On solving parametric polynomial systems (extended abstract)

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Many authors have contributed to the study of parametric polynomial systems, and there is a large collection of references, such as [4, 17, 9, 18, 13, 14, 22, 11, 2, 12], to name a few. Various notions have been formulated for investigating the properties of parametric polynomial systems from different aspects. Border polynomial [21, 22, 20, 1], discriminant variety [11], discriminant ideal [18], discriminant set [2] are some of those notions. For (parametric) semi-algebraic systems, methods based on *cylindrical algebraic decomposotion* (CAD) and its variants [5, 6, 7] are applicable. However, these methods may compute much more than what is needed for the purpose of solving.

One central question in the study of parametric polynomial systems is the dependence of the solutions on the parameter values. There are different ways to express the fact that the zeros of a parametric system depends *continuously* on the parameters in a neighborhood of a given parameter value. The notions of a border polynomial and a discriminant variety aims at capturing the parameter values at which this dependence is not continuous.

The main objective of the present work is to study the relations between the notions of a border polynomial and a discriminant variety. A second intention is to gather in a single report key results on these objects, including results previously published in [20, 1]. We stress the fact that most of our results assume that the input parametric system is triangular, since triangular decomposition methods [19, 10, 8, 16, 3] can help reducing the study of general parametric systems to the triangular case.

In Section 2, we revisit the notions of a border polynomial and a discriminant variety in a unified framework. In the context of triangular parametric systems, we show that the two notions essentially coincide, see Theorem 1.

In Section 3, which is dedicated to parametric algebraic systems, we compare the minimal discriminant variety of a regular chain and that of its saturated ideal. This leads us to answer the following question: among all regular chains that have the same saturated ideal as a given one, what is the best choice to make the border polynomial set minimal. Most of the results in this part were presented in [20].

However, the result of Proposition 4 is new and the proof is included. We shall supply the proofs for all the results in the long version of the present paper.

In Section 4, we consider parametric semi-algebraic systems and study the notion of an effective boundary introduced in [1]. The purpose is to obtain properties which can improve the efficiency of triangular decomposition algorithms. The results of this part are essentially new and their proofs will appear in an extended version of [1].

1. Preliminaries

In this paper, a parametric polynomial system S is a system of equations, inequations and/or inequalities given by polynomials in $\mathbb{Q}[U, X]$ where $U = u_1, u_2, \ldots, u_d$ are the parameters and $X = x_1, x_2, \ldots, x_s$ are the unknowns. All variables (parameters and unknowns) hold values from a fixed field $\mathbb{K}$, which is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. In the former case, we say that the system is algebraic[1] and in the latter case, we say that the system is semi-algebraic.

We denote by $Z(S)$ the solution set of S, which is a subset of $\mathbb{K}^{d+s}$. The canonical projection Π_{U} on the parameter space is defined as follows:

$$\Pi_{\mathrm{U}} : Z(S) \subset \mathbb{K}^{s+d} \mapsto \mathbb{K}^d$$
$$\Pi_{\mathrm{U}}(x_1, \ldots, x_s, u_1, \ldots, u_d) = (u_1, \ldots, u_d)$$

Let us denote by E (resp. I) the set of the polynomials of S defining respectively its equations (resp. inequations and strict inequalities). The ideal $\langle E \rangle : (\prod_{h \in I} h)^\infty$ is called the *ideal associated with* S. We say that S is *well-determinate* if the set U is an $\subseteq$-maximal algebraic independent variable set modulo the ideal associated with S. Note that the notion of " well-determinate" is more general than the notion of "well-behaved" used in [11], in the sense that it is less restrictive for E. Indeed, the polynomial set E is not required to have exactly s elements, nor to generate a radical ideal in $\mathbb{Q}(U)[X]$.

Example 1. *Consider a semi-algebraic system*

$$S = \{x(x^2 + ay + b) = x(y^2 + bx + a) = 0, x > 0\}$$

with parameters a, b. The ideal generated by the polynomials defining the equations of S is

$$\langle x \rangle \cap \langle x^2 + ay + b, y^2 + bx + a \rangle.$$

The polynomial system $\{x(x^2 + ay + b) = x(y^2 + bx + a) = 0\}$ with parameters a, b is not well-determinate, since $\{a, b\}$ is not a maximal algebraic independent set modulo $\langle x \rangle$. However, the ideal associated to S is $\mathcal{I} := \langle x^2 + ay + b, y^2 + bx + a \rangle$, and $\{a, b\}$ is a maximal algebraic independent variable set modulo $\mathcal{I}$. Therefore, S is a well-determinate parametric semi-algebraic system.

