ABSTRACT

The Hensel series is a series expansion of multivariate algebraic function at its singular point. The Hensel series is computed by the (extended) Hensel construction, and it is expressed in a well-structured form. In previous papers, we clarified theoretically various interesting properties of Hensel series in restricted cases. In this paper, we present a theory of Hensel series in general case. In particular, we investigate the Hensel series arising from non-squarefree initial factor, and derive a formula which shows “fine structure” of the Hensel series. If we trace a Hensel series along a path passing a divergence domain, the Hensel series often jumps from one branch of the algebraic function to another. We investigate the jumping phenomenon near the ramification point, which has not been clarified in our previous papers.

Categories and Subject Descriptors
G.1.2 [Approximation]: Nonlinear approximation, approximation of surfaces and contours; I.1.2 [Symbolic and Algebraic manipulation]: Algebraic algorithms

General Terms
Theory, Experimentation

Keywords
multivariate algebraic function, series expansion, singular point, convergence, many-valuedness

1. INTRODUCTION

Series expansion is a fundamental tool not only in mathematics but also in numerical as well as algebraic computations. So far, the Taylor series is used mostly but the Taylor expansion breaks down when the expansion point is a singular point. We can expand univariate algebraic functions at singular points (critical points) by Newton-Puiseux’s method; see [21]. Newton-Puiseux’s method can be generalized to multivariate algebraic functions recursively, giving “multivariate Puiseux series” which are sums of monomials of fractional powers w.r.t. each variable; see [10, 11, 1]. However, multivariate Puiseux series are not easy to use practically, in fact, even the convergence property is not known yet.

In a series of papers [17, 18, 13, 5, 14, 15, 16], we have developed a method of expanding multivariate algebraic function into series at singular points, where the algebraic function is defined to be the roots of a given multivariate polynomial. Since our method is based on the Hensel construction, we named the series obtained by our method Hensel series. For bivariate polynomials, our method computes Puiseux series roots simultaneously and efficiently. For multivariate polynomials which define multivariate algebraic functions, the series obtained by our method are very different from multivariate Puiseux series. In this paper, by Hensel series, we always mean multivariate series.

The Hensel series was used so far to the analytic continuation of algebraic functions via singular points [20], solving multivariate algebraic equations in series forms [17, 18], the multivariate polynomial factorization [4], the analytic factorization of polynomials of more than two variables [6, 7], and so on. For some other researches of utilizing series expansions at singular points, see [2, 9, 19, 12, 3].

The Hensel series has following remarkable properties.

1. Each term of Hensel series is well-structured. For example, given \( F(x, u, v) = x^2 - 2(u+v)x + (u^2+v^2) - (u^3-v^3) \), our method computes the following Hensel series (to the 2nd order w.r.t. an auxiliary variable \( \xi \)):

\[
\phi_{\pm}^{(2)}(u, v, \xi) = \alpha_{\pm} \pm \xi \frac{u^3-v^3}{2\sqrt{uv}} + \xi^2 \frac{(u^3-v^3)^2}{16u^2v^2},
\]

where \( \alpha_{\pm} = u \mp v \). The convergence and the divergence domains co-exist in any small neighborhood of the expansion point, and the neighborhood is occupied mostly by the convergence domain.

2. A simple formula has been derived to express the general term of Hensel series, which allows us to clarify the convergence properties of the Hensel series considerably; the convergence and the divergence domains co-exist in any small neighborhood of the expansion point, and the neighborhood is occupied mostly by the convergence domain.

3. The Hensel series often jumps from one branch of algebraic functions to another when it passes through the divergence domain.

The Hensel series is computed by the extended Hensel construction (EHC in short) which is the multivariate Hensel
construction at the singular point \([8, 17, 18]\). The most important concept in the EHC is Newton polynomial. So far, we have investigated mostly the cases where the Newton polynomial is squarefree and the given polynomial is either monic [15] or nonmonic [16].

In Sec. 2, we survey the EHC briefly. In Sec. 3, we explain the method of computing Hensel series in the general case, in particular, the Hensel construction in mass and roots, which we devised recently [14]. Furthermore, we consider the Tschirnhaus transformation in nonmonic case and present a method for expressing the roots of the Newton polynomial in a series form. In Sect. 4, we investigate the convergence property generally, in particular, we investigate Hensel series arising from non-squarefree initial factor. In Sect. 5, we investigate the behavior of Hensel series around the ramification point, including the jumping phenomenon.

2. SURVEY OF EHC IN GENERAL CASE

Let \(F(x, u) \equiv F(x, u_t, . . . , u_{t+1}) \in \mathbb{C}[x, u_t, . . . , u_{t+1}]\) be a given multivariate polynomial. By \(\text{deg}(F)\) and \(l(F)\), we denote the degree and the leading coefficient, respectively, w.r.t. \(x\), of \(F(x, u)\). We put \(\text{deg}(F) = n\) and \(l(F) = f_0(u)\). By \(\text{tdeg}(f)\), with \(f(u) \in \mathbb{C}[u]\), we denote the total-degree of \(f(u)\) w.r.t. sub-variables \(u_t, . . . , u_{t+1}\). For any \(c \in \mathbb{C}\), then \(\text{tdeg}(T) = e_1 + \cdots + e_t\), and \(\text{tdeg}(f)\) is the maximum of the total-degrees of terms of \(f(u)\). By \(\text{ord}(f)\), we denote the order of \(f(u)\), i.e., the minimum of the total-degrees of terms of \(f(u)\). For rational function \(N(u)/D(u)\) we define the order to be \(\text{ord}(N) - \text{ord}(D)\). The order of algebraic function is defined in Subsect. 3.3. By \(\text{res}(F, G)\) and \(\text{rem}(F, G)\), we denote the resultant of \(F\) and \(G\) w.r.t. \(x\) and the remainder of \(F\) divided by \(G\), respectively. By \(\mathbb{C}(u)\) we mean a ring of infinite sum of rational functions such as \(\sum_{n=0}^{\infty} N(u)/D(u)\), where \(N(u)\) and \(D(u)\) are homogeneous polynomials in \(u_t, . . . , u_{t+1}\), satisfying \(\text{ord}(N(u)/D(u)) = k\).

By \(F\) we denote the partial derivative of \(F\) w.r.t. \(x\). By \(\|u\|\), we denote the square norm \((|u_1|^2 + \cdots + |u_t|^2)^{1/2}\) after substituting numbers for \(u_t, . . . , u_{t+1}\). By \(\|f(u)\| = \|l(u)\|^n\) and \(\|f(u)\| = O(\|u\|^n)\) we mean \(\|f(u)\|/\|u\|^n \rightarrow \text{nonzero-number}\) and \(\|f(u)\|/\|u\|^n \rightarrow 0\), respectively, as \(\|u\| \rightarrow 0\).

Without loss of generality, we assume that \(F(x, u)\) is irreducible hence squarefree. Let \(\varphi(u)\) be an algebraic function defined by a root w.r.t. \(x\). of \(F(x, u)\), namely \(\varphi(u) = 0\). Let \(s \equiv (s_1, \ldots, s_t) \in \mathbb{C}^t\) be an expansion point of \(\varphi(u)\). If \(f_s(u) \neq 0\) and \(F(x, u)\) is squarefree then the algebraic function defined by \(F(x, u)\) can be expanded into Taylor series in \(u_t, \ldots, u_{t+1}\). Let the squarefree decomposition of \(F(x, s)\) be \(F(x, s) = F_0(x) \prod_{i=1}^{n} F_i(x)^{m_i}\), where \(F_0(x)\) and \(F_i(x)\) are squarefree, so we have \(\text{gcd}(F_0, F_i) = 1\) for \(1 \leq i \leq n\), and \(m_i \geq 2\) for any \(i\). Then, the Hensel construction shows that \(F(x, u)\) is factored as \(F(x, u) = F(x, u_t) \prod_{i=1}^{n} F_i(u_t)^{m_i}\), where \(F(x, u_t)\) and \(F_i(u_t)\) are \(\mathbb{C}(\{u_t\})\), \(F(x, s) = F_0(x)\) and \(F_i(x) = F_0(x)^{m_i}\) for each \(i\) (the factors of \(l(F)\) may be distributed among \(F_i\), \(F_t\), \(F_i\)). In the rest of this paper, we consider only one factor \(F_i(u_t)\), by renaming it to \(F(x, u_t)\), hence we have \[F(x, s) = [F_0(x)]^{m_1}, \quad F_0(x) \text{ is squarefree and } m_1 \geq 2.\] (2.1)

DEFINITION 1 (SINGULARITY). We call the expansion point \(s \in \mathbb{C}^t\) a singular point of algebraic function, or a singular point in short, if \(F(x, s)\) is not squarefree. If \(f_n(s) = 0\) then we say the leading coefficient is singular at \(s\).

