Computing Moore-Penrose Inverses of Ore Polynomial Matrices

Yang Zhang

Department of Mathematics

University of Manitoba, Canada

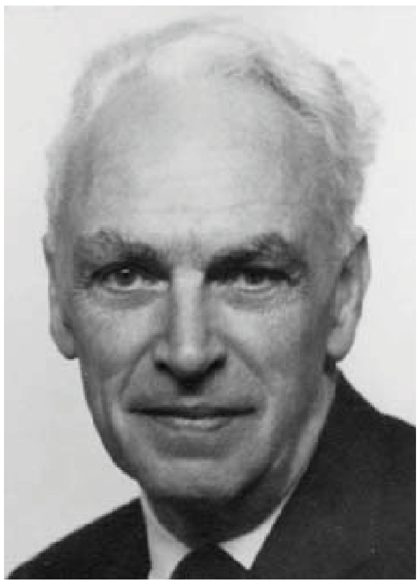
Outline

- History and motivation.
- Theorems and algorithms for quaternion polynomials
- Examples

History

Applying algebraic methods to differential equations: 1930's - 40's.

E. Noether



O. Ore

N. Jacobson

J. Wedderburn

First Question:

differential operators $\overset{?\ correspondence}{\longleftrightarrow}$ rings

Differential Operators: Multiplication

Given differential operators: $\mathcal{D} := \frac{d}{dt}$, $t\mathcal{D}^2 + 1 := t\frac{d^2}{dt^2} + 1$ over $\mathbb{Q}(t)$.

Question: how to write $\mathcal{D} \circ (t\mathcal{D}^2 + 1)$ in the form $\sum_{i=0}^{n} a_i \mathcal{D}^i$?
where $a_i \in \mathbb{Q}(t)$.

? True: $\mathcal{D} \circ (t\mathcal{D}^2 + 1) = t\mathcal{D}^3 + \mathcal{D}$

Differential Operators: Multiplication

Given differential operators: $\mathcal{D} := \frac{d}{dt}$, $t\mathcal{D}^2 + 1 := t\frac{d^2}{dt^2} + 1$ over $\mathbb{Q}(t)$.

Question: how to write $\mathcal{D} \circ (t\mathcal{D}^2 + 1)$ in the form $\sum_{i=0}^{n} a_i \mathcal{D}^i$?
where $a_i \in \mathbb{Q}(t)$.

? True: $\mathcal{D} \circ (t\mathcal{D}^2 + 1) = t\mathcal{D}^3 + \mathcal{D}$

$$(\mathcal{D} \circ (t\mathcal{D}^2 + 1))(t^3) = \mathcal{D} \circ ((t\mathcal{D}^2 + 1)(t^3)) = 12t + 3t^2$$

$$(t\mathcal{D}^3 + \mathcal{D})(t^3) = (t\mathcal{D}^3)(t^3) + \mathcal{D}(t^3) = 6t + 3t^2$$

Need conditions to "swap" the positions of t and $\mathcal{D}$!

Ore Polynomials

- Let σ be an automorphism of a ring R, *i.e.*, σ is 1-1 and

$$\sigma(a+b)=\sigma(a)+\sigma(b) \quad \sigma(ab)=\sigma(a)\sigma(b) \quad \forall a,b\in R.$$

- A σ-derivation δ of R is a mapping $R\to R$ satisfying: $\forall a,b\in R$,

$$\delta(a+b)=\delta(a)+\delta(b), \quad \delta(ab)=\sigma(a)\delta(b)+\delta(a)b$$

- Ore polynomial ring $R[x;\sigma,\delta]$ over R is the set of usual polynomials in x over R, *i.e.*, $\{\sum r_i x^i \mid r_i\in R\}$, with usual "$+$" and

$$xr=\sigma(r)x+\delta(r) \quad \forall r\in R.$$

- Appeared in Noether and Schmeidler (1920). More discussion given in Ore (1933).

Examples: Usual Polynomials

When $\sigma = 1$: identity, and $\delta = 0$: 0-derivative, i.e.,

$$\sigma(r) = r \text{ and } \delta(r) = 0 \quad \text{for any } r \in R$$

we have

$$xr = \sigma(r)x + \delta(r) = rx + 0 = rx, \quad \text{Commutative !}$$

In this case,

Ore polynomial ring $R[x;1,0]$

$||$

Usual polynomial ring $R[x]$

Examples: Differential operators

Consider $\mathbb{Q}(t)[x;\sigma,\delta]$.