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[1] No inequalities are present in that case.

Throughout this paper, and in order to keep the presentation concise, we assume that S is well-determinate. This simplification assumption can be relaxed and the results discussed here can be adapted to more general systems. We leave this for an extended version of this paper.

We rely on triangular decomposition techniques for studying the parametric system S. In the algebraic case, we decompose $Z(S)$ into zero sets of finitely many *squarefree triangular algebraic systems* (STAS); in the semi-algebraic case, we decompose $Z(S)$ into zero sets of finitely many *squarefree triangular semi-algebraic systems* (STSAS).

We refer to [1] for the standard notions and notations on triangular decomposition, such as: regular chain, main variable (mvar), main degree (mdeg), initial (ini), iterated resultant (ires).

An STAS is a pair $[T, H_{\neq}]$ where T is a regular chain of $\mathbb{Q}[U, X]$ and $H_{\neq}$ is a set of non-constant polynomials of $\mathbb{Q}[U, X]$ such that each of those is regular modulo $\mathrm{sat}(T)$, the saturated ideal of T, which is $\langle T \rangle : \mathrm{ini}(T)^{\infty}$. A point of $\mathbb{K}^{d+s}$ is a zero of $[T, H_{\neq}]$ if it is a zero of T not cancelling the polynomials of $H_{\neq}$.

An STSAS is a triple $[T, H_{\neq}, P_{>}]$ such that $[T, H_{\neq}]$ is an STAS and $P_{>}$ is a set of non-constant polynomials of $\mathbb{Q}[U, X]$ such that each of those is regular modulo $\mathrm{sat}(T)$. A point of $\mathbb{K}^{d+s}$ is a zero of $[T, H_{\neq}, P_{>}]$ if it is a zero of $[T, H_{\neq}]$ making each polynomial of $P_{>}$ strictly positive.

2. Border polynomials, discriminant varieties and effective boundaries

Let $\alpha \in \mathbb{K}^d$. As mentioned in the introduction, there are different ways to express the fact that the zeros of the parametric system S depends continuously on the parameters in a neighborhood of α in $\mathbb{K}^d$. In this paper, we focus on two of them. We say that S is *Z-continuous* at α if there exists a neighborhood O_α of α such that for any two parameter values $\alpha_1, \alpha_2 \in O_\alpha$, we have $\#\left(Z(S(\alpha_1)) = \#\left(Z(S(\alpha_2)\right)\right)$. We say that S is *Π_{U}-continuous* at α if there exists a neighborhood O_α of α such that there exists a finite partition $\{C_1, \ldots, C_k\}$ of $\Pi_{\mathrm{U}}^{-1}(O_\alpha) \cap Z(S)$ such that the restriction $\Pi_{\mathrm{U}}|_{C_j} : C_j \xrightarrow{\Pi_{\mathrm{U}}} O_\alpha$ is a diffeomorphism, for each $j \in \{1, \ldots, k\}$.

Example 2. *Consider the semi-algebra algebraic system*

$$S := \{x^2 + ay^2 - x = ax^2 + y^2 - y = 0, x \neq y\}$$

with parameter a. When the parameter takes value in the open interval $(-1, \frac{1}{3})$, there are two solutions, which are given by:

$$x = \frac{a + 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)}, y = \frac{-a - 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)},$$

and

$$x = \frac{-a - 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)}, y = \frac{a + 1 + \sqrt{-3a^2 - 2a + 1}}{2(a^2 - 1)}.$$

Therefore, the system S is Z-continuous as well as Π_{U}-continuous at any point in $(-1, \frac{1}{3})$.

It is obvious that Π_{U}-continuity implies Z-continuity. Moreover, these two kinds of continuity are equivalent in many cases, e.g. for parametric STASes, as we shall see in Section 3. Another notion of continuity (or discontinuity) is *non-properness*. The canonical projection Π_{U} is said to be *not proper* at the point α, if for any compact set $S \subseteq \mathbb{K}^d$ containing α, the set $(\Pi_{\mathrm{U}}^{-1}(S)$ is not compact. We denote by $\mathcal{O}_\infty$ the set of all points where Π_{U} is not proper.