Without loss of generality, we assume that the origin \(u = 0\) is a singular point or \(f_n(u)\) is singular at the origin, and we consider the series expansion at the origin. Note that at least one root \(\varphi(u)\) of \(F(x, u)\) becomes infinity at any zero-point of \(f_n(u)\).

The conventional Hensel construction is unavailable to expand the roots of \(F(x, u)\) given in (2.1), and we are necessary to adopt the EHC. The most important concept in the EHC is the Newton polynomial, defined as follows.

DEFINITION 2 (NEWTON POLYNOMIAL). For each term \(cx^{e_1}u_1^{e_2} \cdots u_{t+1}^{e_{t+1}}\) of \(F(x, u_t)\), where \(c \in \mathbb{C}\) and \(j = j_1 + \cdots + j_{t+1}\), plot a dot at the point \((i, j)\) in the \((e_1, e_2)\)-plane. The Newton polygon \(N\) for \(F(x, u)\) is a convex hull containing all the dots plotted. Let the bottom sides of \(N\), counted clockwise, be \(L_1, \ldots, L_q\). We call \(L_1, \ldots, L_q\) Newton lines. For each \(i \in \{1, \ldots, q\}\), Newton polynomial \(F_{E_i}(x, u)\) is defined to be the sum of all the terms plotted on \(L_i\); see Fig. 1.

Note that \(F_{E_i}(x, u)\) may or may not be squarefree.

![Fig. 1. Newton polygon N and Newton lines L1, ..., Lq.](image-url)
where \((g_1, \ldots, g_m)\) denotes the ideal generated by \(g_1, \ldots, g_m\) and \(t\) is the total-degree w.r.t. \(u_1, \ldots, u_\ell\). Below, we omit the total-degree variable \(t\) in \(F(x, u)\) etc., for simplicity. Note that we have \(F(x, u) \equiv F(z, u) \pmod{I_\alpha}\).

Suppose we have calculated \(F^{(k'_1)}, \ldots, F^{(k'_r)}\) up to \(k' = k - 1\). Then, we calculate

\[
\delta F^{(k)} = F - F^{(k-1)} = F^{(k-1)} \cdots F^{(k-1)} \pmod{I_{k+1}}
\]

and construct \(F^{(k)} = F^{(k-1)} + \delta F^{(k)} \) (i = 0, 1, \ldots, r) as

\[
\delta F^{(k'_i)} = F^{(k'_i)} + \delta F^{(k'_i)} (i \neq j).\]

Precisely, we must distribute the factors of leading coefficient \(f_0(u)\) among \(F^{(k'_1)}, \ldots, F^{(k'_r)}\), but we omit the explanation; see [4] for the distribution.

In order to factor \(F(x, u) \in \mathbb{C}[[u]]\), we must perform the EHC q times from \(L_1\) to \(L_q\). The right end of \(L_1\) be \((n_0, v_0)\) hence \(n_0 = 0\), and the terms plotted on \((n_0, v_0)\) be \(f_0(u)^{x^i}\). The left ends of \(L_1, \ldots, L_q\) be \((n_1, v_1), \ldots, (n_q, v_q)\), respectively, and the terms plotted on these ends be \(f_0(u)^{x^i_n} \cdots f_0(u)^{x^i_n}\) (see Fig. 1). For each \(i \leq l \leq q\), let the Newton polynomial \(F_{\ell_i}(x, u)\) be factored in \(\mathbb{C}[x, u]\) as

\[
F_{\ell_i}(x, u) = x^{n_i} f^{(n_i)}_{\ell_i}(x, u) \cdots F^{(n_j)}_{\ell_i}(x, u, v),
\]

gcd\(F^{(n_j)}_{\ell_i}, F^{(n_j)}_{\ell_i}) = 1 (j \neq i)\). (2.7)

Then, we have the following theorem; see [13] for the proof. Note that, since \(f_0(u)^{x^i_n}\) appears on both the right end of \(L_i\) and the right end of \(L_{i+1}\), \(F_{\ell_i}(x, u)\) is divided by \(f_0(u)^{x^i_n}\).

**Theorem 1** (Sasaki-Inaba 2000). \(F(x, u)\) can be factored in \(\mathbb{C}[[u]][x]\) as

\[
\begin{align*}
\{ & f_0^{(n_1)}(u) \cdots f_0^{(n_q)}(u) \prod_{i=1}^q \left[ F^{(n_i)}_{\ell_i}(x, u) \cdots F^{(n_j)}_{\ell_i}(x, u, v) / f_0^{(n_j)}(u) \right], \\
& F^{(n_j)}_{\ell_i}(x, u) \in \mathbb{C}[[u]][x] (i = 1, \ldots, q; j = 1, \ldots, r), \\
& F^{(n_j)}_{\ell_i}(x, u) \rightarrow F^{(n_j)}_{\ell_i}(x, u) \in \mathbb{C}[x, u] \text{ as } k \rightarrow 0. \}
\end{align*}
\]

(2.8)

The decomposition is unique up to unit factors in \(\mathbb{C}[[u]]\).

**Corollary 1.** We can factor \(F(x, u) \in \mathbb{C}[[u]][x]\) as follows.

\[
\begin{align*}
& f_0^{(n_1)} \cdots f_0^{(n_q)} F(x, u) = \left( F^{(n_1)}_{\ell_1}(x, u) / x^{n_1} \cdots F^{(n_q)}_{\ell_1}(x, u) / x^{n_q} \right), \\
& F^{(n_j)}_{\ell_i}(x, u) \rightarrow F^{(n_j)}_{\ell_i}(x, u) \in \mathbb{C}[x, u] \text{ as } k \rightarrow 0. \}
\end{align*}
\]

(2.9)

Here, for each \(i \in \{1, \ldots, \ell\}\), only products of factors of \(f_0^{(n_i)}(u), \ldots, f_0^{(n_j)}(u)\) appear in the denominators of terms of \(F^{(n_i)}_{\ell_i}(x, u)\).

\[\Box\]

3. Computing Hensel Series

**IN GENERAL CASE**

In this section, we consider computing Hensel series, i.e., factoring each \(F^{(n_i)}_{\ell_i}(x, u)\) in (2.8) into linear factors w.r.t. \(x\), by renaming \(F^{(n_j)}_{\ell_i}(x, u)\) to \(G(x)\) (actually, we treat \(F^{(n_j)}_{\ell_i}(x, u)\) for a sufficiently large \(k\)). The first problem we must solve is how to treat the leading coefficient of \(G(x, u)\), and the second problem is how to compute the Hensel series corresponding to multiple roots of the Newton polynomial. On the other hand, in order to compute and analyze the Hensel series simply, we have developed two techniques, Hensel construction in mass and Hensel construction in roots, see [14, 15].

As for the first problem, if we use the conventional technique for treating the leading coefficient, then the Hensel construction in mass and roots leads us to complicated expressions; see [14]. In [16], we found that, if we redefine the Newton polynomial as follows, then the leading coefficient can be treated simply and reasonably.

**Definition 3.** (Newton polynomial for \(G(x, u)\)).

Following Def 2, construct the homogeneous Newton polynomial \(G_{\text{new}}(x, u)\) for \(G(x, u)\). The Newton polynomial \(G_{\text{new}}(x, u)\) for \(G(x, u)\) is defined by replacing the leading coefficient of \(G_{\text{new}}(x, u)\) by \(lG(G)\).

Let \(G_{\text{new}}(x, u) = g_0^{(0)}(u)x^n + g_1^{(0)}(u)u + \cdots + g_0^{(0)}(u)\). The slope of the Newton line for \(G(x, u)\) is \(-\lambda\), hence each root \(\alpha_j(u)\) of \(G_{\text{new}}(x, u)\) is such that \(|\alpha_j(u)| = 0\) no matter how the coefficients of \(g_0^{(0)}(u)\) and \(g_0^{(0)}(u)\) are small compared with other coefficients. From the viewpoint of applications, however, if \(|\alpha_j(u)| > 1000|\alpha_j(u)|\) for \(\forall i < n\) (this case occurs when \(|g_0^{(0)}(u)|\) is very small) or \(|\alpha_j(u)| < 0.001|\alpha_j(u)|\) (this case occurs when \(|g_0^{(0)}(u)|\) is very small), for example, then treating such roots equally is impractical. Therefore, we define normality and closeness.