Pure differential case: $\sigma(t)=t$: identity mapping;
$\delta(t)=1$: usual derivative.

Commutative Rule: $xt=\sigma(t)x+\delta(t)=tx+1$. Non-commutative!

Set $\mathcal{D}:=\frac{d}{dt}\longleftrightarrow x$ and $t\mathcal{D}^2+1:=t\frac{d^2}{dt^2}+1\longleftrightarrow tx^2+1$.

$$\begin{aligned}\mathcal{D}\circ(t\mathcal{D}^2+1) &= x(tx^2+1)=xtx^2+1=(tx+1)x^2+1=tx^3+x^2+1\\ &= t\mathcal{D}^3+\mathcal{D}^2+1\end{aligned}$$

Examples: Difference Operators

Consider $\mathbb{Q}(t)[x;\sigma,\delta]$.

Pure difference case: $\sigma(t) = t+1$ shift mapping
$\delta(t) = 0$ zero derivative

Commutative rule: $\forall f(t) \in \mathbb{Q}(t)$

$$xf(t) = \sigma(f(t))x + \delta(f(t)) = f(t+1)x$$

In particular, $xt = \sigma(t)x = (t+1)x$.

History: Ore Polynomials

- One of the main research fields in Ring Theory.
  - 1920s – 1950s: Noether, Ore, Jacobson, etc.
  - 1960s – present: Cohn, Goodearl, Lam, etc.
- One of the main research fields in Computer Algebra.
- Applications:
  Differential (difference) equations, Model theory, Coding theory, Control theory, Cryptography.
- Maple packages: Ore, Ore module.

Matrices over Ore Polynomial Rings

Let $R[x;\sigma,\delta]$ be an Ore polynomial ring and $R[x;\sigma,\delta]^{n\times m}$ be the set of all $n \times m$ matrices over $R[x;\sigma,\delta]$.

Questions: compute various generalized inverses in $R[x;\sigma,\delta]^{n\times m}$:

$\{1\}$-inverse, $\{1,2\}$-inverse, Moore-Penrose inverse, etc.

Difficult points:
Ore polynomials are noncommutative algebra, and have a much more complex structure.

Many of the algorithmic breakthroughs in computer algebra over the past three decades do not obviously apply in these domains.

Moore-Penrose Inverses for Quaternion Matrices

In 1843 Sir Rowan Hamilton discovered the algebra $\mathbb{H}$ of real quaternion, which is a four-dimensional non-commutative algebra over $\mathbb{R}$ with canonical basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the conditions:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

that implies

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \text{ and }\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The elements in $\mathbb{H}$ can be written in a unique way:

$$\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

The conjugate of α is defined as $\bar{\alpha} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$, and the norm $|\alpha|$ is $|\alpha| = \sqrt{\alpha\bar{\alpha}}$.

Quaternion Polynomials

The study of quaternion polynomials may go back to Niven in the early 1940's.

A quaternion polynomial $f(x)$ over $\mathbb{H}$ is defined as

$$f(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{H},\ i = 0, \ldots, n,$$

where x commutes element-wise with $\mathbb{H}$.

The conjugate of $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{H}[x]$ is defined as $\overline{f}(x) = \bar{a}_n x^n + \cdots + \bar{a}_0$, and has the following properties:

Properties Let $f, g \in \mathbb{H}[x]$. Then (i) $\overline{fg} = \bar{g}\bar{f}$ (ii) $f\bar{f} = \bar{f}f \in \mathbb{R}[x]$, where $\mathbb{R}$ are reals (iii) If $fg \in \mathbb{R}[x]$, then $fg = gf$.

Definitions

Let $\mathbb{H}[x]^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{H}[x]$.

For $A \in \mathbb{H}[x]^{m\times n}$, the conjugate $\overline{A}$ of A is defined as $\overline{A} = (\overline{A_{ij}})$.

If $A = P + Q\mathbf{j}$ with $P, Q \in \mathbb{C}[x]^{m\times n}$, then $\chi_A = \begin{pmatrix} P & Q \\ -\overline{Q} & \overline{P} \end{pmatrix} \in \mathbb{C}[x]^{2m\times 2n}$ denotes the complex adjoint of A.