The notion of a *border polynomial* is based on the Z-continuity and was proposed in [21] for computing the real root classification of a parametric semi-algebraic system. We reformulate the definition here, for both parametric algebraic systems and parametric semi-algebraic systems.

Definition 1 (Border polynomial). *A polynomial b in $\mathbb{Q}[U]$ is called a* border polynomial *of the parametric polynomial system S if the zero set $V(b)$ of b in $\mathbb{K}^d$ contains all the points at which S is not Z-continuous.*

Example 3. *Consider a polynomial system $S := \{x^2 + bx - 1\}$ with parameter b. Regarding S as an algebraic system, it is easy to check that the sytem has two solutions for $b^2 + 4 \neq 0$ and has only one solution for $b^2 + 4 = 0$; therefore, $b^2 + 4$ is a border polynomial. In fact, it is a minimal border polynomial of S in the sense that it divides any other border polynomials of S.*

Viewing S as a semi-algebraic system, this system always has two real solutions; therefore, 1 is the minimal border polynomial. Indeed, recall that in the semi-algebraic case, the field $\mathbb{K}$ of Definition 1 is $\mathbb{R}$.

The notion of a *discriminant variety* is based on the Π_{U}-continuity and was proposed in [11] for general parametric algebraic systems. We reformulate the definition here, for both parametric algebraic systems and parametric semi-algebraic systems.

Definition 2 (Discriminant variety). *An algebraic set $\mathcal{W} \subsetneq \mathbb{K}^d$ is a* discriminant variety *of the parametric polynomial system S if $\mathcal{W}$ contains all the points at which S is not Π_{U}-continuous.*

Example 4 (Example 2 Cont.). *Consider again the semi-algebraic system*

$$S := \{x^2 + ay^2 - x = ax^2 + y^2 - y = 0, x \neq y\}$$

with parameter a. It is not hard to show that when either $a < -1$ or $a > \frac{1}{3}$ holds the system has no real solutions. So $\{-1, \frac{1}{3}\}$ is a (indeed, the minimal) discriminant variety of S and $(a+1)(a - \frac{1}{3})$ is a (again, the minimal) border polynomial of S. Now, viewing S as a parametric algebraic system, the minimal discriminant variety is $\{-1, \frac{1}{3}, 1\}$. Note that $(a^2 - 1)(a - \frac{1}{3})$ is the minimal border polynomial of S.

Remark 1. *The following facts can be easily deduced from the above definitions.*

(i) One can form a discriminant variety of S by taking the intersection of all discriminant varieties, which is the minimal discriminant variety *of S.*
(ii) If the hypersurface of a polynomial contains the minimal discriminant variety, then this polynomial is a border polynomial.
(iii) In general, there is no "minimal border polynomial". This will typically happen when the minimal discriminant variety of S is not the zero set of a single polynomial. However, we call a border polynomial quasi-minimal *if none of its proper factors is a border polynomial.*
(iv) In the algebraic case, the set of points where the Π_{U}-continuity of S does not hold is just the minimal discriminant variety of S; in the semi-algebraic case, the points at which the Π_{U}-continuity of S does not hold form a semi-algebraic set, which is not algebraic in general.

For each type of continuity (namely the Z-continuity and the Π_{U}-continuity), there are essentially two steps in solving the parametric system S:
(1) describe the parameter values where the continuity does not hold,
(2) describe the (groups of) regions where the continuity is maintained.
Step (1) is achieved by computing a border polynomial or a discriminant variety, depending on the continuity notion. It is not hard to show that the computation of a border polynomial or a discriminant variety of S in both algebraic and semi-algebraic cases can be reduced to the computation of a discriminant variety in the algebraic case. For simplicity with Step (2), let us assume that a border polynomial of S is a polynomial whose hypersurface is also a discriminant variety of S. In the algebraic case, the complement of an algebraic set in $\mathbb{C}^d$ has only one connected component, thus Step (2) is rather simple. However, in the semi-algebraic case, there are usually more than one connected components in the complement of an algebraic set in $\mathbb{R}^d$ and the description of those connected components is more challenging.

The following notion of an *effective boundary*, was originally defined in [1], is dedicated to help the description of complement of the hypersurface of a border polynomial in $\mathbb{R}^d$, see [1] for its usage.