**Definition 4.** (Normality, Closeness).

Let \(N = \max\{|g_0^{(0)}(u)|/|u|^{n+1}\} | i = 0, 1, \ldots, n\) and \(S\) be a given small positive number, 0 < \(S < \varepsilon\), specifying the practical measure of smallness. If \(|g_0^{(0)}(u)|/|u|^{n+1} > S\) and \(|g_0^{(0)}(u)|/|u|^{n+1} > S\) for generic \(u\) then we say \(G_{\text{new}}(x, u)\) is normal. Let \(i_1(u)\) and \(i_1(u)\) (\(i \neq j\)) be roots of \(G_{\text{new}}(x, u)\) or \(G_{\text{new}}(x, u)\). We say \(i_1(u)\) and \(i_1(u)\) are mutually close if, for generic \(u\) such that \(|u| < S\), we have \(|i_1(u) - i_1(u)| < 0.001|\alpha_j(u)|\). Furthermore, we say \(u\) is close to \(u\) if \(|u - u| < S|u|\).

The second problem can be solved by the Tschirnhaus transformation, see Sect. 3.2 for details. However, if the Newton polynomial has close roots then the situation becomes complicated. In Def 3, we have defined the Newton polynomial \(G_{\text{new}}(x, u)\) so that we have \(lG(G)\). If \(lG(G)\) has higher order terms then \(G_{\text{new}}(x, u)\) is squarefree in most cases. The reason is as follows. Let \(R = \text{res}(G_{\text{new}}, G_{\text{new}})\) be considered as a polynomial in \(g_0\). Among the coefficients of \(G_{\text{new}}(x, u)\), only \(g_0\) has higher order terms, hence \(R\) becomes 0 identically iff every coefficient w.r.t. \(g_0\), of \(R\) is 0, which is very rare. On the other hand, we are interested in the behaviors of Hensel series near the expansion point, where the Hensel series are determined mostly by \(G_{\text{new}}(x, u)\) defined in Def 3. If \(G_{\text{new}}(x, u)\) has multiple roots then \(G_{\text{new}}(x, u)\) will have close roots. Therefore, we confine ourselves to investigating the following two cases in this paper.

Case 1) \(G_{\text{new}}(x, u)\) has neither multiple nor close root.

Case 2) \(G_{\text{new}}(x, u)\) has multiple roots but no close root.

We do not investigate the case that \(G_{\text{new}}(x, u)\) has close roots, which will be a theme in approximate algebra.
3.1 Case 1) Hensel construction in mass and roots

The case 1) has been studied in [16], by assuming that $G_{\text{New}}(x, u)$ is squarefree. In order to make this paper self-contained, we describe previous theories briefly.

Let the roots of $G_{\text{New}}(x, u)$ be $\alpha_1(u), \ldots, \alpha_n(u)$, which we often write as $\alpha_i(u)\neq \alpha_j(u) (V i \neq j)$. The $\alpha_i(u)$ is usually an algebraic function. We define $\overline{G}(x, u, \xi)$ by introducing an auxiliary variable $\xi$, as follows (we put $\xi = 1$ after finishing the Hensel construction).

$$ G(x, u) = G_{\text{New}}(x, u) + G_0(x, u), $$

The $G_0(x, u)$ is the Hensel series corresponding to $\alpha_i(u)$. Actual Hensel construction is performed as follows. Suppose we computed $G^{(k)}$ and $G^{(k)}$ up to $k = k - 1$. Then, we compute the $k$-th order residual $\overline{G}^{(k)}$ as follows.

$$ \overline{G}(x, u, \xi) = G^{(k)}(x, u, \xi) \cdot \xi^{k+1} (\mod \xi^{k+1}), $$

$$ \phi^{(k)}(u, \xi) = x - \alpha^{(k)}(u, \xi), \quad \phi^{(k)}(u, \xi) = \alpha^{(k)}(u). $$

The $\phi^{(k)}$ is the Hensel series corresponding to $\alpha_i(u)$. We express $G_{\text{New}}(x, u)$ and $G_0(x, u, \xi)$ as initial factors, we perform the Hensel construction of $G(x, u, \xi)$ w.r.t. modulus $\xi$ (Hensel construction in mass), obtaining

$$ \overline{G}(x, u, \xi) \equiv G^{(k)}(x, u, \xi) \cdot \xi^{k+1} (\mod \xi^{k+1}), $$

$$ G(x, u, \xi) = \alpha^{(k)}(u, \xi) + \xi^{k+1} + \delta^{(k)}(u). $$

The $\phi^{(k)}(u, \xi) = \alpha^{(k)}(u, \xi)$ is given as follows.

$$ \phi^{(k)}(u, \xi) = \alpha^{(k)}(u, \xi) = \alpha^{(k)}(u). $$

We compute the $k$-th order Hensel factors $G^{(k)}$ and $G^{(k)}$ by the well-known formula

$$ G^{(k)}(x, u, \xi) = \alpha^{(k)}(x, u, \xi) + \xi^{k+1} \sum_{l=0}^{k} A_l \delta^{(l)}(u), $$

$$ G^{(k)}(x, u, \xi) = \alpha^{(k)}(x, u, \xi) + \xi^{k+1} \sum_{l=0}^{k} B_l \delta^{(l)}(u). $$

(3.4)

Now, we can find the remainder of $G_{\text{New}}(x, u)$ w.r.t. modulus $\xi^{k+1}$.

$$ A_l(x, u) = A_l(x, u) \sum_{l=0}^{k} A_l(x, u, \xi) = \xi^{k+1}, $$

$$ B_l(x, u) = B_l(x, u) \sum_{l=0}^{k} B_l(x, u, \xi) = \xi^{k+1}. $$

(3.5)

We express $A_l(x, u)$ and $B_l(x, u)$ in $\alpha_i, \ldots, \alpha_n$. Substituting $\alpha_i$ and $\alpha_j (j \geq 2)$ for $x$ in (3.6), we obtain $A_{(\alpha_i, u)} = \alpha_i / \prod_{j=1}^{\infty} (x - \alpha_j)$ and $B_{(\alpha_i, u)} = \alpha_i / \prod_{j=1}^{\infty} (x - \alpha_j)$. By these, $A_l(x, u)$ and $B_l(x, u)$ are determined uniquely as follows.

$$ A_l(x, u) = A_{(\alpha_i, u)} = \frac{\alpha_i}{G_{\text{New}}(x, \alpha_i, u)}, $$

$$ B_l(x, u) = B_{(\alpha_i, u)} = \frac{\alpha_i}{G_{\text{New}}(x, \alpha_i, u)} = \frac{G_{\text{New}}(x, u, (x - \alpha_i, u))}{G_{\text{New}}(x, u, \alpha_i, u)}. $$

(3.7)

Since $\delta^{(k)}(x, u) = G_0(x, u)$, substituting the above expressions of $A_l(x, u)$ and $B_l(x, u)$ for those in (3.5), we obtain the following theorem (Hensel construction in roots).

**Theorem 2** (Sasaki-Nakada 2009). If $G_{\text{New}}(x, u)$ is squarefree then the Hensel factors $G^{(\infty)}$ and $G^{(\infty)}$ in (3.3) are expressed as follows.

$$ G^{(\infty)}(x, u, \xi) = G_{\text{New}}(x, u) + \sum_{k=1}^{\infty} \xi^k \frac{G^{(k)}(x, u, \xi)}{G_{\text{New}}(x, u, \alpha_i, u)}, $$

$$ G^{(\infty)}(x, u, \xi) = G_{\text{New}}(x, u) + \sum_{k=1}^{\infty} \frac{G_{\text{New}}(x, u, \xi)}{(x - \alpha_i, u)} \overline{G}^{(k)}(x, u, \xi), $$

where $\delta^{(k)}(x, u) = G_0(x, u)$ and the $k$-th order residual $\overline{G}^{(k)}$ $(k \geq 2)$ is given as follows.

$$ \overline{G}^{(k)}(x, u, \xi) = - \sum_{j=2}^{n} \frac{G_{\text{New}}(x, u, \xi)}{(x - \alpha_1, u)} \overline{G}^{(k-1)}(x, u, \xi). $$

(3.10)

The above formula (3.8) gives (we omit 1 in $\phi^{(1)}(u, 1)$).

$$ \phi^{(\infty)}(u) = \alpha_1 - \sum_{k=1}^{\infty} \frac{G^{(k)}(x, u, \xi)}{G_{\text{New}}(x, u, \alpha_i, u)}. $$

(3.11)

**Definition 5** (Rational, algebraic series). If $\alpha_i(u) \in C(u)$ then the truncated Hensel series $\phi^{(k)}(u)$, $k \geq 1$, is called rational, otherwise it is called algebraic.