Moreover, $A^T, A^* \in \mathbb{H}[x]^{n\times m}$ denote the transpose and the conjugate transpose of A, respectively.

Definitions

$A^\dagger \in \mathbb{H}[x]^{n\times m}$ is called a Moore-Penrose inverse of $A \in \mathbb{H}[x]^{m\times n}$ if it is a solution of the following system of equations:

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA.$$

Note that we require that $A^\dagger$ must be in $\mathbb{H}[x]^{n\times m}$.

$A \in \mathbb{H}^{m\times m}$ is unitary if $AA^* = A^*A = I_m$.

Properties

Let $A \in \mathbb{H}[x]^{m\times n}$ and $B \in \mathbb{H}[x]^{n\times l}$. Then

(i) $(AB)^* = B^*A^*$ and $AA^* = (AA^*)^*$.

(ii) If A has a Moore-Penrose inverse $A^\dagger$, then $(A^*)^\dagger = \left(A^\dagger\right)^*$, $A^\dagger \left(A^\dagger\right)^* A^* = A^\dagger = A^* \left(A^\dagger\right)^* A^\dagger$ and $A^\dagger AA^* = A^* = A^*AA^\dagger$.

(iii) If A has a Moore-Penrose inverse $A^\dagger$, then $A^\dagger$ is unique.

(iv) Let A have the Moore-Penrose inverse $A^\dagger$. If $U \in \mathbb{H}^{m\times m}$ is a unitary matrix, then $(UA)^\dagger = A^\dagger U^*$.

Properties

Lemma If $E \in \mathbb{H}[x]^{m\times m}$ is a symmetric projection, that is, $E = E^2 = E^*$, then $E \in \mathbb{H}^{m\times m}$.

Lemma $A \in \mathbb{H}^{m\times m}$ is hermitian, that is, $A = A^*$, if and only if there exists a unitary matrix $U \in \mathbb{H}^{m\times m}$ such that

$$U^*AU = \text{diag}(d_1,\ldots,d_m),$$

where d_i are the eigenvalues of A.

Theorem

Let $A \in \mathbb{H}[x]^{m\times n}$. Then A has the Moore-Penrose inverse $A^\dagger$ if and only if

$$A = U \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$$

with $U \in \mathbb{H}^{m\times m}$ unitary and $A_1A_1^* + A_2A_2^*$ a unit in $\mathbb{H}[x]^{r\times r}$ with $r \leq \min\{m, n\}$. Moreover,

$$A^\dagger = \begin{pmatrix} A_1^*\left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \\ A_2^*\left(A_1A_1^* + A_2A_2^*\right)^{-1} & 0 \end{pmatrix} U^*.$$

Leverrier-Faddeev algorithm

Given $A \in \mathbb{H}[x]^{m\times m}$. An element $\lambda \in \mathbb{H}$ is called an eigenvalue of A if there exists a vector $X \in \mathbb{H}^{m\times 1}[x]$ such that $AX = X\lambda$.

Lemma Let $A \in \mathbb{H}[x]^{m\times n}$. Then eigenvalues of AA^* are real.

Let $A \in \mathbb{H}[x]^{m\times n}$ and set $B = AA^*$. Then $f_B(\lambda) = \det(\lambda I_{2m} - \chi_B)$ is called the characteristic polynomial of A.

Theorem Let $A \in \mathbb{H}[x]^{m\times n}$ and $B = AA^*$. Then $f_B(\lambda) = g(\lambda)^2$ where $g(\lambda) \in (\mathbb{R}[x])[\lambda]$.

Characteristic polynomials

Let $A \in \mathbb{H}[x]^{m\times n}$, $B = AA^*$ and $f_B(\lambda) = g(\lambda)^2$. Then $g(B) = 0$. We will call $g(\lambda)$ the generalized characteristic polynomial of A.

Lemma Let $A \in \mathbb{H}[x]^{m\times n}$ have the Moore-Penrose inverse $A^\dagger$. Set $B = AA^*$. Then

(i) $B^\dagger = (A^*)^\dagger A^\dagger$ and $B^\dagger B = AA^\dagger$.

(ii) $B^\dagger B = BB^\dagger$ and $(B^\dagger B)^2 = B^\dagger B$.

(iii) $(B^\dagger)^k = (B^k)^\dagger$ and $(B^{n-k})^\dagger B^{n-k} = B^\dagger B$, for any $k \in \mathbb{N}$.