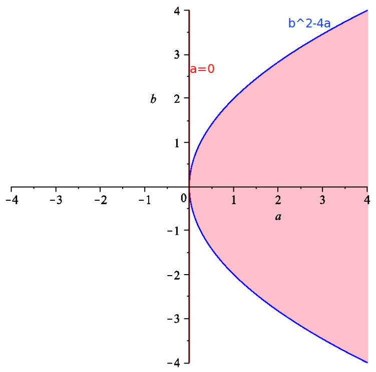
Definition 3 (Effective boundary). *Let $\mathbf{h}$ be a hypersurface defined by an irreducible polynomial in $\mathbb{Q}[U]$. We call $\mathbf{h}$ an* irreducible effective boundary *of the parametric system S if there exists an open ball $O \subset \mathbb{R}^d$ satisfying the following properties:*
(i) $O \setminus \mathbf{h}$ consists of two connected components O_1, O_2,
(ii) for $i = 1, 2$ and any two points $\alpha_1, \alpha_2 \in O_i$ we have $\# Z(S(\alpha_1)) = \# Z(S(\alpha_2))$,
(iii) for any $\beta_1 \in O_1, \beta_2 \in O_2$ we have $\# Z(S(\beta_1)) \neq \# Z(S(\beta_2))$.
The union of all irreducible effective boundaries of S is called the effective boundary *of S, denoted by $\mathbf{EB}(S)$.*

Let $\mathcal{W}$ be either a discriminant variety of S or the hypersurface of a border polynomial of S. It is easy to show that $\mathbf{EB}(S) \subseteq \mathcal{W}$ holds. Therefore, $\mathbf{EB}(S)$ itself is a hypersurface. We define $\mathbf{ebf}(S)$ to be the set of the irreducible factors of the monic polynomial defining $\mathbf{EB}(S)$.

The concepts of border polynomial, discriminant variety and effective boundary are illustrated in the following example.

Example 5. *Consider the semi-algebraic system* $S = \{ax^2 + bx + 1 = 0\}$ *with parameters* a, b. *Its minimal border polynomial factors to* $a(b^2 - 4a)$. *One can verify from Figure* 1 *that* $Z_{\mathbb{R}}(b^2 - 4a = 0)$ *is an irreducible effective boundary of* S, *but* $Z_{\mathbb{R}}(a = 0)$ *is not. Indeed, all* (a, b)*-values in the blank area will specialize* S *to have* 2 *real solutions while all* (a, b)*-values in the filled region will specialize* S *to have no real solutions.*

FIGURE 1. Effective and non-effective boundary



3. Parametric algebraic systems

In this section, we study the minimal discriminant variety of an STAS, regarded as a parametric system in the free variables of its regular chain. We show that for this type of parametric systems the notions of Z-continuity and Π_U-continuity coincide. Then, we compare the minimal discriminant variety of a regular chain T and that of its saturated ideal, both regarded as a parametric system in the free variables of T. Finally, we show that among all regular chains having the same saturated ideal as T, the canonical regular chain associated with T has a $\subseteq$-minimal border polynomial set.

3.1. The minimal discriminant variety of a parametric STAS

In this subsection, we focus on the characterization of the minimal discriminant variety of an STAS $R = [T, H]$, as defined in Section 1. We view an STAS as a parametric algebraic system with the free variables of T as parameters. The following notations are related to the triangular structure of an STAS.

Notation 1 ([20, 1])**.** *We denote by* $B_{sep}(T)$*,* $B_{ini}(T)$*,* $B_{ie}([T,H])$ *respectively the set of the irreducible factors of*

$$\prod_{t\in T}\mathrm{ires}(\mathrm{discrim}(t,\mathrm{mvar}(t)),T),\ \prod_{t\in T}\mathrm{ires}(\mathrm{ini}(t),T),\ and\ \prod_{f\in H}\mathrm{ires}(f,T).$$

The set $B_{sep}(T)\cup B_{ini}(T)\cup B_{ie}([T,H])$ *is called the* border polynomial set *of* R*, denoted by* $\mathbf{BPS}(R)$*.*

Lemma 1 and Theorem 1 imply that the notions of Z-continuity and Π_{U}-continuity coincide for STASes. In particular, Theorem 1 shows that the minimal discriminant variety of R can be characterized by $BPS(R)$. Moreover, Lemma 1 justifies the terminology introduced above. That is, with those notations, the polynomial $b=\prod_{f\in\mathbf{BPS}(R)} f$ is indeed a border polynomial of R.

