particular those around the notion of GCD (Greatest Common Divisor).
- More on this later.

Expanded multivariate data-types (1/2)

- Each polynomial can be viewed as a linear combination of monomials (with coefficients in $\mathbb{A}$).
- Then the polynomial

$$p = a_1 m_1 + \cdots + a_t m_t. \tag{12}$$

  where the m_i are pairwise different monomials and the a_i are nonzero coefficients, can be represented as an aggregate of terms $[a_i, m_i]$.
- Once an order is chosen on X, say $x_1 > x_2 > \cdots > x_n$, a monomial $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ is given by the *exponent vector* $[e_1, e_2, \ldots, e_n]$.
- Assume that we monomials are totally ordered. Assume also that the aggregate is *linear*, that is, defining a 1-to-1 map from $[1, t] \cap \mathbb{N}$ to the terms of p. So that we can speak of the *first* term, the *second* term, ... Finally, assume that this map sorts terms decreasingly. Then, this provides us with a canonical representation for $\mathbb{A}[X]$.
- The most commonly used aggregate is *linked list*.
- One can also consider alternating arrays (thus alternating one coefficient and one monomial on consecutive memory words).

Expanded multivariate data-types (2/2)

Two types of monomial orderings are frequently used.

- The lexicographical ordering. With $X = \{x > y > z\}$ we have

$$1 < z < \cdots < z^n \cdots < y < yz < \cdots < yz^n \cdots < y^2 < y^2 z < \cdots < y^2 z^n \cdots \quad (13)$$

- The degree-lexicographical ordering. With $X = \{x > y > z\}$ we have

$$1 < z < y < x < z^2 < zy < y^2 < zx < xy < x^2 < \cdots < \quad (14)$$

Sparse multivariate addition (1/2)

Compute $C := A + B$ where

- $A = \sum_{i=1}^{i=n} a_i\, x^{\alpha_i}$,
- $B = \sum_{j=1}^{j=m} b_j\, x^{\beta_j}$,
- $C = \sum_{k=1}^{k=K} c_k\, x^{\gamma_k}$,

where

- $\alpha_i, \beta_j, \gamma_k$ are exponent vectors,
- a_i, b_j, c_k are coefficients,
- m, n, K are the number of terms of A, B, C respectively,
- terms are **sorted decreasingly** in the above expressions.

The algorithm on the right-hand side performs $\mathcal{O}(m + n)$ comparisons of exponent vectors and additions in $\mathbb{A}$.

Sparse multivariate addition (1/2)

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where

- $\alpha_i, \beta_j, \gamma_k$ are exponent vectors,
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- terms are **sorted decreasingly** in the above expressions.

The algorithm on the right-hand side performs $\mathcal{O}(m + n)$ comparisons of exponent vectors and additions in $\mathbb{A}$.

```
k = 0
i = 1
j = 1
while ( i ≤ n  and  j ≤ m ) {
      k = k + 1
      if ( α_i < β_j ) {
            c_k = b_j
            γ_k = β_j
            j = j + 1
            }
      else if ( α_i = β_j ) {
            c_k = a_i + b_j
            γ_k = α_i
            if ( c_k = 0 ) k = k − 1
            i = i + 1
            j = j + 1
            }
      else if ( α_i > β_j ) {
            c_k = a_i
            γ_k = α_i
            i = i + 1
            }
      }
while ( i ≤ n ) {
      k = k + 1
      c_k = a_i
      γ_k = α_i
      i = i + 1
      }
while ( j ≤ m ) {
      k = k + 1
      c_k = b_j
      γ_k = β_j
      j = j + 1
      }
K = k
```

Sparse multivariate addition (2/2)

```
k = 0
i = 1
j = 1
while ( i ≤ n and j ≤ m ) {
    k = k + 1
    if ( α_i < β_j ) {
        c_k = b_j
        γ_k = β_j
        j = j + 1
        }
    else if ( α_i = β_j ) {
        c_k = a_i + b_j
        γ_k = α_i
        if ( c_k = 0 ) k = k - 1
        i = i + 1
        j = j + 1
        }
    else if ( α_i > β_j ) {
        c_k = a_i
        γ_k = α_i
        i = i + 1
        }
    }
while ( i ≤ n ) {
    k = k + 1
    c_k = a_i
    γ_k = α_i
    i = i + 1
    }
while ( j ≤ m ) {
    k = k + 1
    c_k = b_j
    γ_k = β_j
    j = j + 1
    }
K = k
```

Sparse multivariate multiplication (1/4)

Now, we want to compute $C := AB$ with, as above,

$$A = \sum_{i=1}^{i=n} a_i \, x^{\alpha_i}, \;\; B = \sum_{j=1}^{j=m} b_j \, x^{\beta_j}, \quad \text{and} \quad C = \sum_{k=1}^{k=K} c_k \, x^{\gamma_k}. \qquad (15)$$

What is the cost of a *plain multiplication?*

- generating all terms costs $\mathcal{O}(nm)$,
- sorting them all costs $\mathcal{O}(nm \log(nm))$,
- combining terms of all equal exponent vectors $\mathcal{O}(nm)$.
- This yields an arithmetic complexity of $\mathcal{O}(nm \log(nm))$,
- with a space complexity of $\Theta(nm)$, which can be improved.