Formula

Theorem Let $A \in \mathbb{H}[x]^{m\times n}$ have the Moore-Penrose inverse $A^\dagger$ and $B = AA^*$. Suppose the generalized characteristic polynomial of A is

$$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m,$$

where $a_i \in \mathbb{R}[x]$.
If k is the largest integer such that $a_k \neq 0$, then the generalized inverse of A is given by

$$A^\dagger = -\frac{1}{a_k}A^*\left[B^{k-1} + a_1B^{k-2} + \cdots + a_{k-1}I\right].$$

If $a_i = 0$ for all $1 \leq i \leq m$, then $A^\dagger = 0$.

Fadeev-Leverrier's method

Lemma Let $A \in \mathbb{H}[x]^{m\times n}$ have the Moore-Penrose inverse $A^{\dagger}$ and set $B = AA^{*}$. Then for $1 \leq k \leq m$,

$$\mathrm{tr}\left[\left(B^{k} + a_1 B^{k-1} + \cdots + a_{k-1} B\right)\right] = -k a_k,$$

where the a_i arise from the generalized characteristic polynomial of A:

$$g(\lambda) = \lambda^{m} + a_1 \lambda^{m-1} + \cdots + a_k \lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m.$$

Laverrier-Faddeev algorithm

Let $A \in \mathbb{H}[x]^{m\times n}$ have the generalized inverse $A^{\dagger}$ and $B = AA^*$. Suppose the generalized characteristic polynomial of A is

$$g(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_k\lambda^{m-k} + \cdots + a_{m-1}\lambda + a_m,$$

where $a_i \in \mathbb{R}[x]$. Define $a_0 = 1$. If p is the largest integer such that $a_p \neq 0$ and we construct the sequence $A_0, \cdots, A_p$ as follows:

$$\begin{array}{llllll}
A_0 & = 0 & -1 & = q_0 & B_0 & = I \\
A_1 & = AA^*B_0 & \frac{\mathrm{tr}A_1}{1} & = q_1 & B_1 & = A_1 - q_1 I \\
 & \vdots & & \vdots & & \vdots \\
A_{p-1} & = AA^*B_{p-2} & \frac{\mathrm{tr}A_{p-1}}{p-1} & = q_{p-1} & B_{p-1} & = A_{p-1} - q_{p-1} I \\
A_p & = AA^*B_{p-1} & \frac{\mathrm{tr}A_p}{p} & = q_p & B_p & = A_p - q_p I
\end{array}$$

then $q_i(x) = -a_i(x), i = 0, \cdots, p$.

Laverrier-Faddeev algorithm

Leverrier-Faddav algorithm for quaternion polynomial matrices

Input: $A \in \mathbb{H}[x]^{m\times n}$
Output: The Moore-Penrose inverse $A^\dagger$ of A in $\mathbb{H}[x]^{n\times m}$ if exists.

1. $B_0 \leftarrow I_m,\ a_0 \leftarrow 1$
2. for $i = 1,\ldots,m$ do
   $A_i \leftarrow AA^*B_{i-1},\ a_i \leftarrow -\frac{\mathrm{tr}_{A_i}}{i},\ B_i \leftarrow A_i + a_i I_m$
3. Find the maximal index p such that $a_p \neq 0$.
4. Return $A^\dagger = \begin{cases} -\frac{1}{a_p}A^*B_{p-1}, & p > 0, \\ 0, & p = 0. \end{cases}$

Finding Moore-Penrose inverses by interpolation

Let f, g and $h \in \mathbb{H}[x]$, $f = gh$ and $r \in \mathbb{H}$.

If $h(r) = 0$, then $f(r) = 0$.

Otherwise, set $\beta = h(r) \neq 0$. Then the evaluation of $f(x)$ at $x = r$ is defined as

$$f(r) = g\left(\beta r \beta^{-1}\right) h(r).$$

In particular, if r is a root of f but not of h, then $\beta r \beta^{-1}$ is a root of g.

In 1965, Gordon proved that if $f \in \mathbb{H}[x]$ is of degree n, then the roots of f lie in at most n conjugacy classes of $\mathbb{H}$.

Interpolation

Theorem Let $c_1, \cdots, c_n$ be n pairwise non-conjugate elements of $\mathbb{H}$. Then there is a unique monic polynomial $n \in \mathbb{H}[x]$ of degree n such that $g_n(c_1) = \cdots = g_n(c_n) = 0$.
Moreover, $c_1, \ldots, c_n$ are the only roots (up to conjugacy classes) of g_n in $\mathbb{H}$.