Lemma 1. *Let* $b=\prod_{f\in\mathbf{BPS}(R)} f$*; let* $N:=\prod_{f\in T}\mathrm{mdeg}(f)$*. Then for each parameter value* $\alpha\in\mathbb{C}^d$*:*

1. *if* $b(\alpha)\neq 0$*, then* $\#\,Z(R(\alpha))=N$ *holds;*
2. *if* $b(\alpha)=0$*, then* $\#\,Z(R(\alpha))$ *is either infinite or less than* N*.*

Theorem 1 ([20])**.** *Let* $b=\prod_{f\in\mathbf{BPS}(R)} f$*. Then* $V(b)$ *in* $\mathbb{C}^d$ *is the minimal discriminant variety of* R*.*

3.2. The minimal discriminant variety of a saturated ideal

As before, let us denote by $U=u_1,u_2,\ldots,u_d$ and $X=x_1,x_2,\ldots,x_s$ the set of free and algebraic variables of our regular chain T. Since $\mathrm{sat}(T)$ is a strongly equidimensional ideal[2] it is natural to view it as a parametric system with U as parameters and compare its minimal discriminant variety with that of T, also regarded as a parametric system in U.

In this subsection, we perform this comparison. We shall also show, with Theorem 2, that among all regular chains having $\mathrm{sat}(T)$ as saturated ideal, the discriminant variety of the *canonical regular chain associated with* T is the smallest under inclusion. We denote by DV_T (resp. $DV_{\mathrm{sat}(T)}$) the minimal discriminant variety of T (resp. $\mathrm{sat}(T)$).

Proposition 1 ([20])**.** *The minimal discriminant variety* $[\mathrm{sat}(T),B_{ini}(T)_{\neq}]$ [3] *equals to* DV_T*. In particular, we have*

$$DV_T=V(\prod_{f\in B_{ini}(T)\cup B_{sep}(T)} f)=DV_{\mathrm{sat}(T)}\cup V(\prod_{f\in B_{ini}(T)} f).$$

The following proposition gives an upper bound on the set theoretic difference $DV_T\setminus DV_{\mathrm{sat}(T)}$.

[2] More precisely, $\mathrm{sat}(T)$ is an equidimensional ideal of dimensional d such that U a maximal algebraic independent variable set modulo each associated prime of $\mathrm{sat}(T)$.

[3] Here, $[\mathrm{sat}(T),B_{ini}(T)_{\neq}]$ denotes the parametric algebraic system with equations defined by any basis of $\mathrm{sat}(T)$ and with inequations defined by $B_{ini}(T)_{\neq}$.

Proposition 2 ([20]). *We have*

$$DV_T \setminus DV_{\mathrm{sat}(T)} \subseteq V(\prod_{f \in B_{ini}(T)} f) \setminus \mathcal{O}_\infty(\mathrm{sat}(T)).$$

The next proposition shows that the difference of $DV_T \setminus DV_{\mathrm{sat}(T)}$ is actually dominated by the difference of the non-properness loci of T and that of $\mathrm{sat}(T)$, respectively denoted by $\mathcal{O}_\infty(T)$ and $\mathcal{O}_\infty(\mathrm{sat}(T))$.

Proposition 3 ([20]). *We have* $\mathcal{O}_\infty(T) = V(\prod_{f \in B_{ini}(T)} f)$.

One can find an algorithm for computing the $\mathcal{O}_\infty$ set of a parametric algebraic system in [11]; an algorithm for computing the non-properness loci of a general polynomial map can be found in [15]. With the next proposition, we show a nice construction of $\mathcal{O}_\infty(\mathrm{sat}(T))$, which can be exploited to design new algorithms to compute the $\mathcal{O}_\infty$ set of a parametric polynomial system.

Recall that T squarefree regular chain T with U and $X = x_1, x_2, \ldots, x_s$ as free variables and algebraic variables respectively and let $I = \mathrm{sat}(T)$.

Lemma 2. *For each $i = 1 \cdots s$, the ideal $I \cap \mathbb{Q}[U, x_i]$ is a principal ideal generated by a polynomial $g_i \in \mathbb{Q}[U, x_i]$ whose content w.r.t. x_i belongs to $\mathbb{Q}$.*

Proof. Let $\{P_j | j = 1, 2, \ldots, e\}$ be the set of the associated primes of I. Then for each j, the set U is $\subseteq$-maximal algebraic independent variable set modulo P_j. For each $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, e$, denote by $Q_{j,i}$ the ideal $P_j \cap \mathbb{Q}[U, x_i]$.

