Regular Chains under Linear Changes of Coordinates and Applications

Parisa Alvandi[‡], Changbo Chen[†], Amir Hashemi[§¶], Marc Moreno Maza[‡]

[†] Chongqing Key Laboratory of Automated Reasoning and Cognition, Chongqing Institute of Green and Intelligent Technology, Chinese Academy of Sciences
[‡] ORCCA, University of Western Ontario
[§] Department of Mathematical Sciences, Isfahan University of Technology Isfahan, 84156-83111, Iran
[¶] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, 19395-5746, Iran
palvandi@uwo.ca, changbo.chen@hotmail.com, amir.hashemi@cc.iut.ac.ir, moreno@csd.uwo.ca

Abstract. Given a regular chain, we are interested in questions like computing the limit points of its quasi-component, or equivalently, computing the variety of its saturated ideal. We propose techniques relying on linear changes of coordinates and we consider strategies where these changes can be either generic or guided by the input.

1 Introduction

Applying a change of coordinates to the algebraic or differential representation of a geometrical object is a fundamental technique to obtain a more convenient representation and reveal properties. For instance, random linear change of coordinates are performed in algorithms for solving systems of polynomial equations and inequations in the algorithms of Krick and Logar [11], Rouillier [15], Verschelde [17] and Lecerf [12].

For polynomial ideals, one desirable representation is Noether normalization, which was introduced by Emmy Noether in 1926. We refer to the books [9,8] for an account on this notion and, to the articles [16,10] for deterministic approaches to compute Noether normalization, when the input ideal is given by a Gröbner basis.

In regular chain theory, one desirable and challenging objective is, given a regular chain T, to obtain the (non-trivial) limit points of its quasi-component $W(T)$, or equivalently, computing the variety of its saturated ideal $\mathrm{sat}(T)$. The set $\lim(W(T))$ of the non-trivial limit points of $W(T)$ satisfies $V(\mathrm{sat}(T)) = \overline{W(T)} = W(T) \cup \lim(W(T))$. Hence, $\lim(W(T))$ is the set-theoretic difference $V(\mathrm{sat}(T)) \setminus W(T)$. Deducing $\lim(W(T))$ or $V(\mathrm{sat}(T))$ from T is a central question which has theoretical applications (like the so-called *Ritt Problem*) and practical ones (like removing redundant components in triangular decomposition).

Of course $V(\mathrm{sat}(T))$ can be computed from T via Gröbner basis techniques. But this approach is of limited practical interest. In fact, considering the case where the base field is $\mathbb{Q}$, we are looking for approaches that would run in polynomial time w.r.t. the degrees and coefficient heights of T. Thanks to the work of [7], algorithms for change of variable orders (and more generally, algorithms for linear changes of coordinates) are good candidates.

Returning to Noether normalization[1], we ask in Section 4 how "simple" can T be if we assume that $\mathrm{sat}(T)$ is in Noether position. Unfortunately, an additional hypothesis is needed in order to obtain a satisfactory answer like "all initials of T are constant", see Theorem 1 and Remark 1.

In Section 5 and 6, we develop a few criteria for computing $\lim(W(T))$ or $V(\mathrm{sat}(T))$. Our techniques (see Proposition 2, Theorem 2, Theorem 3, Theorem 4 and Algorithm 1) rely on linear changes of coordinates and allow us to relax the "dimension one" hypothesis in our previous paper [1], where $\lim(W(T))$ was computed via Puiseux series.

Therefore, the techniques proposed in this paper can be used to compute $\lim(W(T))$ or $V(\mathrm{sat}(T))$ without Gröbner basis or Puiseux series calculations. Moreover, these new techniques can handle cases where the results of our previous paper [1] could not apply. One of the main ideas of our new results (see for instance Theorem 2) is to use a linear change of coordinates so as to replace the description of $\overline{W(T)}$ by one for which $\overline{W(T)} \cap V(h_T)$ can be computed by means of standard operations on regular chains. Nevertheless, our proposed techniques do not cover all possible cases and the problem of finding a "Gröbner-basis-free" general algorithm for $\lim(W(T))$ or $V(\mathrm{sat}(T))$ remains unsolved.

2 Preliminaries

Throughout this paper, polynomials have coefficients in a field $\mathbf{k}$ and variables in a set $\mathbf{x}$ of n ordered variables $x_1 < \cdots < x_n$. The corresponding polynomial ring is denoted by $\mathbf{k}[\mathbf{x}]$. Let F be a subset of $\mathbf{k}[\mathbf{x}]$. We denote by $\langle F \rangle$ the ideal generated by F in $\mathbf{k}[\mathbf{x}]$. Recall that a polynomial $f \in \mathbf{k}[\mathbf{x}]$ is *regular* modulo the ideal $\langle F \rangle$ whenever f does not belong to any prime ideals associated with $\langle F \rangle$, thus, whenever f is neither null nor a zero-divisor modulo $\langle F \rangle$. Further, $\overline{\mathbf{k}}$ stands for the algebraic closure of $\mathbf{k}$ and $V(F) \subset \overline{\mathbf{k}}^n$ for the algebraic set consisting of all common zeros of all $f \in F$. For a set $W \subset \overline{\mathbf{k}}^n$, we denote by $\overline{W}$ the *Zariski closure* of W, that is, the intersection of all algebraic sets containing W.

We briefly review standard notions and concepts related to regular chains and we refer to [2,6] for details. For a non-constant $f \in \mathbf{k}[\mathbf{x}]$, we denote by $\mathrm{mvar}(f)$, $\mathrm{mdeg}(f)$ and $\mathrm{init}(f)$, the variable of greatest rank appearing in f, the degree of f w.r.t. that variable and the leading coefficient of f w.r.t. that same variable. The quantities $\mathrm{mvar}(f)$, $\mathrm{mdeg}(f)$ and $\mathrm{init}(f)$ are called respectively the *main variable*, *main degree* and *initial* of f. A set T of non-constant polynomials from $\mathbf{k}[\mathbf{x}]$ is called *triangular* if no two polynomials from T have the same main variable. Let $T \subset \mathbf{k}[\mathbf{x}]$ be a triangular set. Observe that T is necessarily finite and that every subset of T is itself triangular. For a variable $v \in \mathbf{x}$, if there exists

[1] Section 4 contains a brief review of Noether normalization which makes our paper self-contained.