Theorem Let $c_1, \cdots, c_{n+1} \in \mathbb{H}$ be pairwise non-conjugate and let $d_1, \cdots, d_{n+1} \in \mathbb{H}$. Then there exists a unique lowest degree polynomial $f \in \mathbb{H}[x]$, of degree $p \leq n$, such that $f(c_i) = d_i$ for all $1 \leq i \leq n+1$.

Degree Bonds

For a given $A \in \mathbb{H}[x]^{m\times n}$, the degree

$$\deg A = \max\{\deg(A_{ij}) \mid 1 \le i \le m, 1 \le j \le n\}.$$

Lemma Let $A \in \mathbb{H}[x]^{m\times n}$ have the Moore-Penrose inverse $A^{\dagger}$. Then

$$\deg A^{\dagger} \le (2m-1)\deg A.$$

Proposition Let $c_1, \cdots, c_{k+1} \in \mathbb{H}$ be pairwise non-conjugate and let $A_1, \cdots, A_{k+1} \in \mathbb{H}^{n\times m}$. Then there is a unique lowest degree matrix $A \in \mathbb{H}[x]^{n\times m}$ of degree $p \le k$, such that $A(c_i) = A_i$ for all $1 \le i \le k+1$.

Interpolation method

Let $A \in \mathbb{H}[x]^{m\times n}$ have the Moore-Penrose inverse $A^{\dagger}$, and set $B = AA^*$. Let p be the largest integer such that $a_p \neq 0$. We construct the sequence $A_0, \cdots, A_p$ as follows:

$$\begin{array}{llllll}
A_0 & = 0 & -1 & = q_0 & B_0 & = I \\
 & \vdots & & \vdots & & \vdots \\
A_{p-1} & = AA^*B_{p-2} & \frac{\mathrm{tr}A_{p-1}}{p-1} & = q_{p-1} & B_{p-1} & = A_{p-1} - q_{p-1}I \\
A_p & = AA^*B_{p-1} & \frac{\mathrm{tr}A_p}{p} & = q_p & B_p & = A_p - q_pI.
\end{array}$$

Algorithm

Theorem In the above setting, let $k = (2m-1)\deg A$ and $c_1, \cdots,$ $c_{k+1} \in \mathbb{R}$ be $k+1$ distinct real numbers such that $q_p(c_{s'}) \neq 0$ for any $1 \leq s' \leq k+1$. Let $S = \{1, \cdots, k+1\} \setminus \{s'\}$. Then

$$A^{\dagger} = \sum_{s'=1}^{k+1} A(c_{s'})^{\dagger} g_S$$

where

$$A(c_{s'})^{\dagger} = \frac{1}{q_p(c_{s'})} A(c_{s'})^{*} \left[B(c_{s'})^{p-1} - q_1(c_{s'}) B(c_{s'})^{p-2} - \cdots - q_{p-1}(c_{s'}) I \right]$$

and

$$g_S(c_\alpha) = \begin{cases} 0 & \alpha \in S, \\ 1 & \alpha = s'. \end{cases}$$

Example

$$A = \begin{pmatrix} 14x+14+76\mathbf{i}+70\mathbf{j}+56\mathbf{k} & 56-28\mathbf{i}-70\mathbf{j}+70\mathbf{k} & 28\mathbf{j}-56\mathbf{k} & 14x-56-8\mathbf{i}-14\mathbf{j}-56\mathbf{k} \\ -2x-2-43\mathbf{i}-10\mathbf{j}-8\mathbf{k} & -8+4\mathbf{i}+10\mathbf{j}-10\mathbf{k} & -4\mathbf{j}+8\mathbf{k} & -2x+8-31\mathbf{i}+2\mathbf{j}+8\mathbf{k} \\ -3x-3+3\mathbf{i}-15\mathbf{j}-12\mathbf{k} & -12+6\mathbf{i}+15\mathbf{j}-15\mathbf{k} & -6\mathbf{j}+12\mathbf{k} & -3x+12+21\mathbf{i}+3\mathbf{j}+12\mathbf{k} \\ -4x-4+4\mathbf{i}-20\mathbf{j}-16\mathbf{k} & -16+8\mathbf{i}+20\mathbf{j}-20\mathbf{k} & -8\mathbf{j}+16\mathbf{k} & -4x+16+28\mathbf{i}+4\mathbf{j}+16\mathbf{k} \end{pmatrix} \in \mathbb{H}^{4\times 3}[x]$$