Clearly, the ideal $Q_{j,i}$ is prime and U is a $\subseteq$-maximal algebraic independent variable set modulo $Q_{j,i}$.

Consider two distinct polynomials $f, g \in Q_{j,i}$. Since their resultant lies in $Q_{j,i}$ and has degree zero in x_i, this latter polynomial must be null. Thus the GCD of $h := \gcd(f, g)$ has a positive degree w.r.t. x_i. Since $Q_{j,i}$ is prime, either h or f/h must belong to $Q_{j,i}$. From there, it is routine (proceeding by contradiction) to show that $Q_{j,i}$ is a principal ideal. Moreover, the fact that $Q_{j,i}$ is prime, implies that $Q_{j,i}$ is generated by an irreducible polynomial, say $g_{j,i}$.

Denote by g_i the polynomial $\prod_{j=1}^{e} g_{j,i}$. Note that $I \cap \mathbb{Q}[U, x_i] = \bigcap_{j=1}^{e} Q_{j,i}$ holds. Therefore, $I \cap \mathbb{Q}[U, x_i] = \langle g_i \rangle$. And it is obvious that g_i is content free. □

Proposition 4. *For each $i = 1, \ldots, s$, let g_i be a polynomial generating the principal ideal* $\mathrm{sat}(T) \cap \mathbb{Q}[U, x_i]$. *Then we have*

$$\mathcal{O}_\infty(\mathrm{sat}(T)) = \cup_{i=1}^{s} V(\mathrm{ini}(g_i)).$$

To some sense, Proposition 4 proposes an algorithmic independent description of the $\mathcal{O}_\infty$ set. To prove Proposition 4, we recall a description of the $\mathcal{O}_\infty$ set by Gröbner basis in [11].

Lemma 3 (Theorem 2 in [11]**).** *Given a parametric ideal I with parameters $U = u_1, u_2, \ldots, u_d$ and variables $X = x_1, x_2, \ldots, x_s$, let $\mathcal{G}$ be a reduced Gröbner basis* *of I w.r.t a product ordering $\prec_{U,X}$ where $\prec_X$ is a degree reverse lexicographic ordering. For $i = 1, \ldots, s$, define*

$$\Xi_i^\infty = \{\mathrm{lc}_{\prec_X}(g) \mid g \in \mathcal{G}, \mathrm{lm}_{\prec_X}(g) = x_i^m \text{ for some } m \geq 0\}.$$

Then we have $\mathcal{O}_\infty(I) = \cup_{i=1}^s V(\Xi_i^\infty)$.

Proof (**Proposition** 4). Let $\mathcal{I} = \langle g_1, g_2, \ldots, g_s\rangle$. Let $\alpha \notin \cup_{i=1}^s V(\mathrm{ini}(g_i))$ be a parameter value. Let N be the number of preimages (i.e. $\Pi_U^{-1}(\alpha) \cap V(\mathcal{I})$). Let O_α be a compact neighborhood of α not intersecting $\cup_{i=1}^s V(\mathrm{ini}(g_i))$. Then, if O_α is sufficiently small, there exists N sets $O_1, O_2, \cdots, O_N$, such that

$$\Pi_U^{-1}(O_\alpha) \cap V(\mathcal{I}) = \cup_{k=1}^N O_k$$

holds and such that each O_k is diffeomorphic to O_α. Therefore, we have $\alpha \notin \mathcal{O}_\infty(\mathcal{I})$, which implies

$$\mathcal{O}_\infty(\mathcal{I}) \subseteq \cup_{i=1}^s V(\mathrm{ini}(g_i)).$$

Using the notations in Lemma 3, consider $\prec_X = x_1 \prec \cdots \prec x_s$. It is easy to deduce that $\Xi_1^\infty = \{\mathrm{ini}(g_1)\}$ holds. This implies $V(\mathrm{ini}(g_1)) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T))$. Since $\mathcal{O}_\infty(\mathrm{sat}(T))$ can be defined independently of the variable ordering on $x_1, x_2, \ldots, x_s$, we have $\cup_{i=1}^s V(\mathrm{ini}(g_i)) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T))$. Since $V(\mathrm{sat}(T)) \subseteq V(\mathcal{I})$ holds, we have $\mathcal{O}_\infty(\mathrm{sat}(T)) \subseteq \mathcal{O}_\infty(\mathcal{I})$.