$f \in T$ such that $\mathrm{mvar}(f) = v$, we denote this polynomial by T_v and say that v is *algebraic* w.r.t. T, otherwise we say that v is *free* w.r.t. T; in all cases, we define $T_{<v} := \{g \in T \mid \mathrm{mvar}(g) < v\}$ and denote by $\mathsf{free}(T)$ the set of the variables from $\mathbf{x}$ which are free w.r.t. T. We denote by h_T the product of the polynomials $\mathrm{init}(f)$, for $f \in T$. We say that T is *strongly normalized* if all variables occurring in h_T are in $\mathsf{free}(T)$; when this holds, it is easy to check that T is a Gröbner basis of the ideal that T generates in $\mathbf{k}(\mathbf{u})[\mathbf{x} \setminus \mathbf{u}]$ where $\mathbf{u} := \mathsf{free}(T)$ and $\mathbf{k}(\mathbf{u})$ is the field of rational functions over $\mathbf{k}$ and with variables in $\mathbf{u}$. Moreover, we say that T is *monic* whenever $h_T \in \mathbf{k}$ holds. The *saturated ideal* of T, written $\mathrm{sat}(T)$, is defined as the column ideal $\mathrm{sat}(T) = \langle T \rangle : h_T^{\infty}$. The *quasi-component* of T is the basic constructible set given by $W(T) := V(T) \setminus V(h_T)$. The following two properties are easy to prove:

$$\overline{W(T)} \ = \ V(\mathrm{sat}(T)) \quad \text{and} \quad \overline{W(T)} \ = \ W(T) \ \cup \ \lim(W(T)), \tag{1}$$

where $\lim(W(T)) := \overline{W(T)} \cap V(h_T)$ holds and the points of that latter set are called the *(non-trivial) limit points* of $W(T)$, for reasons explained in [1]. We say that T is a *regular chain* whenever T is empty or $T_{<w}$ is a regular chain and the initial of T_w is regular modulo $\mathrm{sat}(T_{<w})$, where w is the largest main variable of a polynomial in T. If T consists of $n - d$ polynomials, for $0 \leq d < n$, then $\mathrm{sat}(T)$ has dimension d and either $\lim(W(T))$ is empty or has dimension $d - 1$; moreover, we have $\mathbf{k}[\mathbf{u}] \cap \mathrm{sat}(T) \ = \ \langle 0 \rangle$, where $\mathbf{u} := \mathsf{free}(T)$.

Let $F \subset \mathbf{k}[\mathbf{x}]$ be finite. Let $T_1, \ldots, T_e$ be finitely many regular chains of $\mathbf{k}[\mathbf{x}]$. We say that $\{T_1, \ldots, T_e\}$ is a *Kalkbrener triangular decomposition* of $V(F)$ if we have $V(F) = \cup_{i=1}^{e} \overline{W(T_i)}$. We say that $\{T_1, \ldots, T_e\}$ is a *Lazard-Wu triangular decomposition* of $V(F)$ if we have $V(F) = \cup_{i=1}^{e} W(T_i)$.

We call *linear change of coordinates in* $\overline{\mathbf{k}}^n$ any bijective map A of the form

$$\begin{array}{rrcl} A: & \overline{\mathbf{k}}^n & \rightarrow & \overline{\mathbf{k}}^n \\ & \mathbf{x} & \longmapsto & (A_1(\mathbf{x}), \ldots, A_n(\mathbf{x})) \end{array} \tag{2}$$

where $A_1, \ldots, A_n$ are linear forms over $\overline{\mathbf{k}}$. Hence $A(\mathbf{x})$ can be written as $M\mathbf{x}$ where M is an invertible matrix over $\overline{\mathbf{k}}$. For the algebraic set $V(F)$, we denote $V^A(F) \ := \ V(\{f^A \ \mid \ f \in F\})$, where $f^A(\mathbf{x}) := f(A_1(\mathbf{x}), \ldots, A_n(\mathbf{x}))$. Observe that if $V(F)$ is irreducible, then so is $V^A(F)$. Similarly, the image of $W(T)$ under A is $W^A(T) \ = \ V^A(T) \setminus V^A(h_T)$.

3 Algorithm for linear change of coordinates

The goal of this section is to explain how to obtain a practically efficient algorithmic solution to the following problem.

Problem 1 *Given a regular chain $T \subset \mathbf{k}[\mathbf{x}]$ and given a linear change of coordinates A in $\overline{\mathbf{k}}^n$, compute finitely many regular chains $C_1, \ldots, C_e$ such that*

$$\overline{W^A(T)} \ = \ \overline{W(C_1)} \cup \cdots \cup \overline{W(C_e)}.$$

In the literature, see [3,4,7], the following related problem has been addressed.

Problem 2 *Given two total orderings $\mathcal{R}$ and $\overline{\mathcal{R}}$ on $\{x_1, \ldots, x_n\}$, given $T \subset \mathbf{k}[x_1, \ldots, x_n]$, assuming that*

1. *T is a regular chain for the ordering $\mathcal{R}$ on $\{x_1, \ldots, x_n\}$ and,*
2. *the saturated ideal $\mathrm{sat}(T, \mathcal{R})$ (which is an alias of $\mathrm{sat}(T)$ with a second argument recalling the ordering) of T of $\mathbf{k}[x_1, \ldots, x_n]$ is prime,*

compute $C \subset \mathbf{k}[x_1, \ldots, x_n]$ such that

3. *C is a regular chain for the ordering $\overline{\mathcal{R}}$ on $\{x_1, \ldots, x_n\}$ and,*
4. *the saturated ideal $\mathrm{sat}(C, \overline{\mathcal{R}})$ of C in $\mathbf{k}[x_1, \ldots, x_n]$ is equal to $\mathrm{sat}(T, \mathcal{R})$.*

We call this second problem *change of variable order*. The articles [3,4] are actually dedicated to the case of differential regular chains, where a differential counterpart of Problem 2 is termed *ranking conversion*. However, these articles suggest that, from the differential case, a solution to Problem 2 could be derived and they call it `PALGIE`, which is an acronym for Prime ALGebraic IdEal.

Next, towards Problem 1, we consider the following extension of Problem 2 where the primality assumption is relaxed.

Problem 3 *Given two total orderings $\mathcal{R}$ and $\overline{\mathcal{R}}$ on $\{x_1, \ldots, x_n\}$, given $T \subset \mathbf{k}[x_1, \ldots, x_n]$, assuming that T is a regular chain for the ordering $\mathcal{R}$ on $\{x_1, \ldots, x_n\}$, compute finitely many regular chains $C_1, \ldots, C_e$ such that the radical of the saturated ideal $\mathrm{sat}(T, \mathcal{R})$ of T in $\mathbf{k}[x_1, \ldots, x_n]$ is equal to the intersection of the radicals of the saturated ideals $\mathrm{sat}(C_i, \overline{\mathcal{R}})$ of C_i in $\mathbf{k}[x_1, \ldots, x_n]$, for $1 \leq i \leq e$.*

Extending the `PALGIE` algorithm (as suggested in [3]) to a solution of Problem 3 can be achieved by standard techniques from regular chain theory, see [6].

Before further extending the `PALGIE` algorithm to a solution of Problem 1, we argue that Problem 2 deals with a special case of Problem 1, that is, ranking conversions are, indeed, a special case of linear change of coordinates.

As in the statement of Problem 2, consider two total orderings $\mathcal{R}$ and $\overline{\mathcal{R}}$ on $\{x_1, \ldots, x_n\}$ as well as a regular chain $T \subset \mathbf{k}[x_1, \ldots, x_n]$ for the order $\mathcal{R}$ such that its saturated ideal $\mathrm{sat}(T, \mathcal{R})$ is prime. W.l.o.g. we can assume that the order $\mathcal{R}$ on $\{x_1, \ldots, x_n\}$ is given by $x_1 < \cdots < x_n$. Then, the change of variable order from $\mathcal{R}$ to $\overline{\mathcal{R}}$ can be interpreted as a permutation σ of the sequence $(x_1, \ldots, x_n)$. Let A be the linear change of coordinates replacing the column vector $(x_1, \ldots, x_n)^t$ with $M_\sigma(y_1, ..., y_n)^t$ where $(y_1, \ldots, y_n)$ stand for the new coordinates and M_σ is the matrix of σ w.r.t. the canonical basis of $\overline{\mathbf{k}}^n$ as a vector space over $\overline{\mathbf{k}}$. Running the extended version of the `PALGIE` algorithm solving Problem 1 we obtain a regular chain C such that we have

$$\mathrm{sat}(C) \ = \ \mathrm{sat}(T)^A.$$

Then simply renaming y_i with $x_{\sigma(i)}$, for $1 \leq i \leq n$, in C produces a regular chain D satisfying the output specifications of the original version of the `PALGIE` algorithm whose purpose is to perform change of variable order. To make the proof strict, requiring that T and D be strongly normalized (and reduced Gröbner bases over the field of rational functions $\mathbf{k}(\mathsf{free}(T))$) make them unique which completes the proof.