Sparse multivariate multiplication (2/4)

Using distributivity naively

- Write $C = \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=m} a_i b_j x^{\alpha_i + \beta_j} \right)$
- Suppose we add the n summands, one after another.
- Then the i-th summation may cost $\mathcal{O}(im + n)$ arithmetic operations
- This yields a total arithmetic complexity of $\mathcal{O}(n^2 m)$, butt reduces space complexity when cancellations happens

Divide-and-conquer approach

- Instead of summing the summands, one after another, proceed in a divide-and-conquer manner
- Assume $m \leq n$ and let $C(n)$ be the cost of adding n polynomials of size m. We have

$$C(n) = 2C(n/2) + \left(\frac{mn}{2} + \frac{mn}{2} \right). \tag{16}$$

- This yields $C(n) \in \mathcal{O}(nm \log(nm))$.

Sparse multivariate multiplication (3/4)

- Consider now algorithms attempting to generate terms by decreasing order of exponent vectors.
- The term with exponent $\alpha_i + \beta_j$ appears in the product before the term with exponent $\alpha_i + \beta_{j+1}$
- Thus, at each step of those algorithms, and for each $1 \leq i \leq n$, there exists an index f_i such that terms with exponent $\alpha_i + \beta_j$ have (resp. have not) been included in the answer for $j < f_i$ (resp. $j \leq f_i$).
- The exponent of the next term to be included in the answer will be the largest of the $\alpha_i + \beta_{f_i}$ where i ranges from 1 to n.
- The f_i are decreasing with i; thus if $f_i > m$ for some index i, then $f_j > m$ holds for all $j \leq i$
- In the algorithm, I is the smallest i such that $f_i \leq m$ holds.

Sparse multivariate multiplication (3/4)

- Consider now algorithms attempting to generate terms by decreasing order of exponent vectors.
- The term with exponent $\alpha_i + \beta_j$ appears in the product before the term with exponent $\alpha_i + \beta_{j+1}$
- Thus, at each step of those algorithms, and for each $1 \leq i \leq n$, there exists an index f_i such that terms with exponent $\alpha_i + \beta_j$ have (resp. have not) been included in the answer for $j < f_i$ (resp. $j \leq f_i$).
- The exponent of the next term to be included in the answer will be the largest of the $\alpha_i + \beta_{f_i}$ where i ranges from 1 to n.
- The f_i are decreasing with i; thus if $f_i > m$ for some index i, then $f_j > m$ holds for all $j \leq i$
- In the algorithm, I is the smallest i such that $f_i \leq m$ holds.

```
if ( m=0 or n=0 ) {
      K=0
      return
      }
k=1
c_1 =0
γ_1 =α_1 +β_1
for i=1 to n do f_i =1
I=1
while ( I ⩽ n ) {
      { Find an s with I ⩽ s ⩽ n which
      maximizes α_s +β_{f_s} }
      if ( γ_k ≠ α_s +β_{f_s} ) {
            if ( c_k ≠ 0 ) {
                  k=k+1
                  c_k =0
                  }
            γ_k =α_s +β_{f_s}
            }
      c_k =c_k + a_s b_{f_s}
      f_s =f_s +1
      if ( f_s > m ) I=I+1
      }
K=k
```

Sparse multivariate multiplication (4/4)

```
if ( m=0 or n=0 ) {
      K=0
      return
      }
k=1
c_1 =0
γ_1 =α_1 +β_1
for i=1 to n do f_i =1
l=1
while ( l ≤ n ) {
      { Find an s with l ≤ s ≤ n which
      maximizes α_s +β_{f_s} }
      if ( γ_k ≠ α_s +β_{f_s} ) {
            if ( c_k ≠ 0 ) {
                  k=k+1
                  c_k =0
                  }
            γ_k =α_s +β_{f_s}
            }
      c_k =c_k + a_s b_{f_s}
      f_s =f_s +1
      if ( f_s > m ) l=l+1
      }
K=k
```

This algorithm runs in $\mathcal{O}(mn\log(n))$. Explain why! What is its space complexity?

Plan

Polynomials and the Fast Fourier Transform (FFT)

- http://web.cecs.pdx.edu/~maier/cs584/Lectures/lect07b-11-MG.pdf

Plan

- Horner's method
  https://en.wikipedia.org/wiki/Horner%27s_method
- SLP in Wikopedia
  https://en.wikipedia.org/wiki/Straight-line_program
- SLP in Chaptter 2 of http://www.csd.uwo.ca/~moreno/
  /Publications/Liyun.Li-MasterThesis-2010.pdf

Plan

- Course notes:
  - https://people.eecs.berkeley.edu/~fateman/282/F%20Wright%20notes/week4.pdf
- Additional course notes:
  - http://www.dei.unipd.it/~geppo/DA2/DOCS/arithmetic.pdf
- Demo codes:
  - http://faculty.cse.tamu.edu/djimenez/ut/utsa/cs3343/lecture20.html
  - https://www3.cs.stonybrook.edu/~skiena/392/programs/bignum.c
- Profesional codes:
  - GMP https://gmplib.org/
  - NTL http://www.flintlib.org/
  - FLINT http://www.shoup.net/ntl/