The upper bound of the degree of $A^\dagger$ is less than $(2m-1)\deg A = (2\times 4-1)\cdot 1 = 7$. Choose $c_1 = 0$ and $c_2 = 1$.

$$A(c_1) = \begin{pmatrix} 14+76\mathbf{i}+70\mathbf{j}+56\mathbf{k} & 56-28\mathbf{i}-70\mathbf{j}+70\mathbf{k} & 28\mathbf{j}-56\mathbf{k} & -56-8\mathbf{i}-14\mathbf{j}-56\mathbf{k} \\ -2-43\mathbf{i}-10\mathbf{j}-8\mathbf{k} & -8+4\mathbf{i}+10\mathbf{j}-10\mathbf{k} & -4\mathbf{j}+8\mathbf{k} & 8-31\mathbf{i}+2\mathbf{j}+8\mathbf{k} \\ -3+3\mathbf{i}-15\mathbf{j}-12\mathbf{k} & -12+6\mathbf{i}+15\mathbf{j}-15\mathbf{k} & -6\mathbf{j}+12\mathbf{k} & 12+21\mathbf{i}+3\mathbf{j}+12\mathbf{k} \\ -4+4\mathbf{i}-20\mathbf{j}-16\mathbf{k} & -16+8\mathbf{i}+20\mathbf{j}-20\mathbf{k} & -8\mathbf{j}+16\mathbf{k} & 16+28\mathbf{i}+4\mathbf{j}+16\mathbf{k} \end{pmatrix}$$

and

$$A(c_2) = \begin{pmatrix} 28+76\mathbf{i}+70\mathbf{j}+56\mathbf{k} & 56-28\mathbf{i}-70\mathbf{j}+70\mathbf{k} & 28\mathbf{j}-56\mathbf{k} & -42-8\mathbf{i}-14\mathbf{j}-56\mathbf{k} \\ -4-43\mathbf{i}-10\mathbf{j}-8\mathbf{k} & -8+4\mathbf{i}+10\mathbf{j}-10\mathbf{k} & -4\mathbf{j}+8\mathbf{k} & 6-31\mathbf{i}+2\mathbf{j}+8\mathbf{k} \\ -6+3\mathbf{i}-15\mathbf{j}-12\mathbf{k} & -12+6\mathbf{i}+15\mathbf{j}-15\mathbf{k} & -6\mathbf{j}+12\mathbf{k} & 9+21\mathbf{i}+3\mathbf{j}+12\mathbf{k} \\ -8+4\mathbf{i}-20\mathbf{j}-16\mathbf{k} & -16+8\mathbf{i}+20\mathbf{j}-20\mathbf{k} & -8\mathbf{j}+16\mathbf{k} & 12+28\mathbf{i}+4\mathbf{j}+16\mathbf{k} \end{pmatrix}.$$

we calculate and obtain:

$$A\left(c_1\right)^{\dagger} = A\left(0\right)^{\dagger} = \frac{1}{230175} \times$$

$$\begin{pmatrix} 140 - 560\mathbf{i} - 228\mathbf{j} - 342\mathbf{k} & 355 + 1730\mathbf{i} - 96\mathbf{j} + 81\mathbf{k} & -255 - 870\mathbf{i} + 1 \\ 276 + 88\mathbf{i} + 426\mathbf{j} - 382\mathbf{k} & 282 + 416\mathbf{i} - 93\mathbf{j} - 149\mathbf{k} & -252 - 276\mathbf{i} - 7 \\ 32 + 16\mathbf{i} - 176\mathbf{j} + 292\mathbf{k} & -176 - 88\mathbf{i} + 68\mathbf{j} + 194\mathbf{k} & 96 + 48\mathbf{i} + 12\mathbf{j} \\ -140 - 122\mathbf{i} + 228\mathbf{j} + 342\mathbf{k} & -355 + 2021\mathbf{i} + 96\mathbf{j} - 81\mathbf{k} & 255 - 1176\mathbf{i} - 1 \end{pmatrix}$$

and

$$A\left(c_2\right)^{\dagger} = A\left(1\right)^{\dagger} = \frac{1}{230175} \times$$