3.2 Case 2) Tschirnhaus transformation

We can assume that the factorization in (2.7) is the squarefree decomposition. The squarefree factors of $F(x, u)$ have been treated in Case 1). Therefore, in Case 2), we have only to consider such an $H(x, u) \equiv F^{(\infty)}(x, u)$ that its homogeneous Newton polynomial $H_{\text{New}}(x, u)$ is of the form $H_{\text{New}}(x, u) = [H_0(x, u)]^m$, where $m \geq 2$ and $H_0(x, u)$ is irreducible in $C[x, u]$. Let $\deg(H_0) = r$ and the roots of $H_0(x, u)$ be $\alpha_1, \ldots, \alpha_r$.

$$ H_0(x, u) = h_0^{(0)}(x) (x - \alpha_1(u)) \ldots (x - \alpha_r(u)). $$

(3.12)

We first define the order of algebraic function $\alpha(u)$.

**Definition 6**. Let $\alpha(u)$ be a root of $H_{\text{New}}(x, u)$ or $H_{\text{New}}(x, u)$. We define the order of $\alpha(u)$ to be $\deg(x)$, where $\deg(x)$ is the slope of the Newton line. We plot $\alpha(u)$ in the $x$-plane defined in Def. 2.

Let $\deg(H) = n = mr$. Since $H_{\text{New}}(x, u)$ has $m$ multiple roots, $H_{\text{New}}(x, u)$ will have $m$ clusters of $m$ close roots. Hence, we express the roots of $H_{\text{New}}(x, u)$ as follows.

$$ H_{\text{New}}(x, u) = h_n(u) \sum_{i=1}^{m} (x - \alpha_i(u)) \ldots (x - \alpha_m(u)), $$

$$ \|\alpha_j(u) - \alpha_i(u)\| = o(\|u\|^k) (i = 1, \ldots, m; j = 1, \ldots, m). $$

(3.13)

In the monic case, a procedure to handle $H(x, u)$ was established in [18]. We generalize the procedure to nonmonic case.

If $r = 1$ then go to the following Tschirnhaus transformation. If $r \geq 2$ then, putting $H^{(k)}(x, u) = \prod_{j=1}^{\infty} (x - \alpha_j(u))$
\((i = 1, \ldots, r)\), we perform the EHC of \(H(x, u)\), with initial factors \(H_{i}^{(0)}(x, u)\), obtaining (actually, we perform the EHC only up to a finite order)

\[
\begin{align*}
\{ H(x, u) &= h_{0}(u) H_{i}^{(0)}(x, u) \cdots H_{i}^{(0)}(x, u), \\
H_{i}^{(0)}(x, u) &\in \mathbb{C} \{ (u) \}[x, \alpha_{1}, \ldots, \alpha_{m}] \quad (i = 1, \ldots, r). \tag{3.14}
\end{align*}
\]

Let \(H_{i}^{(0)}(x, u) = x^{n} + h_{1,m-1} x^{m-1} + \cdots + h_{0,0}\). We apply the following Tschirnhaus transformation to each factor \(H_{i}^{(0)}(x, u)\).

\[
H_{i}^{(0)}(x, u) \mapsto \tilde{H}_{i}(x, u) = H_{i}^{(0)}(x - \frac{h_{1,m-1}}{m}, u). \tag{3.15}
\]

Since \(\alpha_{1}, \ldots, \alpha_{m}\) are the roots of Newton polynomial for \(H_{i}^{(0)}(x)\), the above transformation maps the Newton polynomial to \(x^{n}\). Therefore, we can apply the EHC to \(\tilde{H}_{i}(x, u)\); let the slope of the Newton line for \(\tilde{H}_{i}\) be \(-\lambda\); then we have \(\lambda^{' \prime} > \lambda\). If the homogeneous Newton polynomial for \(\tilde{H}_{i}\) is still not squarefree, we apply the Tschirnhaus transformation to \(\tilde{H}_{i}\) again.

**Remark 1.** The above formulation is, although consistent with Def. 3, quite complicated from the viewpoint of computation. For easier computation, we may use the homogeneous Newton polynomial \(H_{\text{New}} = h_{0}(u) \prod_{i=1}^{r} (x - \alpha_{i}(u))^{m}\) for the initial factors of Hensel construction, and perform the Hensel construction to satisfy

\[
\begin{align*}
\{ H(x, u) &= h_{0}(u) H_{i}^{(0)}(x, u) \cdots H_{i}^{(0)}(x, u), \\
H_{i}^{(0)}(x, u) &\in \mathbb{C} \{ (u) \}[x, \alpha_{1}] \quad (i = 1, \ldots, r). \tag{3.16}
\end{align*}
\]

In the next sections, we consider only the homogeneous Newton polynomials.

### 3.3 Computing \(\alpha_{1}, \ldots, \alpha_{n}\) in a series form

In the nonmonic case, we define the Newton polynomial \(G_{\text{New}}(x, u)\) so that \(k(G_{\text{New}}) = k(G) = g_{n}(u)\). If \(g_{n}(u)\) contains higher order terms then the homogeneity of \(G_{\text{New}}(x, u)\) is destroyed by \(g_{n}(u)\). The behavior of the Hensel series near the expansion point is determined mostly by \(G_{\text{New}}(x, u)\) and we want to know the behaviors of \(\alpha_{1}(u), \ldots, \alpha_{n}(u)\) near the expansion point. So, we consider a method to express the roots in a series form.

Let \(g_{n}(u)\) be split into lower and higher terms:

\[
g_{n}(u) = g_{n}(u) + g_{\text{ord}}(g_{n}) < \text{deg}(g_{n}). \tag{3.17}
\]

Then, we can express \(G_{\text{New}} \overset{\text{df}}{=} g_{n}(u) G_{\text{New}}\) as follows.

\[
\begin{align*}
\tilde{G}_{\text{New}} &= \tilde{G}_{\text{New}, 1} + \tilde{G}_{\text{New}, h}, \\
G_{\text{New}, 1} &= g_{n}(u) x^{n} + g_{n-1}(u) x^{n-1} + \cdots + g_{0}(u), \\
G_{\text{New}, h} &= \left(-g_{n}/g_{n}(u) (g_{n-1}(u) x^{n-1} + \cdots + g_{0}(u)). \tag{3.18}
\end{align*}
\]

Note that \(\tilde{G}_{\text{New}, 1}\) is \((\lambda, 1)\)-homogeneous and \(\tilde{G}_{\text{New}, h}\) can be regarded to be higher order than \(\tilde{G}_{\text{New}, 1}\), because \(g_{n}/g_{n} = g_{n}/(g_{n}+g_{n}) = (g_{n}/g_{n})^{2} (1 - (g_{n}/g_{n}^{2} + (g_{n}/g_{n})^{2})^{2})\).

Let the roots of \(\tilde{G}_{\text{New}, 1}\) be \(\alpha_{1}(u), \ldots, \alpha_{n}(u)\):

\[
\tilde{G}_{\text{New}} = g_{n}(u) (x - \alpha_{1}(u)) \cdots (x - \alpha_{n}(u)). \tag{3.19}
\]

Applying the EHC to \(\tilde{G}_{\text{New}}\), with initial factors \((x - \alpha_{1}(u))\) and \(\tilde{G}_{\text{New}, 1}/(x - \alpha_{1}(u))\), we can compute \(\alpha_{1}(u)\) in a series form, just similarly as we have computed the power-series root \(\phi_{1}(u)\) in the previous subsection. We show the resulting formula.

**Theorem 3.** If \(\tilde{G}_{\text{New}, i}(x, u)\) is squarefree then \(\alpha_{1}(u)\) can be expressed as follows.

\[
\alpha_{1}(u) = \alpha_{1}(u) - \sum_{k=1}^{\infty} \tilde{G}_{\text{New}, k}^{(1)}(\alpha_{1}(u), u), \tag{3.20}
\]

where \(\tilde{G}_{\text{New}, 1}^{(1)} = \tilde{G}_{\text{New}, 1}(x, u)\) and the \(k\)-th order residual \(\tilde{G}_{\text{New}, k}^{(k)}(k \geq 2)\) is given as follows.

\[
\tilde{G}_{\text{New}, k}^{(k)}(x, u) = - \sum_{j=0}^{k-1} \tilde{G}_{\text{New}, k}^{(k-1)}(x - \alpha_{1}(u)) \tilde{G}_{\text{New}, j}^{(k-1)}(\alpha_{j}(u), u) G_{\text{New}}(\alpha_{j}(u), u). \tag{3.21}
\]

### 4. CONVERGENCE PROPERTY NEAR THE EXPANSION POINT

In this section, we study the convergence property of Hensel series near the expansion point \((= \text{the origin})\). After surveying some previous results in 4.1, we study the Hensel series computed on different Newton lines in 4.2, and study the Hensel series corresponding to multiple roots of the Newton polynomial in 4.3. We will show that, in both Case 1) and Case 2) mentioned in Sect. 3, the divergence domain of any Hensel series occupies only a small part of the neighborhood of the expansion point.