Finally, we have

$$\cup_{i=1}^s V(\mathrm{ini}(g_i)) \subseteq \mathcal{O}_\infty(\mathrm{sat}(T)) \subseteq \mathcal{O}_\infty(\mathcal{I}) \subseteq \cup_{i=1}^s V(\mathrm{ini}(g_i)),$$

which yields the conclusion. □

Since different regular chains may have the same saturated ideal, a natural question to ask is: which regular chain(s) will be the best choice in the sense that the set theoretic difference of DV_T and $DV_{\mathrm{sat}(T)}$ is minimal. This question is answered by Proposition 2 and Theorem 2.

For the notion of a canonical regular chain in Theorem 2, one can find a definition in [20] or [1].

Theorem 2 ([20, 1])**.** *Let T^* be another regular chain satisfying* $\mathrm{sat}(T) = \mathrm{sat}(T^*)$*. If T^* is canonical, then we have $B_{ini}(T^*) \subseteq B_{ini}(T)$.*

Proposition 5 ([20])**.** *Let T_1 and T_2 be two regular chains satisfying* $\mathrm{sat}(T_1) = \mathrm{sat}(T_2)$*. If $B_{ini}(T_1) \subseteq B_{ini}(T_2)$ holds, then $\mathbf{BPS}(T_1) \subseteq \mathbf{BPS}(T_2)$ holds.*

4. Parametric semi-algebraic systems

As we mentioned earlier in Section 2, the computation of a discriminant variety of a semi-algebraic system reduces to that of the discriminant variety of an algebraic system, which is the way we follow in practice. For this reason, we only exhibit here several properties of effective boundaries and do not discuss border polynomials or discriminant varieties in the semi-algebraic case.

As mentioned in Section 2, effective boundaries were introduced in [1] as a tool for improving the computation of triangular decomposition of semi-algebraic systems. These improvements are based on the properties below which essentially state that effective boundaries behave well under the "transformations" that occur when computing triangular representations, in particular splitting, see Theorem 4.

Let $\mathbf{I}$ be the ideal associated with the parametric system S. Recall that we assume that S is well-determinate. Let $\mathfrak{p}$ be an associated prime ideal of $\mathbf{I}$. The ideal $\mathfrak{p}$ is called a *main prime component* of $\mathbf{I}$ (or S) if U is a maximal algebraically independent set modulo $\mathfrak{p}$. Theorem 3 extends Corollary 1 which appears in [1].

Theorem 3. *Given two parametric semi-algebraic systems S_1 and S_2 defined by polynomials in $\mathbb{Q}[U, X]$. Suppose S_1 and S_2 have the same set of inequalities and the same set of main prime components. Then we have $\mathbf{EB}(R_1) = \mathbf{EB}(R_2)$.*

Next, we shall discuss the effective boundary in the context of triangular decomposition.

Corollary 1 ([1]). *For any two STSASes $R_1 = [T_1, H_{1>}, P_>]$ and $R_2 = [T_2, H_{2>}, P_>]$ satisfying $\mathrm{sat}(T_1) = \mathrm{sat}(T_2)$, we have $\mathbf{EB}(R_1) = \mathbf{EB}(R_2)$.*

Theorem 4. *Consider three STSASes $R = [T, H_{\neq}, P_>]$, $R_1 = [T_1, H_{1\neq}, P_>]$ and $R_2 = [T_2, H_{2\neq}, P_>]$ satisfying $\mathrm{sat}(T) = \mathrm{sat}(T_1) \cap \mathrm{sat}(T_2)$. Assume that $\mathbf{ebf}(R_1) \cap \mathbf{ebf}(R_2) = \emptyset$ holds. Then, we have $\mathbf{ebf}(R) = \mathbf{ebf}(R_1) \cup \mathbf{ebf}(R_2)$.*

Theorem 5. *Given a parametric STSAS $R = [T, H_{\neq}, P_>]$, we have $\mathbf{ebf}(R) \subseteq B_{sep}(T) \cup B_{ie}([T, P])$.*

Let B be the set of the irreducible factors of a border polynomial of a parametric STSAS $R = [T, H_{\neq}, P_>]$. The above theorem suggests how to remove from B polynomials which are not defining irreducible effective boundaries.

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