We turn our attention back to Problem 1 and suggest how a solution of Problem 3 can lead to a solution of Problem 1. Let $T \subset \mathbf{k}[\mathbf{x}]$ be a regular chain and let A be a linear change of coordinates in $\overline{\mathbf{k}}^n$. We denote by d the dimension of $\mathrm{sat}(T)$. W.l.o.g. we assume that the variables $x_1 < \cdots < x_d$ are algebraically independent modulo $\mathrm{sat}(T)$, that is, $\mathsf{free}(T) = \{x_1, \ldots, x_d\}$. Let us write $T = \{t_{d+1}, \ldots, t_n\}$ such that t_i has main variable x_i and initial h_i. We apply the extended version of the `PALGIE` algorithm (that is, the one solving Problem 3) to the solving of the polynomial system S below

$$\begin{cases} t_n^A(\mathbf{x}) = 0 \\ \vdots \; \vdots \; \vdots \\ t_{d+1}^A(\mathbf{x}) = 0 \\ h_{d+1}^A(\mathbf{x}) \cdots h_n^A(\mathbf{x}) \neq 0 \end{cases} \tag{3}$$

We denote by $Z(S) \subset \overline{\mathbf{k}}^n$ the zero set of S. Observe that for all polynomials $f \in \mathbf{k}[\mathbf{x}]$, we have

$$f \in \langle Z(S) \rangle \iff f^{A^{-1}} \in \sqrt{\mathrm{sat}(T)}. \tag{4}$$

where $\langle Z(S) \rangle$ is the ideal of $\mathbf{k}[\mathbf{x}]$ consisting of all polynomials vanishing on $Z(S)$. Relation (4) allows one to easily adapt the *master - student relationship* described in Section 3.2 of [4] and thus to adapt the (extended version of the) `PALGIE` algorithm so as to solve Problem 1.

4 Noether normalization and regular chains

In this section, we study the relation between Noether normalization and regular chains. Our initial quest was to determine whether, for a prime ideal $\mathcal{P} \subset \mathbf{k}[\mathbf{x}]$ in Noether position, one could find a monic regular chain T whose saturated ideal is precisely $\mathcal{P}$. For this purpose, we start by reviewing basic properties of Noether normalization, following Logar's paper [14].

Let $\mathcal{P} \subset \mathbf{k}[\mathbf{x}]$ be a (proper) prime ideal and G the reduced lexicographical Gröbner basis of $\mathcal{P}$. Recall that $\mathbf{x}$ counts n variables ordered as $x_1 < \cdots < x_n$. We assume that $\mathbf{k}$ is an infinite field. We denote by $T_{\mathcal{P}}$ the set defined by $T_{\mathcal{P}} = \{v \in \mathbf{x} \mid (\forall g \in G)\ \mathrm{mvar}(g) \neq v\}$. This set satisfies two important properties:

- $T_{\mathcal{P}}$ is algebraically independent modulo $\mathcal{P}$ that is, $\mathcal{P} \cap \mathbf{k}[T_{\mathcal{P}}] = \langle 0 \rangle$,
- the number of elements in $T_{\mathcal{P}}$ gives the dimension of $\mathcal{P}$, that is, $\dim(\mathcal{P}) = \mathrm{card}(T_{\mathcal{P}})$.

A variable $x_s \in \mathbf{x}$ is said *integral* over $\mathbf{k}[x_1, \ldots, x_{s-1}]$ modulo $\mathcal{P}$ if there exists $f \in \mathcal{P} \cap \mathbf{k}[x_1, \ldots, x_{s-1}, x_s]$ such that $\mathrm{mvar}(f) = x_s$ and $\mathrm{init}(f) \in \mathbf{k}$. Integral variables satisfy two important properties:

- A variable $x_s \in \mathbf{x}$ is integral over $\mathbf{k}[x_1, \ldots, x_{s-1}]$ modulo $\mathcal{P}$ if and only if there exists $g \in G$ such that $\mathrm{lm}(g) = x_s^{d_s}$ for some positive integer d_s,

– if a variable $x_s \in \mathbf{x}$ is integral over $\mathbf{k}[x_1, \ldots, x_{s-1}, \mathbf{u}]$ modulo $\mathcal{P}$, with $\mathbf{u} \subseteq T_{\mathcal{P}}$ disjoint from $\{x_1, \ldots, x_s\}$, then x_s is also integral over $\mathbf{k}[x_1, \ldots, x_{s-1}]$ modulo $\mathcal{P}$.

Thanks to the above properties, we may assume w.l.o.g. that if $d = \dim(\mathcal{P})$ then we have $T_{\mathcal{P}} = \{x_1, \ldots, x_d\}$. Consider a linear change of coordinates A in $\overline{\mathbf{k}}^n$ defined by a matrix M of the following form:

$$M = \left(\begin{array}{c|ccc} & a_{1,d+1} & \cdots & a_{1,n} \\ \mathrm{I}_{d\times d} & \vdots & \vdots & \vdots \\ & a_{d,d+1} & \cdots & a_{d,n} \\ \hline \mathbf{0} & \multicolumn{3}{c}{\mathrm{I}_{(n-d)\times(n-d)}} \end{array} \right) \tag{5}$$

where $a_{i,j} \in \mathbf{k}$. We denote by $\mathcal{P}^A$ the ideal generated by f^A for all $f \in \mathcal{P}$. Then, by Noether normalization lemma, for a generic choice of $a_{1,d+1}, \ldots, a_{d,n}$ the following properties hold:

1. $x_1, \ldots, x_d$ are algebraically independent modulo $\mathcal{P}^A$,
2. x_{d+i} is integral over $\mathbf{k}[x_1, \ldots, x_d]$ modulo $\mathcal{P}^A$ for all $i = 1, \ldots, n-d$.

In this case, we say that $\mathcal{P}^A$ is in *Noether position.*

We turn our attention to the regular chain representation of the prime ideal $\mathcal{P}$. To this end, using Theorem 3.3 of [2], one can extract, in an algorithmic fashion, a subset T of G such that T is a regular chain whose saturated ideal is precisely $\mathcal{P}$. Let H be the reduced lexicographical Gröbner basis of $\mathcal{P}^A$ and C be the regular chain extracted from H using the same theorem from [2].

Theorem 1 *If T generates its saturated ideal, then the regular chain C is monic, that is, for each polynomial $f \in C$ we have* $\mathrm{init}(f) \in \mathbf{k}$.

Proof. Assume by contradiction that there exists $f \in C$ such that $\mathrm{init}(f) \notin \mathbf{k}$ and let us choose such an f with minimum main variable. Since $x_1, \ldots, x_d$ are algebraically independent modulo $\mathcal{P}^A$ and since C is a regular chain, one can compute a polynomial f' such that $\mathrm{init}(f') \in \mathbf{k}[x_1, \ldots, x_d]$ and $\mathrm{sat}(C') = \mathrm{sat}(C)$ holds with $C' = C \setminus \{f\} \cup \{f'\}$.