#### 4.1 When the Newton polynomial is squarefree

We consider \(G(x, u)\) defined at the beginning of Sect. 3, the Newton polynomial which is factored as in (3.1). Theorem 2 tells us that the Hensel series \(\phi_{1}(u)\) corresponding to \(\alpha_{1}(u)\) is expressed as in (3.11). Therefore, we can see the convergence property of \(\phi_{1}^{(k)}(u)\) by investigating the behaviors of \(\tilde{G}_{\text{New}, k}^{(k)}(\alpha_{1}(u), u)\) and \(G_{\text{New}}(\alpha_{1}(u), u)\). Such investigations were done in [15, 16], and we show several important results (Theorem 6 is new). We put

\[
G(x, u) = g_{n}(u) x^{n} + g_{n-1}(u) x^{n-1} + \cdots + g_{0}(u). \tag{4.1}
\]

Note that at least one algebraic function diverges at the zero-points of \(g_{n}(u)\) and becomes 0 at the zero-points of \(g_{0}(u)\). We first show an important lemma; below, by \(o(||u||^{\alpha})\) we mean either \(O(||u||^{\alpha} + 1)\) or less.

**Lemma 1.** We have the following order estimations near the expansion point, so long as \(u\) is generic and not close to the zero-points of \(g_{n}(u)\) and \(\alpha_{1}(u) - \alpha_{j}(u) (\forall i \neq j)\).

\[
\begin{align*}
|\alpha_{1}|, |\alpha_{1} - \alpha_{i}| &= O(||u||^{\lambda}) \quad (j = 2, \ldots, n), \tag{4.2} \\
|G_{\text{New}}(\alpha_{i}, u)| &= O(||u||^{\lambda+n-1}) \quad (i = 1, \ldots, n), \tag{4.3} \\
|G_{\text{New}}(\alpha_{i}, u)| &= o(||u||^{\lambda+n}) \quad (i = 1, \ldots, n). \tag{4.4}
\end{align*}
\]

Let \(\delta_{\phi_{1}^{(b)}}(u)\) be the coefficient of \(u^{k}\)-term of Hensel series \(\phi_{1}^{(k)}(u, \xi)\). We have the following order estimation near the expansion point, so long as \(u\) is not close to the zero-points of \(g_{n}(u)\), \(g_{0}(u)\) and \(\alpha_{1}(u) - \alpha_{j}(u) (\forall i \neq j)\).

\[
|\delta_{\phi_{1}^{(b)}}(u)| = O(||u||^{\lambda+k-1}). \tag{4.5}
\]
Theorem 4. Assume that $G_{Nw}(x, u)$ is normal and has no close roots. Except near the zero-points of $g_n(u)$, any divergence domain of Hensel series of $\phi_1^{(\infty)}(u)$ starts from the expansion point and runs outside radially along the zero-points of $\alpha_j(u) - \alpha_j(u)$, $2 \leq j \leq n$.

Theorem 5. Let $S_r$ be the surface of the hypersphere $\|u\|^2 = r^2$, where $r$ is a small real positive number. Suppose $S_r$ contains zero-points of $g_n(u)$ on which $\alpha_j(u)$ diverges, and let $\delta S_r$ be small neighborhood of the zero-points, on $S_r$. Let $S_r$ be $S_r - \delta S_r$. Then, we have

$$\frac{\text{divergence area of } \phi_1^{(\infty)}(u) \text{ on } S_r}{\text{convergence area of } \phi_1^{(\infty)}(u) \text{ on } S_r} \to 0 \quad \text{as } r \to 0.$$  \hspace{1cm} (4.6)

Theorem 6. Assume that the Newton polynomial $G_{Nw}(x, u)$ is normal and has no close roots. Let $\alpha_j(u)$ be any root of $G_{Nw}(x, u)$. Let $\mathcal{D}_i$ be the divergence domain of $\phi_1^{(\infty)}(u)$ running along the zero-points of $\alpha_j(u) - \alpha_j(u)$ (i $\neq j$). Then, the spread of $\mathcal{D}_i$ on the surface $\|u\| = r$, $0 < r \ll 1$, is of order $\|G_{Nw}(\alpha_i, u)\|/\|G_{Nw}(\alpha_j, u)\|$ for generic $u$. Therefore, if $\|u\|/\|\alpha_i(u) - \alpha_j(u)\|$ becomes small then $\mathcal{D}_i$, the divergence domain on the surface $\|u\| = r$, spreads in proportion to $1/\|\alpha_i(u) - \alpha_j(u)\|$ approximately.

Proof Consider formulas in Theorem 2. The initial term of $\phi_1^{(\infty)}(u)$ is $\alpha_j(u)$ which does not diverge, because $G_{Nw}$ is normal. The order-estimations in (4.5) tells us that $\mathcal{D}_i$ is determined mostly by the second term $G_{Nw}(\alpha_i, u)/G_{Nw}(\alpha_j, u)$ because $u$ is near the origin. Hence, the theorem follows.

It is remarkable that the Hensel series converges not only in any small neighborhood of the expansion point but also at almost all points of the small neighborhood.

4.2 Hensel series on different Newton lines

We consider the Hensel factors in (2.9) in details, by separating the Hensel factor on $L_i$, for example, from $F(x, u)$, as follows,

$$\begin{cases} F(x, u) = F_1^{(\infty)}(x, u) F_2^{(\infty)}(x, u), \\ \hat{F}^{(0)}(x, u) = x^{n_1}, \quad F^{(0)}(x, u) = F_2(x, u)/x^{n_1}. \end{cases}  \hspace{1cm} (4.7)$$

We use the Hensel construction in mass for this separation; below, we omit the auxiliary variable $\xi$ for simplicity. Moses-Yun's polynomials are expressed concisely as follows.

Lemma 2. Moses-Yun's polynomials $A^{(l)}(0)$ and $B^{(l)}(0)$ satisfying $A^{(l)} F^{(0)}(x, u) + B^{(l)} x^{n_1} = x^l$ ($l = 0, 1, \ldots, n_1 - 1$), with deg$(A^{(l)} < n_1$ and deg$(B^{(l)} < n_1 - m$, are given uniquely as follows.

$$\begin{align*} \text{for } l \geq n_1: & \quad A^{(l)} = 0, \quad B^{(l)} = x^{l - n_1}, \\ \text{for } l < n_1: & \quad A^{(l)} = \hat{F}^{(l - n_1)}(x, u), \quad B^{(l)} = [1 - \hat{F}^{(l - n_1)}(0)] x^{n_1 - l}. \end{align*}  \hspace{1cm} (4.8)$$

Here, $\hat{F}^{(l-1)}(x, u)$ is a polynomial in the main variable $x$, satisfying $\text{deg}(\hat{F}^{(l-1)}(x, u)) < m$ and $\hat{F}^{(l-1)}(0) \equiv 1 \pmod{x^m}$, i.e., it is the inverse of $F^{(0)}(x, u)$ modulo $x^m$.

We can compute $\hat{F}^{(l-1)}(x, u)$ by the power-series division of 1 by $F^{(0)}$: note that $F^{(0)}$ has a nonzero $x^{l_0}$-term. Therefore, only $F_1^{(l_0)}(u)$ (= the $x^0$-term of $F^{(0)}$) appears in the denominators of $A^{(l)}(0)$ and $B^{(l)}(0)$ ($l < n_1$).

We investigate the Hensel series arising from $F^{(\infty)}(x, u)$ and $F^{(\infty)}(x, u)$ near the expansion point. We put $1 - F^{(l-1)}(0) = Q(x, u)x^{l_1 - 1}$. Note that $F^{(l-1)}(0)$ has 1/$F_1^{(l_0)}(u)$ as the constant term w.r.t. $x$. Since $F^{(l_1)}(x, u)$ is $(\lambda, 1)$-homogeneous, $\hat{F}^{(l_1)}(x, u)$ in (4.8) is $(\lambda, 1)$-homogeneous, and $Q(x, u)$ is also $(\lambda, 1)$-homogeneous. Let

$$(4.10)$$

$$(4.11)$$

Put $F^{(l_1)} = F^{(l_1)} + \delta F^{(l_1)}$ and $F^{(l_1)} = F^{(l_1)} + \delta F^{(l_1)}$. We can compute $\delta F^{(l_1)}$ and $\delta F^{(l_1)}$ as $\delta F^{(l_1)} = \sum_{l_i=0}^1 A^{(l_1)}(0)$ and $\delta F^{(l_1)} = \sum_{l_i=0}^1 B^{(l_1)}(0)$. Substituting $A^{(l_1)}(0)$ and $B^{(l_1)}(0)$ into these expressions and noting that $F^{(l_1)} = F^{(l_1)} (mod x^{m'})$, which is valid for any $m < n_1$, we obtain

$$(4.12)$$

Let the slope of Newton line $L_i$ be $-\lambda_i$ and put $\text{deg}(F^{(l_1)}(x, u)/x^{m'}) = d_i$ ($i = 1, \ldots, q$). Let $\alpha_{i,1}, \ldots, \alpha_{i,d_i}$ be the roots of $F^{(l_1)}(x, u)$. The following lemma is essentially the same as Corollary 1.