$$\begin{pmatrix} 152 - 550\mathbf{i} - 244\mathbf{j} - 330\mathbf{k} & 289 + 1675\mathbf{i} - 8\mathbf{j} + 15\mathbf{k} & -219 - 840\mathbf{i} + 7 \\ 268 + 104\mathbf{i} + 406\mathbf{j} - 402\mathbf{k} & 326 + 328\mathbf{i} + 17\mathbf{j} - 39\mathbf{k} & -276 - 228\mathbf{i} - 13 \\ 32 + 16\mathbf{i} - 160\mathbf{j} + 300\mathbf{k} & -176 - 88\mathbf{i} - 20\mathbf{j} + 150\mathbf{k} & 96 + 48\mathbf{i} + 60\mathbf{j} \\ -152 - 132\mathbf{i} + 244\mathbf{j} + 330\mathbf{k} & -289 + 2076\mathbf{i} + 8\mathbf{j} - 15\mathbf{k} & 219 - 1206\mathbf{i} - 7 \end{pmatrix}$$

$$A^{\dagger} = \sum_{s'=1}^{2} A\left(c_{s'}\right)^{\dagger} g_S = A\left(0\right)^{\dagger}\left(1-x\right) + A\left(1\right)^{\dagger} x$$

$$= \frac{1}{230175} \times$$

$$\left(\begin{array}{ll}
(12+10\mathbf{i}-16\mathbf{j}+12\mathbf{k})x+140-560\mathbf{i}-228\mathbf{j}-342\mathbf{k} & (-66-55\mathbf{i}+8 \\
(-8+16\mathbf{i}-20\mathbf{j}-20\mathbf{k})x+276+88\mathbf{i}+426\mathbf{j}-382\mathbf{k} & (44-88\mathbf{i}+11 \\
(16\mathbf{j}+8\mathbf{k})x+32+16\mathbf{i}-176\mathbf{j}+292\mathbf{k} & (-88\mathbf{j}- \\
(-12-10\mathbf{i}+16\mathbf{j}-12\mathbf{k})x-140-122\mathbf{i}+228\mathbf{j}-342\mathbf{k} & (66+55\mathbf{i}-8
\end{array} \right.$$

$$\begin{array}{ll}
(36+30\mathbf{i}-48\mathbf{j}+36\mathbf{k})x-255-870\mathbf{i}+126\mathbf{j}+54\mathbf{k} & (48+40\mathbf{i}-64\mathbf{j} \\
(-24+48\mathbf{i}-60\mathbf{j}-60\mathbf{k})x-252-276\mathbf{i}-72\mathbf{j}+204\mathbf{k} & (-32+64\mathbf{i}-80 \\
(48\mathbf{j}+24\mathbf{k})x+96+48\mathbf{i}+12\mathbf{j}-204\mathbf{k} & (64\mathbf{j}+32 \\
(-36-30\mathbf{i}+48\mathbf{j}-36\mathbf{k})x+255-1176\mathbf{i}-126\mathbf{j}-54\mathbf{k} & (-48-40\mathbf{i}+64\mathbf{j}
\end{array}$$

Example

Let $A = \begin{pmatrix} 1 & \mathbf{i}+2\mathbf{k} & 3 \\ \mathbf{i} & 6+\mathbf{j} & 7 \end{pmatrix}_{2\times 3}$.

Then its Moore-Penrose inverse can be found by using our Maple package as follows:

$$A^{\dagger} = \begin{pmatrix} \frac{47}{347}+\frac{21}{694}\mathbf{i}+\frac{11}{694}\mathbf{j} & -\frac{21}{694}-\frac{11}{347}\mathbf{i}-\frac{11}{694}\mathbf{k} \\ -\frac{63}{347}-\frac{28}{347}\mathbf{i}+\frac{21}{694}\mathbf{j}-\frac{101}{694}\mathbf{k} & \frac{61}{694}+\frac{21}{694}\mathbf{i}-\frac{6}{347}\mathbf{j}+\frac{21}{347}\mathbf{k} \\ \frac{57}{347}+\frac{49}{694}\mathbf{i}+\frac{77}{694}\mathbf{k} & \frac{21}{347}-\frac{21}{694}\mathbf{i}-\frac{33}{694}\mathbf{k} \end{pmatrix}_{3\times 2}.$$