Let $\mathrm{mvar}(f) = x_r$. Since $\mathcal{P}^A$ is in Noether position, it follows from [14] that there exists a polynomial $H_{x_r} \in H$ whose leading monomial is of the form $x_r^{d_r}$. Since $\mathrm{init}(H_{x_r}) \in \mathbf{k}$, we have $\deg(f', x_r) = \deg(f, x_r) < d_r = \deg(H_{x_r}, x_r)$. Indeed, otherwise the polynomial H_{x_r} would have been selected as an element of the regular chain C.

From the choice of f and the assumption on T, the regular chain $C' \cap \mathbf{k}[x_1, \ldots, x_r]$ is a basis of $\mathcal{P}^A \cap \mathbf{k}[x_1, \ldots, x_r]$. Therefore, the polynomial H_{x_r} reduces to zero through multivariate division by $C' \cap \mathbf{k}[x_1, \ldots, x_r]$ and thus by $C \cap \mathbf{k}[x_1, \ldots, x_r]$. This contradicts the fact that H is a reduced Gröbner basis. □

Remark 1 *Theorem 1 states that if T generates* $\mathrm{sat}(T)$ *and $\mathcal{P}^A$ is in Noether position, then C is monic. Unfortunately, if T does not generate* $\mathrm{sat}(T)$*, then the previous conclusion may not hold as shown by the following example.*

Example 1 *Consider the regular chain* $T := \{x_2^5 - x_1^4, x_1x_3 - x_2^2\} \subset \mathbb{Q}[x_1 < x_2 < x_3]$ *which does not generate its saturated ideal. Consider also the linear change of coordinates* A *defined by the matrix below*

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\langle T \rangle^A$ *is in Noether position and under this new change of coordinates we can compute the regular chain* $C = \{c_1, c_2\}$ *such that* $\sqrt{\mathrm{sat}(C)} = \sqrt{\mathrm{sat}(T)}^A$ *where* $c_1 = x_2^5 - 2x_2^4 + x_2^3 + 4x_1^2x_2^2 - x_1^4$ *and* $c_2 = \left(-x_1^3 + 2x_2^2x_1\right)x_3 + x_1^2x_2^2 - x_2^4 + x_2^3$. *As you can see* $\mathrm{init}(c_2) \notin \mathbb{Q}$.

5 Applications of random linear changes of coordinates

Let $T \subset \mathbf{k}[\mathbf{x}]$ be a regular chain whose saturated ideal has dimension d. Let $\mathbf{u}$ be the free variables of T. Recall that h_T stands for the product of the $\mathrm{init}(f)$ for $f \in T$. Let A be a linear change of coordinates in $\overline{\mathbf{k}}^n$. Assume that the extended version of the `PALGIE` algorithm (see Problem 3 in Section 3) applied to T and A produces a single regular chain $C \subset \mathbf{k}[\mathbf{x}]$, thus satisfying $\overline{W^A(T)} = \overline{W(C)}$. Let h_T and h_C be the products of the initials of T and C, respectively. Let r_T^A and r_C be the iterated resultants (see [6] for this term) of h_T^A and h_C w.r.t. C.

Proposition 1 gathers elementary properties of r_T^A and r_C. Proposition 2 provides conditions for deriving a basis of $\mathrm{sat}(T)$ from the calculation of C while Theorem 2 provides a condition for deriving $\lim(W(T)) = \overline{W(T)} \cap V(h_T)$ from the calculation of C. The basic idea of Theorem 2 is to use a linear change of coordinates so as to replace the description of $\overline{W(T)}$ by one for which $\overline{W(T)} \cap V(h_T)$ can be computed by set-theoretic operations on constructible sets (represented by regular chain as in [5]). Moreover, Corollary 1 shows that, if T generates $\mathrm{sat}(T)$, then the computation of $\lim(W(T))$ can always be achieved by the techniques of [5].

Proposition 1 *The following properties hold:*

(*i*) *the polynomial* h_T^A *is regular w.r.t.* $\mathrm{sat}(C)$,
(*ii*) *the polynomials* r_T^A *and* r_C *belong to* $\mathbf{k}[\mathbf{u}]$ *and are non-zero.*

Proof. Property (*i*) is by construction, that is, following the extended `PALGIE` algorithm applied to T and A. Property (*ii*) follows from (*i*) and the relations between regular chains and iterated resultants, see [6]. □

Proposition 2 *The following properties hold:*

(*i*) *if* $\mathrm{sat}(T)$ *is radical and if the ideal* $\langle h_T, (h_C^{A^{-1}})\rangle$ *equals the whole ring* $\mathbf{k}[\mathbf{x}]$, *then* $T \cup C^{A^{-1}}$ *generates* $\mathrm{sat}(T)$,
(*ii*) *if the regular chain* C *is monic, then* $C^{A^{-1}}$ *generates* $\mathrm{sat}(T)$.

PROOF. We prove Property (i). Since $\operatorname{sat}(T)$ is radical, the relations $\overline{W^A(T)} = \overline{W(C)}$ implies $C^{A^{-1}} \subset \operatorname{sat}(T)$. Hence, we "only" need to prove that if a polynomial f belongs to $\operatorname{sat}(T)$, then f is generated by $T \cup C^{A^{-1}}$. So let $f \in \operatorname{sat}(T)$. On one hand, there exists a non-negative integer e such that $h_T^e f \in \langle T \rangle$. On the other, there exists a non-negative integer d such that $(h_C^{A^{-1}})^d f \in \langle C^{A^{-1}} \rangle$. Since the ideal $\langle h_T^e, (h_C^{A-1})^d \rangle$ is the whole ring $\mathbf{k}[\mathbf{x}]$, then we can write f as an element of $\langle T, C^{A^{-1}} \rangle$. Now we prove (ii). Since C is monic, it is a Gröbner basis of $\operatorname{sat}(C)$, and, from the specifications of the `PALGIE` algorithm, a basis of $\operatorname{sat}(T)^A$ as well. Thus $C^{A^{-1}} := \{f^{A^{-1}} \mid f \in C\}$ is a basis of $\operatorname{sat}(T)$. $\square$

From now on, we assume that the coefficients of the matrix $M = (m_{ij})$ are pairwise different variables. We view the coefficients of M, as well as the coefficients of all polynomials, as elements of the field of rational functions $\mathbf{k}(m_{ij})$. Moreover, the base field $\mathbf{k}$ is either $\mathbb{R}$ or $\mathbb{C}$ so that the affine space $\overline{\mathbf{k}}^n$ is endowed with the Euclidean topology. In this context, we recall from [1] that the quasi-component $W(T)$ has the same closure in both the Euclidean and the Zariski topologies.