Lemma 3. For each $i \in \{1, \ldots, q\}$, let $\phi_i^{(\infty)}(j = 1, \ldots, d_i)$ be $\bar{d}_i$, Hensel series obtained by the EHC on the Newton line $L_i$, then they have $\alpha_{i,1}(u), \ldots, \alpha_{i,d_i}(u)$ as initial terms.

Proof Let $F(x, u) = F_{C_i}(x, u) + F_{N_i}(x, u)$, and consider the above $\delta F^{(l_1)}$ and $\delta F^{(l_1)}$. Since $\delta F^{(l_1)} = F_{C_i}(x, u)$, all the terms of $\delta F^{(l_1)}$ and $\delta F^{(l_1)}$ contain coefficients of $F_{N_i}(x, u)$, and so are $\delta F^{(l_1)}$ and $\delta F^{(l_1)}$ ($\forall \lambda \geq 2$). Hence, the homogeneous Newton line for $\phi_i^{(\infty)}$ is $F_{C_i}(x, u)$. Since $F_{N_i}^{(l_1)}$ is $(\lambda, 1)$-homogeneous and $F_{N_i}^{(l_1+1)}$ has the constant term 1/$F_1^{(l_0)}(u)$, the Newton polynomial for $\phi_i^{(\infty)}$ is determined by the lowest order terms of $\sum_{l_i=0}^1 \delta F_i^{(l_1)}(x, u)$. Hence, the Newton polynomial in the range $n_2 < x < n_1$ is $F_{C_i}(x, u)$. This discussion is available for $F_{C_i}(x, u)$.
series $\phi_i(\infty)(u)$ may diverge, see (3.8). Furthermore, if $u$ is close to a zero-point of $f_{n−1}(u)$ (or $f_{n}(u)$) then some root $\alpha_{j, i} (1 \leq j \leq d_i)$ will be very large (resp. very small), but such cases are excluded in the theorem. Then, the Hensel series $\phi_{i,j}(u)$ is dominated by its initial term $\alpha_{j, i}(u)$ for $\|u\| < \epsilon_1$, and we have $|\alpha_{j, i}(u)| = O(\|u\|^\|\lambda\|)$ near the expansion point.

The Hensel series $\phi_{i,j}(u)$ is dominated by its initial term $\alpha_{j, i}(u)$ and $|\alpha_{j, i}(u)| = O(\|u\|^\|\lambda\|)$ near the expansion point. Since $\lambda_i$ and $\lambda_{i'}$ are rational numbers satisfying $\lambda_i \neq \lambda_{i'}$, Theorem 6 with order-estimations in Lemma 1 assures that, near the expansion point, divergence domains are well separated one another.

4.3 When the Newton polynomial is not squarefree

We consider Hensel series arising from the Hensel factor $H_i(\infty)(x, u)$ in (3.16). We will see that the Hensel series is of the form $\tilde{\alpha}_i + \text{higher-order-terms showing "fine structure"}.$

First, we consider denominators appearing in $H_i(\infty)(x, u)$, which is constructed by the EHC with initial factors $H_i(0) = (x - \tilde{\alpha}_i)^m$ and $H_i(0) = [H_0(x)u]^m/(x - \tilde{\alpha}_i)$, where $H_0(x, u) = (x - \tilde{\alpha}_1) \cdots (x - \tilde{\alpha}_r)$.

**LEMMA 4.** Except for denominators introduced by the separation of $H \overset{\text{def}}{=} F_{i,j}(x, u)$ in (2.8), the denominators appearing in the terms of $H_i(\infty)(x, u)$ are only powers of res$(H_0, H_0)$.

**Proof.** Let $A^0$ and $B^0$ ($0 \leq m - 1$) be Moser-Yun’s polynomials satisfying $A^0H_i(0) + B^0H_i(0) = x^r, \deg(A^0) < m$ and $\deg(B^0) < m(r - 1)$. We can compute $A^0$ and $B^0$ by the extended Euclidean algorithm and $A^0$ and $B^0$ ($l \geq 1$) as $A^{l} = \text{rem}(x'A^0H_i(0))$ and $B^{l} = \text{rem}(x'B^0H_i(0))$. The extended Euclidean algorithm allows us to express $A^0$ and $B^0$ in $C(u, \tilde{\alpha}_i)[x]$, where the denominators are res$(H_i, H_i) = \{\text{res}(x - \tilde{\alpha}_1, (x - \tilde{\alpha}_2) \cdots (x - \tilde{\alpha}_r))\}^m = [(\tilde{\alpha}_1 - \tilde{\alpha}_2) \cdots (\tilde{\alpha}_1 - \tilde{\alpha}_r)]^m = [H_0'(\tilde{\alpha}_1, u)]^m$. We can compute the inverse of the extended Euclidean algorithm: let $C(x, u)$ and $D(x, u)$ be in $C(u)[x]$, satisfying $C(x, u)H_0(x, u) + D(x, u)H_0(x, u) = 1$, deg$C) < \text{deg}(H_0)$ and deg$(D) < \text{deg}(H_0)$, then we have $D(\tilde{\alpha}_i, u) = [H_0'(\tilde{\alpha}_1, u)]^{-1}$.

The extended Euclidean algorithm tells us that the denominator of $D(x, u)$ is res$(H_0, H_0)$.

Next, we consider the Hensel-series roots of $H_i(\infty)(x, u)$, by renaming $H_i(\infty), \lambda_i, \lambda_0$ to $H, \lambda, \tilde{\alpha}$, respectively. The Tschirnhaus transformation for $H(x, u)$ now is $H(x, u) \rightarrow H(x, u) = H(x + \tilde{\alpha}, \tilde{\alpha})$. (4.13)

For simplicity, we assume that the Newton polynomial for $H(x, u)$ is squarefree, if not so then we separate the non-squarefree factor again.

Let the slope of Newton line for $H(x, u)$ be $-\tilde{\lambda}$, and the roots of the Newton polynomial for $H(x, u)$ be $\tilde{\alpha}_m(u), \ldots, \tilde{\alpha}_0(u)$. It is obvious that $\tilde{\lambda} > \lambda$.

**LEMMA 5.** Each root $\tilde{\alpha}_i(u)$ ($1 \leq i \leq m$) is such that $\|\tilde{\alpha}_i(u)\| = O(\|u\|^\|\lambda\|)$ for generic $u$. Furthermore, $\tilde{\alpha}_i(u)$ may diverge on the zero-points of res$(H_0, H_0)$ and $\tilde{\alpha}_i - \tilde{\alpha}_j$, ($j \geq 2$).

**Proof.** The first claim is obvious because the Newton polynomial for $H(x, u)$ is $(\lambda, 1)$-homogeneous. Since res$(H_0, H_0)$ appears in the denominators of the terms of $\hat{H}(x, u)$ except for term $x^n$, it may appear in some denominators of the Newton polynomial for $H(x, u)$. The same is true for the EHC in (3.16).

**THEOREM 8.** If the Newton polynomial for $H(x, u)$ is squarefree and has no close root, then $m$ Hensel-series $\phi_i(\infty)$ ($i = 1, \ldots, m$) of $H(x, u)$ are expressed as

$$
\phi_i(\infty)(u) = \tilde{\alpha}_i(u) + \tilde{\alpha}_i(u) + \sum_{h=1}^{\infty} \delta_i^{(k)}(u).
$$

(4.14)

where $\delta_i^{(k)}$ is the $k$-th order term which is computed just similarly as in 3.1. The divergence domains of $\phi_i(\infty)(u)$ near the expansion point start from the expansion point and run outside radially along the zero-points of $\tilde{\alpha}_i(u) - \tilde{\alpha}_j(u) (j \neq i)$. The ratio defined by (4.6), showing the spread of the divergence domain, is specified by $\|u\|^\lambda$ (not by $\|u\|^\|\lambda\|$).

**Proof.** Eq. (4.14) is obvious from the transformation in (4.13). Lemma 4 tells us that $H(x, u) (= \text{the Hensel factor} H_i(\infty)(x, u))$ is in $C(u)[x]$ and the coefficients of higher terms contain res$(H_0, H_0)$ and its powers in the denominators. Thus, the spread of divergence domain of $\phi_i(\infty)(u)$ is specified by both $\|u\|^\lambda$ and $\|u\|^\|\lambda\|$. Since $\lambda < \tilde{\lambda}$, the spread is specified mostly by $\|u\|^\lambda$ near the expansion point.

**Example 1** (fine structure of Hensel series)

$$
F(x, u, v) = (x^2 + u^2 + u^2)(x - v) + u^4 + v^4.
$$

The Newton polynomial for $F(x, u, v)$ is $F_{1,0} = F_{1,0}^0F_{2,0}^0$, where $F_{1,0}^0 = (x - u)^2$ and $F_{2,0}^0 = x - v$, hence $\tilde{\alpha}_1 = u$. We put $D = u - v$ and $F_1 = u^4 + v^4$. Performing the EHC of $F(x, u, v)$ with initial factors $F_{1,0}^0$ and $F_{2,0}^0$, up to order 5, we obtain

$$
F_1 = (x - u)^2 - (x - 2u + v)F_1/D^2 + (2x - 3u + v)F_1/D^3 + (143x - 198u + 55v)F_1/D^4 + \cdots,
$$

$$
F_2 = (x - v) + F_1/D + 2F_1/D^2 + 7F_1/D^3 - 30F_1/D^4 + 143F_1/D^4 + \cdots.
$$

Applying the transformation $x \mapsto x + u$ to $H(x, u, v) \overset{\text{def}}{=} F_{1,0}(x, u, v)$, we obtain

$$
\hat{H}(x, u, v) = x^2 - (x + v)F_1/D^2 + (x - u)F_1/D^3 + \cdots - (143x - 55u + 55v)F_1/D^4 + \cdots.
$$

Note that $D$ appears as denominator factor in $\hat{H}(x, u, v)$. The Newton polynomial for $H(x, u, v)$ and its roots are $x^2 + (u-v)F_1/D^2 = (x - \tilde{\alpha}(x + \tilde{\alpha}))$, $\tilde{\alpha} = \sqrt{(u^2 + v^2)/(v - u)}$.

Applying the EHC, we can compute the Hensel-series roots $\phi_{\pm}(\infty)(u)$ of $\hat{H}(x, u, v)$, as follows.

$$
\phi_{\pm}(\infty)(u) = \frac{u^4 + v^4}{2(u - v)^2} - \frac{(u^4 + v^4)}{(u - v)^5} + \frac{7(u^4 + v^4)}{(u - v)^8} \pm \tilde{\alpha}
$$

for $\tilde{\alpha} = \sqrt{(u^2 + v^2)/(v - u)}$.

We put $\tilde{\alpha} = \pm \tilde{\alpha}$. We have $\text{res}(F_{1,0}^0, F_{2,0}^0) = (u - v)^2$, hence Theorem 8 tells us that the divergence domain of $\phi_{\pm}(\infty)(u, v)$
runs along the line $u = v$, where three branches of the algebraic function become equal as $u = v = 0$.

We check the convergence/divergence domains of $\phi_2^{(5)}(u, v)$. Since we have now no formula of convergence domain, we compute the following domains $C$ and $D$ for simulating the convergence and divergence domains, respectively.

$$
C_k^k \stackrel{\text{def}}{=} \{ (u, v) \in \mathbb{R}^2 \mid d(u, v) / (|u|^2 + |v|^2)^{k/2} < T \},
$$

$$
d_k(u, v) \stackrel{\text{def}}{=} \min_{i=1, 2, 3} |\alpha_i + \phi_k^{(5)}(u, v) - \phi_i(u, v)|,
$$

$$
D_k^k \stackrel{\text{def}}{=} \{ (u, v) \in \mathbb{R}^2 \mid |\phi_k^{(5)}(u, v)| / (|u|^2 + |v|^2)^{k/2} > T \}.
$$

Here, $T$ is a positive number denoting the threshold, $\phi_i(u, v)$ is a branch of the algebraic function defined by $F(x, u, v)$, and $\lambda = 1$ and $\lambda = 3/2$ in this example. Since $\hat{\alpha} = O(|u|^1)$ and $\hat{\alpha}_3 = O(|u|^{3/2})$, we divide $d(u, v)$ and $|\phi_k^{(5)}(u, v)|$ by $N$ and $N^{3/2}$, respectively, where $N = (|u|^2 + |v|^2)$. We have computed $\phi_1(u), \ldots, \phi_3(u)$ exactly by Mathematica.

![Fig. 2a Domain C_{0.01}^1](image)

![Fig. 2b Domain D_{0.1}^{3/2}](image)

Fig. 2a shows $C_{0.01}^1$ (gray area), $-0.3 < u, v < 0.3$. We see that the divergence domain runs along $u = v = 0$ and it is very thin in a small neighborhood of the origin, as the theory predicts. Fig. 2b shows $D_{0.1}^{3/2}$ (gray area), $-0.3 < u, v < 0.3$. (The thin line $u = v$ appears because $\phi_1^{(5)}$ diverges on the line). Noting that the gray area shows “divergence domain”, we may say that both figures are consistent with each other.

$F(x, u, v)$ tells us that $F(x, u, v) = F_{\text{New}}(x, u, v)$ on the lines determined by $u + v = 0$. So, we compare numerically $u + \phi_1^{(5)}(u, v)$ and $\varphi_i(u, v)$ by transforming $(u, v)$ to $(u', v')$ as $(u, v) = (\exp(i \pi/8)u', \exp(-i \pi/8)v')$. Table I shows the comparison, where we set $v' = 0.05$ and $u' \in \{-0.20, -0.15, -0.10, -0.05, 0.05, 0.10, 0.15, 0.20\}$. We see that the Hensel series with Tschirnhaus transformation approximates the algebraic function pretty well in the region where we have investigated.

<table>
<thead>
<tr>
<th>$u'$</th>
<th>$u + \phi_1^{(5)}(u, v)$</th>
<th>$\varphi_{i0} \varphi_{i1} \varphi_{i2}(u, v)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.20$</td>
<td>$-0.11719 - 0.08263i$</td>
<td>$-0.11862 - 0.08461i$</td>
</tr>
<tr>
<td>$-0.15$</td>
<td>$-0.09730 - 0.01490i$</td>
<td>$-0.09746 - 0.01511i$</td>
</tr>
<tr>
<td>$-0.10$</td>
<td>$-0.07259 - 0.01833i$</td>
<td>$-0.07259 - 0.01834i$</td>
</tr>
<tr>
<td>$-0.05$</td>
<td>$-0.06864 - 0.04768i$</td>
<td>$-0.06864 - 0.04768i$</td>
</tr>
<tr>
<td>$0.00$</td>
<td>$0.00000 + 0.00000i$</td>
<td>$0.00000 + 0.00000i$</td>
</tr>
<tr>
<td>$0.05$</td>
<td>$0.00461 + 0.01913i$</td>
<td>$0.00461 + 0.01913i$</td>
</tr>
<tr>
<td>$0.10$</td>
<td>$0.01094 + 0.06997i$</td>
<td>$0.01125 + 0.08227i$</td>
</tr>
<tr>
<td>$0.15$</td>
<td>$0.17599 + 0.00132i$</td>
<td>$0.17686 + 0.00974i$</td>
</tr>
<tr>
<td>$0.20$</td>
<td>$0.25453 - 0.00440i$</td>
<td>$0.24339 + 0.00897i$</td>
</tr>
</tbody>
</table>

The results in this section may be summarized as follows.

1. The Hensel series are classified by Newton polynomials $F_{C_1}, \ldots, F_{C_l}$; see Fig. 1. The Hensel series $\phi^{(\infty)}_j (j = 1, \ldots, \deg(F_{C_j}))$ arising from $F_{C_j}(x, u)$ are characterized as $\|\phi^{(\infty)}_j\| = O(\|u\|^\lambda)$ for generic $\|u\| < 1$, where $-\lambda_i$ is the slope of the Newton line $L_i$.

2. When the Newton polynomial is not squarefree but contains $m$ multiple roots, the corresponding $m$ Hensel series are characterized by the second dominant terms $\alpha_1, \ldots, \alpha_m$ which are the roots of reconstructed Newton polynomial. Divergence domains of $m$ Hensel series are nearly the same; they start the expansion point and run radially along the zero-points of $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_3$, and their spreads are specified by the slope of Newton line before the Tschirnhaus transformation.

3. The most part of small neighborhood of the expansion point is occupied by the convergence domain.

5. HENSEL SERIES AROUND THE RAMIFICATION POINT

If $F(x, u)$ is irreducible over $C$ then the algebraic function defined by $F(x, u)$ has mutually conjugate $n$-brances, hence it is $n$-valued. On the other hand, if the Newton polynomial is factored as $F_{\text{New}} = F_{\text{New}, 1}F_{\text{New}, 2}$, for example, with $F_{\text{New}, 1}$ and $F_{\text{New}, 2}$ irreducible in $C[x, u]$ and $\deg(F_{\text{New}, 1}) = m < n$, then the Hensel series are divided into two classes, mutually conjugate series of $m$-valuedness and $m$-mutually conjugate ones of $(n - m)$-valuedness; see [17]. Then, one may think that there occurs discrepancy between algebraic functions and Hensel series on many-valuedness. Actually, no discrepancy seems to occur: the Hensel series often jumps from one branch of algebraic function to another when it passes a divergence domain, removing the discrepancy. The jumping has been studied considerably in [15, 16], however the jumping is still a mysterious phenomenon, in particular, around ramification points. In this section, we investigate the jumping near the ramification point. For simplicity, we assume that the leading coefficient has no zero-point near the expansion point.