Theorem 2 *For all values of (m_{ij}) such that $V(r_T^A, r_C)$ is empty, we have*

$$\lim(W(T)) \quad = \quad \{A^{-1}(\mathbf{y}) \ \mid \ \mathbf{y} \in V(h_T^A) \cap W(C)\}. \tag{6}$$

PROOF. Observe first that $V(r_T^A, r_C)$ is empty if and only if $V(r_T, r_c^{A^{-1}})$ is empty. Observe next that any zero $\zeta \in \overline{\mathbf{k}}^n$ of h_T extends a zero $\zeta' \in \overline{\mathbf{k}}^d$ of r_T, see [5]. Therefore, for any choice of the parameters (m_{ij}) such that $V(r_T^A, r_C)$ is empty, one can let $(x_1, \ldots, x_n)$ approach a given root of h_T while staying within a bounded open set of $W^{A^{-1}}(C)$ leading to finitely many (possibly zero) finite limits for $(x_1, \ldots, x_n)$. Since, by construction, the constructible sets $W^{A^{-1}}(C)$ and $W(T)$ have the same Zariski closure, it follows that the points of $V(h_T^A) \cap W(C)$ are the images by A of the desired limit points of $W(T)$. $\square$

Example 2 *Consider the regular chain $T := \{x_4, x_2x_3 + x_1^2\} \subset \mathbb{Q}[x_1 < x_2 < x_3 < x_4]$ and the linear change of coordinates A corresponding to the matrix*

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Using the extended of `PALGIE`*, we can compute $C := \{x_4, x_3^2 + x_2x_3 + x_1^2\}$ and consequently, $r_T^A = x_1^2$ and $r_C = 1$. Then $\langle r_T^A, r_C \rangle = \langle 1 \rangle$ holds. Using* `Triangularize` *command of Maple, one can get*

$$\langle C, h_T^A \rangle^{A^{-1}} \quad = \quad \langle x_4, x_2, x_1 \rangle \quad = \quad \lim(W(T)).$$

Corollary 1 *Assume that T generates* $\operatorname{sat}(T)$*. Then we have*

$$\lim(W(T)) = V(T) \setminus W(T) \tag{7}$$

Hence, $\lim(W(T))$ *can be obtained by set-theoretic operations on constructible sets. Moreover, generically, the set* $\lim(W(T))$ *is determined by* $V(h_T^A) \cap W(C)$*.*

Proof. We prove the first claim. The hypothesis implies $V(T) = \overline{W(T)}$. Since $V(T) = W(T) \cup (V(T) \cap V(h_T))$, the conclusion follows. The second claim follows immediately from Theorems 2 and 1. $\square$

6 On the computation of $\lim(W(T))$ and $\mathrm{sat}(T)$

Let T be a regular chain whose saturated ideal has dimension d. A driving application of this paper is the computation of $\lim(W(T))$. Section 5 was primarily dedicated to the case where T is a basis of its saturated ideal, while in the present section we replace this assumption by others. Recall that we have the follow equalities:

$$V(\mathrm{sat}(T)) = \overline{W(T)} = \left(\overline{W(T)} \cap V(h_T)\right) \cup W(T) = \lim(W(T)) \cup W(T).$$

Therefore, computing $\lim(W(T))$ and computing $V(\mathrm{sat}(T))$ are equivalent problems. Theorems 3, 4 and Lemma 2 below deal with the latter problem while Proposition 3 is concerned with the former. All these results make some assumption on T and we do not know a general procedure for computing either $\lim(W(T))$ or $V(\mathrm{sat}(T))$ that would avoid Gröbner basis calculation.

Lemma 1 *Let* $\mathcal{I}$ *be a radical ideal of* $\mathbf{k}[\mathbf{x}]$*. Let* $h \in \mathbf{k}[\mathbf{x}]$*. Assume that the dimension of any associated prime* $\mathfrak{p}$ *of* $\mathcal{I}$ *is at least* d*. Then* $\dim(V(\mathcal{I}, h)) < d$ *implies that* h *is regular modulo* $\mathcal{I}$*. If the dimension of any associated prime* $\mathfrak{p}$ *of* $\mathcal{I}$ *is* d*, that is, if* $\mathcal{I}$ *is an unmixed ideal of dimension* d*, then* $\dim(V(\mathcal{I}, h)) < d$ *holds if and only if* h *is regular modulo* $\mathcal{I}$*.*

Proof. Let $\mathcal{I} = \cap_{i=1}^{s} \mathfrak{p}_i$, where $\mathfrak{p}_i$ are the associated prime of $\mathcal{I}$. Assume that $\dim(V(\mathcal{I}, h)) < d$, it is enough to show that h does not belong to any $\mathfrak{p}_i$. On the other hand, we have $V(\mathcal{I}, h) = \cup_{i=1}^{s} V(\mathfrak{p}_i, h)$. If h belongs to some $\mathfrak{p}_i$, then $V(\mathfrak{p}_i, h) = V(\mathfrak{p}_i)$. Since $\dim(\mathfrak{p}_i) \geq d$, we know that $\dim(V(\mathcal{I}, h)) \geq d$, which is a contradiction to the assumption that $\dim(\mathcal{I}, h) < d$.

If $\mathcal{I}$ is an unmixed ideal of dimension d, by the above argument, $\dim(V(\mathcal{I}, h)) < d$ implies that h is regular modulo $\mathcal{I}$. On the other hand, if h is regular modulo $\mathcal{I}$, then h does not belong to any $\mathfrak{p}_i$. Thus $\dim(V(\mathcal{I}, h)) = \max(\dim(V(\mathfrak{p}_i, h))) < \max(\dim(\mathfrak{p}_i)) = d$. $\square$

Theorem 3 *Let* $T \subset \mathbf{k}[\mathbf{x}]$ *be a regular chain with free variables* $x_1, \ldots, x_d$*. Let* h_T *be the product of the initials of the polynomials in* T*. Then, we have* $\sqrt{\langle T \rangle} = \sqrt{\mathrm{sat}(T)}$ *if and only if* $\dim(V(T, h_T)) < d$ *holds.*

Proof. First we claim that for any associated prime $\mathfrak{p}$ of $\sqrt{\langle T \rangle}$, we have $\dim(\mathfrak{p}) \geq n - d$. To prove this, we first notice that the associated primes $\mathfrak{p}$ of $\sqrt{\langle T \rangle}$ are exactly the minimal associated primes $\mathfrak{p}$ of $\langle T \rangle$. On the other hand, since $\langle T\rangle \subseteq \mathrm{sat}(T)$ and $V(\mathrm{sat}(T)) \neq \emptyset$ hold, we know that $\langle T\rangle$ generates a proper ideal. By Krull's principle ideal theorem, for any minimal associated prime $\mathfrak{p}$ of $\langle T\rangle$, the height of $\mathfrak{p}$ is less than or equal to $|T|$. Since $|T| = n - d$, we have $\dim(\mathfrak{p}) \geq n - d$. The claim is proved.

Now we prove that we have $\sqrt{\langle T\rangle} = \sqrt{\mathrm{sat}(T)}$ if and only if $\dim(V(T, h_T)) < d$ holds. First, we show that the condition is sufficient. If $\dim(V(T, h_T)) < d$ holds, with the previous claim and Lemma 1, we deduce that h_T is regular modulo $\sqrt{\langle T\rangle}$. Thus, we have $\sqrt{\mathrm{sat}(T)} = \sqrt{\langle T\rangle : h_T^\infty} = \sqrt{\langle T\rangle} : h_T^\infty = \sqrt{\langle T\rangle}$. Next, we show that the condition is necessary. If $\sqrt{\langle T\rangle} = \sqrt{\mathrm{sat}(T)}$, then $\sqrt{\langle T\rangle}$ is an unmixed ideal and h_T is regular modulo $\sqrt{\langle T\rangle}$. Thus, $\dim(V(T, h_T)) < d$ holds by Lemma 1. □

Remark 2 *As an immediate corollary, we have $V(T) = \overline{W(T)}$ if and only if $\dim(V(T, h_T)) < d$. There are many ways to compute the dimension of an algebraic set. In particular, this dimension can be determined by computing a Kalkbrener triangular decomposition. We denote by* IsClosure *a procedure to test $V(T) = \overline{W(T)}$, by applying Theorem 3.*

Example 3 *Consider the regular chain $T := \{x_1x_2 + x_1, x_1x_3 + 1\}$ of $\mathbb{Q}[x_1 < x_2 < x_3]$. Since the first polynomial is not primitive w.r.t. x_2, T is not a primitive regular chain in the sense of [13]. Since $V(T, x_1) = \emptyset$ holds, applying Theorem 3, we have $\sqrt{\langle T\rangle} = \sqrt{\mathrm{sat}(T)}$. Actually $\langle T\rangle = \mathrm{sat}(T)$ also holds.*

Theorem 4 *Let T be a regular chain of $\mathbf{k}[\mathbf{x}]$ with free variables $x_1, \ldots, x_d$. Let $C_1, \ldots, C_s \subset \mathbf{k}[\mathbf{x}]$. Assume that $\langle C_i\rangle \subseteq \sqrt{\mathrm{sat}(T)}$ holds, for all $i = 1, \ldots, s$. Let $\mathcal{I} = \langle T, C_1, \ldots, C_s\rangle$. Then $\sqrt{\mathrm{sat}(T)} = \sqrt{\mathcal{I}}$ if and only if there exist regular chains T_i, $i = 1, \ldots, t$, such that each of the following properties hold:*

(*i*) $\sqrt{\mathcal{I}} = \cap_{i=1}^t \sqrt{\mathrm{sat}(T_i)}$,
(*ii*) $|T_1| = \cdots = |T_t| = n - d$,
(*iii*) h_T *is regular modulo all* $\sqrt{\mathrm{sat}(T_i)}$.

Proof. The direction "⇒"obviously holds. Next we prove the direction "⇐". By (*i*) and (*ii*), we know that $\sqrt{\mathcal{I}}$ is an unmixed ideal of dimension d. Since h_T is regular modulo all $\sqrt{\mathrm{sat}(T_i)}$, by Lemma 1, we have $\dim(V(h_T, \mathrm{sat}(T_i))) < d$. Thus $\dim(V(\mathcal{I}, h_T)) < d$ holds. Applying Lemma 1 again, we know that h_T is regular modulo $\sqrt{\mathcal{I}}$. Thus $\sqrt{\mathcal{I}} = \sqrt{\mathcal{I}} : h_T^\infty = \sqrt{\mathcal{I} : h_T^\infty}$ holds. On the other hand, we have $\langle T\rangle \subseteq \mathcal{I}$, thus we deduce that $\sqrt{\mathrm{sat}(T)} \subseteq \sqrt{\mathcal{I}}$. Since $\mathcal{I} = \langle T, C_1, \ldots, C_s\rangle$ and $\langle C_i\rangle \subseteq \sqrt{\mathrm{sat}(T)}$, we also have $\mathcal{I} \subseteq \sqrt{\mathrm{sat}(T)}$. The theorem is proved. □

Remark 3 *In Theorem 4, if $s = 0$, then the theorem trivially holds for $t = 1$ and $T_1 = T$. In practice, for example in Algorithm 1, the polynomial sets C_i, for all $i = 1, \ldots, s$, are regular chains for different orderings such that $\sqrt{\mathrm{sat}(C_i)} = \sqrt{\mathrm{sat}(T)}$ holds. Let $T_1, \ldots, T_t$ be regular chains in the output of* Triangularize($\mathcal{I}$). *Then (i) automatically holds. If condition (ii) is satisfied, then $\overline{W(T)} = V(\mathcal{I})$ holds if and only if (iii) holds, which is easy to check by computing iterated resultants of h_T w.r.t. the regular chains T_i. Thus, this theorem provides an algorithmic recipe which may compute $\overline{W(T)}$ in somes cases, see Algorithm 1.*

Algorithm 1: Closure(T)

Input: A non-empty regular chain T of $\mathbf{k}[x_1 < \cdots < x_n]$.

Output: Return $\emptyset$ or a polynomial set G such that $\overline{W(T)} = V(G)$. If $\emptyset$ is returned, this means that the algorithm fails to compute $\overline{W(T)}$.

```
begin
    G := ∅;
    for i from 1 to n do
        if i = 1 then
            C := T;
        else
            let R be the ordering x_i < x_{i+1} < ··· < x_n < x_1 ··· < x_{i-1};
            D := PALGIE(T, R);
            if |D| ≠ 1 then
                return ∅
            else
                let C be the only regular chain in D;
        if IsClosure(C) then
            return C;
        else
            G := G ∪ C;
            D := Triangularize(G, mode = K)// compute a Kalkbrener
                triangular decomposition of V(G)
            if all regular chains in D have dimension d and h_T is regular w.r.t.
            each of them then
                return G
    return ∅;
```

Example 4 *We illustrate Algorithm 1 on one example. Consider the regular chain* $T := \{x_2^5 - x_1^2, x_1x_3 - x_2^2(x_2+1)\}$ *of* $\mathbb{Q}[x_1 < x_2 < x_3]$. *Then* $V(T, x_1) := \{(x_1, x_2, x_3) \mid x_1 = x_2 = 0\}$, *whose dimension is* 1. *By Theorem 3, we know that* $V(T) \neq V(sat(T))$. *Let* $C := \{x_2x_3^2 - x_2^2 - 2x_2 - 1, x_3x_1 - x_2^3 - x_2^2\}$ *be another regular chain of* $\mathbb{Q}[x_2 < x_3 < x_1]$. *One can verify that* $\mathrm{sat}(C) = \mathrm{sat}(T)$ *holds. Let* $\mathcal{I} := \langle C, T\rangle$. *A Kalkbrener triangular decomposition of* $\mathcal{I}$ *w.r.t. the order* $x_1 < x_2 < x_3$ *consists only of one regular chain, which is* T *itself. Thus by Theorem 4, we have* $V(\mathrm{sat}(T)) = V(\mathcal{I})$.

Remark 4 *We selected* 22 *one-dimensional non-primitive regular chains to test Algorithm 1. For* 10 *of them, the algorithm could successfully compute* $\overline{W(T)}$. *We also tested some random examples. The random regular chains are generated as follows. We choose a pair of random polynomials with* 4 *variables and of total degree* 2. *Then we apply* Triangularize *to this pair, thus obtaining* 2*-dimensional regular chains. In this way, we generated* 20 *regular chains, out of which* 16 *turned out to be non-primitive regular chains. Algorithm 1 successfully computed* $\overline{W(T)}$ *for* 10 *of those* 16 *examples.*

Lemma 2 *Let* $T = \{t_2(x_1, x_2), t_3(x_1, x_3), \ldots, t_s(x_1, x_s)\}$ *be a regular chain of* $\mathbf{k}[x_1 < \cdots < x_s]$. *Assume that for all* $i = 2, \ldots, s$, *the polynomial* t_i *is a primitive polynomial w.r.t. its main variable* x_i. *Then, the regular chain* T *generates its saturated ideal.*