We first show an example of jumping; the next example is the same as that given in [15], where we could not explain how the jumping occurs actually.

**Example 2** (jumping near the ramification point)

$$
F(x, u, v) = (x - u)(x^2 - 2ux + v^2) + u^4 + v^4.
$$

The Newton polynomial is $F_{\text{New}} = (x - u)(x^2 - 2ux + v^2)$, and we have one rational Hensel series $\phi^{(b)}_2(u, v)$ with initial term $\alpha_1 = u$ and two algebraic Hensel series $\phi^{(b)}_1(u, v)$ with initial terms $\alpha_1 = u \pm \sqrt{w^2 - v^2}$:

$$
\phi^{(\infty)}_i = u + \frac{u^4 + v^4}{u^2 - v^2} + \frac{(u^4 + v^4)}{(u^2 - v^2)^2} + \cdots,
$$

$$
\phi^{(\infty)}_2 = u - \frac{u^4 + v^4}{2(u^2 - v^2)} - \cdots \pm \xi \left[ \frac{3(u^4 + v^4)^2}{8(u^2 - v^2)^3} - \cdots \right],
$$

where $\xi = \sqrt{w^2 - v^2}$.

We have $\text{res}(x - u, (x^2 - 2ux + v^2)) = v^2 - u^2$, hence the Hensel series diverges along the lines $u = v = 0$. 

\[41\]
Let $C_{0.1} : (u, v) = 0.1 \times (\cos t, \sin t)$ be a path on which we trace the real parts of two Hensel series and three branches of the algebraic function. In each of the following figures, black curves show a Hensel series and three gray curves show three algebraic functions. We change the parameter $t$ in the range $0 \leq t \leq 2\pi$.

Fig. 3a $\mathcal{Re}(\phi_1^{(10)})$ traced along $C_{0.1}$

Fig. 3b $\mathcal{Re}(\phi_2^{(10)})$ traced along $C_{0.1}$

Observe that the lower gray curve is separated from upper two. Hence, it is clear that the jumping occurs actually. The gray curves show that the algebraic function on $C_{0.1}$ has a ramification point near the ramification points of algebraic Hensel series, and it seems to be quite difficult to know from which branch to another the Hensel series jumps. The figures suggest that two upper gray curves and lower gray one seem to correspond to $\phi_3^{(10)}$ and $\phi_4^{(10)}$, respectively, but it is wrong. Then, how the jumping occurs? ♦

A key idea to answer the above question is to deform the path along which we trace the Hensel series. Observation A: the Hensel series is a continuous function in its convergence domain, and the algebraic function is also continuous except at the zero-points of the leading coefficient. Therefore, by deforming the path so that it does not pass the divergence domain, we can know the correspondence between the Hensel series and the branches of algebraic function. Another useful idea is to choose the deformed path near the expansion point, where the theoretical analysis becomes simple because the Hensel series (and the algebraic function, too) can be approximated well by lower order terms of the Hensel series. Furthermore, we can choose the deformed path easily there so that it does not pass the divergence domain, because most area in the small neighborhood of the expansion point is the convergence domain.

In the neighborhood of the expansion point, we can find one-to-one correspondence among $\alpha_1, \ldots, \alpha_n$ and $\varphi_1, \ldots, \varphi_n$. So, let $\varphi_i$ corresponds to $\alpha_i$ ($i = 1, \ldots, n$). The item 2 at the end of Sect. 4 says that any divergence domain near the expansion point runs along a line on which two roots of the Newton polynomial are equal. So, let $D$ be a divergence domain running along the zero-points of $\alpha_i(u) - \alpha_j(u)$ ($i \neq j$) near the expansion point. Let $C$ be a path passing $D$. We assume that path $C$ passes only the domain $D$. Let $D$ be a path which is obtained by deforming $C$ so that it does not pass $D$. Since we assumed that the leading coefficient has no zero-point around the expansion point, we have three cases:

**case 1** $\alpha_i(u)$ and $\alpha_j(u)$ cross each other,

**case 2** $\alpha_i(u)$ and $\alpha_j(u)$ are tangent to each other, and

**case 3** $\alpha_i(u)$ and $\alpha_j(u)$ are conjugate to each other.

What happens when we trace a Hensel series along path $C$? The above Observation A leads us to the followings. Case 1): Jumping occurs between $\varphi_i$ and $\varphi_j$. The reason is as follows. When we are on path $D$, there is one-to-one correspondence between the Hensel series and the $n$ branches: $\phi_i^{(k)} \leftrightarrow \varphi_i (i = 1, \ldots, n)$. Consider that we trace the branch $\varphi_i$ along path $C$. Since $\varphi_i(u)$ is continuous even in $D$, we are always on $\varphi_i$ during this tracing. On the other hand, $\varphi_i$ and $\varphi_j$ cross in $D$. Hence, if we look $\varphi_i$ and $\varphi_j$ on the path $D$, they look as if they represent a single Hensel series. In other words, if we think that we are tracing the same Hensel series, we must regard that the branch $\varphi_i$ is renamed to be $\varphi_j$ after passing $D$. By the same reason, we have the next. Case 2): Jumping does not occur.

In case 3), the situation is complicated because, at the ramification point $u_0$, we have $\alpha_1(u_0) = \alpha_2(u_0) = \cdots = \alpha_m(u_0)$, where $\alpha_1, \ldots, \alpha_m$ are mutually conjugate roots of the Newton polynomial. For example, in Example 2, we have $\alpha_i - \alpha_1 = \sqrt{n^2 - 1} = (\alpha_i - \alpha_1)$). Observation A tells us that, in this case too, jumping will occur to a branch which is closest to $\phi_1^{(k)}$ after passing $D$. We can determine which branch is the closest by the numerical comparison of $\phi_i^{(k)}(u)$ and $\varphi_i(u)$ ($i = 1, \ldots, m$). In Example 2, $\phi_1^{(k)}$ is real on path $C_{1.1}$ while only one branch $\varphi_3$ is real in the range $\pi/4 \leq t \leq 3\pi/4$ of the path. Therefore, $\phi_1^{(k)}$ jumps to $\varphi_3$ after passing $D$.

So far, we have displayed the convergence domain and jumping for curves which are rather smooth. The algebraic function changes rapidly near the ramification point: two real curves become complex conjugate just after passing the point. So, one may wonder: does the Hensel series approximate algebraic function well around the ramification point? Example 3 below will answer to this question.

**Example 3** (Hensel series around ramification point) We consider $F(x, u)$ given in Example 2, and show behaviors of the Hensel series and three branches of the algebraic function, by tracing them on the following path $C_s$:

$$C_s : (u, v) = 0.1 \times (\cos t, s i + \sin t), \quad 0 \leq t \leq 2\pi.$$ 

Figs. 4a, 4b, 4c and 4d (in the next page) show $\mathcal{Re}(\phi_i^{(10)})$, $i \in \{1, +,-, -\}$, traced along four circles $C_{9.05}$, $C_{10.10}$, $C_{10.15}$ and $C_{9.20}$, respectively. In each figure, black curve shows $\phi_i^{(10)}$ and gray curves show $\phi_j^{(10)}$. Figs. 5a, 5b, 5c and 5d show $\mathcal{Re}(\varphi_i)$, $i \in \{1, 2, 3\}$, traced along four circles $C_{9.05}$, $C_{10.10}$, $C_{10.15}$ and $C_{9.20}$, respectively. In each figure, black curve shows $\phi_i^{(10)}$ and gray curves show $\varphi_i$ ($i = 1, 2, 3$). We see that three Hensel series approximate three branches of the algebraic function pretty well. ♦
The results in this section may be summarized as follows.

1. The correspondence among Hensel series $\phi_{1}^{(k)}, \ldots, \phi_{n}^{(k)}$ and branches $\varphi_1, \ldots, \varphi_n$ of the algebraic function will change if we trace them by passing the divergence domain of the Hensel series. This looks as if the Hensel series jumps from one branch to another.

2. Although the algebraic function behaves complicatedly, the Hensel series approximate all the branches pretty well, in not only the region where algebraic function behaves smoothly but also the region around the ramification points.

6. REFERENCES