PROOF. To prove this lemma, it is enough to prove by induction that $\mathrm{sat}(T_i) = \langle T_i \rangle$, for $i = 2 \ldots, s$, where $T_i := \{t_2, \ldots, t_i\}$. The lemma clearly holds for $i = 2$. Assume that the regular chain T_{i-1} is generating its saturated ideal. If $\mathrm{tail}(t_i)$ is invertible modulo $\langle \mathrm{init}(t_i)\} \cup T_{i-1} \rangle$, then $\langle T_i \rangle = \mathrm{sat}(T_i)$ holds (see [13]). Suppose that $\mathrm{tail}(t_i)$ is not invertible modulo $\langle \{\mathrm{init}(t_i)\} \cup T_{i-1} \rangle$, then $\langle \{\mathrm{init}(t_i)\} \cup T_{i-1} \rangle$ generates a proper zero-dimensional ideal, since $\mathrm{init}(t_i)$ is regular modulo $\langle T_{i-1} \rangle$. Let $\mathfrak{p}$ be an associated prime of this ideal. If $\mathrm{tail}(t_i)$ is not regular modulo $\mathfrak{p}$, then all the coefficients of t_i belong to $\mathfrak{p}$. On the other hand, since $t_s(x_1, x_i)$ is primitive, the ideal formed by the coefficients of t_i is the field $\mathbf{k}$, a contradiction. $\square$

Remark 5 *If a regular chain* T *has the same shape as in Lemma 2, except that the polynomials* t_i *are not necessarily primitive, for* $i = 2, \ldots, s$, *then by making all the polynomials* t_i *primitive, we obtain a new regular chain* T' *such that we have* $\langle T' \rangle = \mathrm{sat}(T') = \mathrm{sat}(T)$.

Example 5 *Let* $T := \{x_3^2 - 2\,x_1, 3\,x_2^3 + 4\,x_1^2\} \subset \mathbb{Q}[x_1 < x_2 < x_3]$ *be a 1-dimensional regular chain. As you can see both elements of* T *are primitive bivariate polynomials. Then Lemma 2 implies that* T *generates its saturated ideal.*

Example 6 *The above lemma clearly does not hold for regular chains with more than one free variable. Consider for example the regular chain* $T := \{x_1x_3 + x_2, x_1x_4 + x_2\}$, *where* $x_1 < x_2 < x_3 < x_4$. *It is clear that* $x_4 - x_3 \notin \langle T \rangle$. *However, one can prove that* $x_4 - x_3 \in \mathrm{sat}(T)$ *because* $x_1x_4 + x_2 = x_1(x_4 - x_3)$ *modulo* $\langle x_1x_3 + x_2 \rangle$.

Lemma 3 *Let* $T \subset \mathbf{k}[\mathbf{x}]$ *be a regular chain with free variable* x_1. *Let* $C_2 = T$ *and let* C_i, *for* $3 \leq i \leq n$, *be regular chains w.r.t. the order* $x_1 < x_i < \mathbf{x} \setminus \{x_1, x_i\}$ *such that* $\sqrt{\mathrm{sat}(C_i)} = \sqrt{\mathrm{sat}(T)}$. *Assume that all the polynomials of* C_i *are primitive w.r.t. their main variables for* $i = 2, \ldots, n$. *Then* $\dim(V(C_2, \ldots, C_n)) = 1$ *holds.*

PROOF. By the fact that $\overline{W(T)} = \overline{W(C_i)}$, we know that $\overline{W(T)} \subseteq V(C_2, \ldots, C_n)$, which implies that $\dim(V(C_2, \ldots, C_n)) \geq 1$. Let c_i be the polynomial in C_i with the main variable x_i. Then the set $C := \{c_2, \ldots, c_n\}$ is clearly a regular chain since $\mathrm{init}(c_i) \in \mathbf{k}[x_1]$ holds for each $i = 2, \ldots, s$. Moreover C generates its saturated ideal by Lemma 2. Thus $\dim(V(C)) = 1$. Since $V(C_2, \ldots, C_n) \subseteq V(C)$, we know that $\dim(V(C_2, \ldots, C_n)) \leq 1$. Thus the lemma holds. $\square$

Example 7 *Let* $T := \{x_2^5 - x_1^4, x_1x_3 - x_2^2\}$ *be a regular chain of* $\mathbb{Q}[x_1 < x_2 < x_3]$. *Let also* $C := \{x_3^5 - x_1^3, x_3^2x_2 - x_1^2\}$ *be a regular chain of* $\mathbb{Q}[x_1 < x_3 < x_2]$ *for which we have* $\mathrm{sat}(C) = \mathrm{sat}(T)$. *One can verify that* $\dim(V(T, C)) = 1$. *Indeed a Kalkbrener triangular decomposition of* $T \cup C$ *computed by the* $\mathsf{Triangularize}$

command of `RegularChains` *library w.r.t. the order* $x_1 < x_2 < x_3$ *is* $\{T, D\}$, *where* $D := \{x_1, x_2, x_3\}$.

It is easy to observe that the decomposition computed by Triangularize *is redundant, that is we have* $\mathrm{sat}(T) \subseteq \mathrm{sat}(D)$ *holds. By Theorem 4, we conclude that* $\sqrt{\langle T, C\rangle} = \sqrt{\mathrm{sat}(T)}$. *However, for this example, Algorithm 1 fails to compute the set* G *such that* $\overline{W(T)} = V(G)$, *since* T *and* D *do not have the same height.*

Lemma 3, Example 7 and Theorem 4 show that it is possible to compute $\mathrm{sat}(T)$ by a change of order of the variables. One might wonder if this is always true. In particular, we ask the following two questions.

Question 1 *Let* $C_1, \ldots, C_n$ *be regular chains of* $\mathbf{k}[\mathbf{x}]$ *w.r.t. the order* $x_i < x_{i+1} < \cdots < x_n < x_1 \cdots < x_{i-1}$, *for* $i = 1, \ldots, n$. *Assume that* $\sqrt{\mathrm{sat}(C_1)} = \cdots = \sqrt{\mathrm{sat}(C_n)}$. *Does* $\sqrt{\mathrm{sat}(C_1)} = \sqrt{\langle \cup_{i=1}^{n} C_i\rangle}$ *always hold?*

Question 2 *Let* $C_1, \ldots, C_n$ *be polynomial sets of* $\mathbf{k}[\mathbf{x}]$ *such that* C_i *is a regular chain for the order* $x_i < x_{i+1} < \cdots < x_n < x_1 \cdots < x_{i-1}$, *for* $i = 1, \ldots, n$. *Assume that* $\sqrt{\mathrm{sat}(C_i)} = \sqrt{\mathrm{sat}(C_j)}$ *for all* $1 \le i < j \le n$. *Let* $P_i \in C_i$ *be the polynomial of least rank. Let* H_1 *be the product of the initials of* C_1. *Does the relation*

$$\lim(W(C_1)) \quad = \quad V(C_1 \cup \{P_1, \ldots, P_n, H_1\})$$

always hold?

To answer the two questions, we investigated over 35 different polynomial systems, and all of them succeeded but two of which failed. Here is one of them.

Example 8 *Suppose* $T := \{t_1, t_2\} \subset \mathbb{Q}[x_1 < x_2 < x_3 < x_4]$ *is a regular chain of dimension two, where* $t_1 = -93\, x_1 x_2^2 + (53\, x_1 - 35)\, x_2 + 93\, x_1^3 - 26\, x_1^2 - 57\, x_1$ *and* $t_2 = 93\, x_1 x_4 + \big((3233\, x_1 - 2135)\, x_2 + 5673\, x_1^3 + 213\, x_1^2 - 3477\, x_1\big)\, x_3 + \big(-530\, x_1^2 - 3091\, x_1\big)\, x_2 - 930\, x_1^4 + 6119\, x_1^3 + 570\, x_1^2 - 1767\, x_1$. *One can verify that* T *does not generate its saturated ideal.*

Following the notations if Question 1, using `PALGIE`, *we will be able to compute regular chains* C_i *for* $i = 1, \ldots, 4$ *w.r.t the orders mentioned in Question 1. To see whether the statement of Question 1 is true or not, on one hand, we can find the Kalkbrener triangular decomposition* $\{C_1, R_1, R_2\}$ *for* $V(\cup_{i=1}^{4} C_i)$ *where* $C_1 = T$, $R_1 := \{x_4 - 19, x_2, x_1\}$, *and* $R_2 := \{961\, x_4^2 + 42428\, x_4 + 279756, x_3, x_2, x_1\}$.

On the other hand, using methods based on Gröbner bases computations to find a generator for $\mathrm{sat}(C_1)$, *one can find the Kalkbrener triangular decomposition* $\{C_1, R_1\}$ *for* $V(\mathrm{sat}(C_1))$.

Therefore, we have

$$V(\mathrm{sat}(C_1)) \;=\; W(C_1) \cup W(R_1) \;\neq\; V(\cup_{i=1}^{4} C_i) \;=\; W(C_1) \cup\, W(R_1) \cup W(R_2).$$

This shows that the statement of Question 1 is not true.

Furthermore,

$$V(C_1 \cup \{P_1, \ldots, P_4, H_1\}) \quad = \quad W(R_1) \quad \cup \quad W(R_2)$$

where H_1 is the product of the initials of C_1 and P_i is the polynomial in C_i with least rank for $i = 1, \ldots, 4$. But the correct limit points are only represented by R_1 which means $\lim(W(C_1)) \neq V(C_1 \cup \{P_1, \ldots, P_4, H_1\})$. Cosequently, for this example, the answer to both Questions 1 and 2 is negative.

In Example 8, as one can see, we computed the limit points plus some extra points. The extra component R_2 in this example is of dimension 0 while the limit points we are expecting are of dimension 1.

Proposition 3 *Let T be a regular chain such that $\langle \mathrm{sat}(T) \rangle$ has dimension d and let $F \subset \langle \mathrm{sat}(T) \rangle$ such that either $V(T \cup F \cup \{h_T\})$ has dimension $d-1$ and is irreducible. Suppose also that $\lim(W(T))$ is not empty. Then, we have $\lim(W(T)) = V(T \cup F \cup \{h_T\})$.*

PROOF. The proof is straightforward. □

Example 9 *Consider the regular chain $T := \{x_1\, x_3 + x_2, x_2\, x_4 + x_1\} \subset \mathbb{Q}[x_1 < x_2 < x_3 < x_4]$. One can consider F to be the regular chain computed by applying* `PALGIE` *to T w.r.t. the variable order $x_3 < x_4 < x_1 < x_2$ and consequently, "fish" the polynomial $x_3\, x_4 - 1 \in \mathrm{sat}(T)$. Then*

$$\begin{aligned} V(T \cup F \cup \{h_T\}) &= V(x_1\, x_3 + x_2, x_2\, x_4 + x_1, x_3\, x_4 - 1, x_1\, x_2) \\ &= V(x_1, x_2, x_3\, x_4 - 1) \\ &= \lim(W(T)). \end{aligned}$$

7 Conclusion

Among all the methods we have considered for computing $\lim(W(T))$ and $\mathrm{sat}(T)$, those based on linear changes of coordinates seem very promising. They are a good trick for finding a subset $F \subset \mathrm{sat}(T)$ such that $F \cup T$ is a basis of $\mathrm{sat}(T)$, see Proposition 3. To develop that direction further, we are currently investigating the following related questions:

– decide whether $\lim(W(T))$ is empty
– decide whether $W(R) \subseteq \lim(W(T))$ for a given regular chain.

Acknowledgements

The authors would like to thank the referees for careful reading of the manuscript and for their helpful suggestions. The research of the second author was partially supported by NSFC (11301524,11471307,61202131). The research of the third author was in part supported by a grant from IPM (No. 93550420).

References

1. P. Alvandi, C. Chen, and M. Moreno Maza. Computing the limit points of the quasi-component of a regular chain in dimension one. In *Computer Algebra in Scientific Computing (CASC 2013)*, volume 8136 of *Lect. Notes Comput. Sci.*, pages 30–45, 2013.
2. P. Aubry, D. Lazard, and M. Moreno Maza. On the theories of triangular Sets. *J. Symb. Comput.*, 28(1-2):105–124, 1999.
3. F. Boulier, F. Lemaire, and M. Moreno Maza. Pardi! In *Proceedings of International Symposium on Symbolic and Algebraic Computation (ISSAC 2001)*, pages 38–47, 2001.
4. F. Boulier, F. Lemaire, and M. Moreno Maza. Computing differential characteristic sets by change of ordering. *J. Symb. Comput.*, 45(1):124–149, 2010.
5. C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, and W. Pan. Comprehensive triangular decomposition. In *Computer Algebra in Scientific Computing (CASC 2007)*, volume 4770 of *Lect. Notes Comput. Sci.*, pages 73–101, 2007.
6. C. Chen and M. Moreno Maza. Algorithms for computing triangular decomposition of polynomial systems. *J. Symb. Comput.*, 47(6):610–642, 2012.
7. X. Dahan, X. Jin, M. Moreno Maza, and É. Schost. Change of order for regular chains in positive dimension. *Theor. Comput. Sci.*, 392(1-3):37–65, 2008.
8. D. Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry.* Springer-Verlag, New York, 1995.
9. G.M. Greuel and G. Pfister. *A Singular Introduction to Commutative Algebra.* Springer-Verlag, Berlin, 2002.
10. A. Hashemi. Effective computation of radical of ideals and its application to invariant theory. In *International Congress on Mathematical Software (ICMS 2014)*, volume 8592 of *Lect. Notes Comput. Sci.*, pages 382–389, 2014.
11. T. Krick and A. Logar. An algorithm for the computation of the radical of an ideal in the ring of polynomials. In *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (AAECC 1991)*, volume 539 of *Lect. Notes Comput. Sci.*, pages 195–205, 1991.
12. G. Lecerf. Computing the equidimensional decomposition of an algebraic closed set by means of lifting fibers. *J. of Complexity*, 19(4):564–596, 2003.
13. François Lemaire, Marc Moreno Maza, Wei Pan, and Yuzhen Xie. When does $\langle T \rangle$ equal $\mathrm{sat}(T)$? *J. Symb. Comput.*, 46(12):1291–1305, 2011.
14. A. Logar. A computational proof of the noether normalization lemma. In *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes (AAECC 1988)*, volume 357 of *Lect. Notes Comput. Sci.*, pages 259–273, 1988.
15. F. Rouillier. Solving zero-dimensional systems through the rational univariate representation. *Appl. Algebra Eng. Commun. Comput.*, 9(5):433–461, 1999.
16. W.M. Seiler. A combinatorial approach to involution and δ-regularity II: Structure analysis of polynomial modules with Pommaret bases. *Appl. Alg. Eng. Comm. Comp.*, 20:261–338, 2009.
17. A.J. Sommese and J. Verschelde. Numerical homotopies to compute generic points on positive dimensional algebraic sets. *J. Complexity*, 16(3):572–602, 2000.