Parallel Scanning

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CS2101

Plan

1. Problem Statement and Applications
2. Algorithms
3. Applications
4. Implementation in Julia

Plan

Parallel scan: chapter overview

Overview

- This chapter will be the first dedicated to the applications of a parallel algorithm.
- This algorithm, called *the parallel scan*, aka *the parallel prefix sum* is a beautiful idea with surprising uses: it is a powerful recipe to turning serial into parallel.
- Watch closely what is being optimized for: this is an amazing lesson of parallelization.
- Application of parallel scan are numerous:
  - it is used in program compilation, scientific computing and,
  - we already met prefix sum with the counting-sort algorithm!

Prefix sum

Prefix sum of a vector: specification

Input: a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$

Ouput: the vector $\vec{y} = (y_1, y_2, \ldots, y_n)$ such that $y_i = \sum_{i=1}^{j=i} x_j$ for $1 \leq j \leq n$.

Prefix sum of a vector: example

The prefix sum of $\vec{x} = (1, 2, 3, 4, 5, 6, 7, 8)$ is $\vec{y} = (1, 3, 6, 10, 15, 21, 28, 36)$.

Prefix sum: thinking of parallelization (1/2)

Remark

So a Julia implementation of the above specification would be:

```
function prefixSum(x)
   n = length(x)
   y = fill(x[1],n)
   for i=2:n
      y[i] = y[i-1] + x[i]
   end
   y
end

n = 10

x = [mod(rand(Int32),10) for i=1:n]

prefixSum(x)
```

Comments (1/2)

- The i-th iteration of the loop is not at all decoupled from the $(i-1)$-th iteration.
- Impossible to parallelize, right?

Prefix sum: thinking of parallelization (2/2)

Remark

So a Julia implementation of the above specification would be:

```
function prefixSum(x)
   n = length(x)
   y = fill(x[1],n)
   for i=2:n
      y[i] = y[i-1] + x[i]
   end
   y
end

n = 10

x = [mod(rand(Int32),10) for i=1:n]

prefixSum(x)
```

Comments (2/2)

- Consider again $\vec{x} = (1, 2, 3, 4, 5, 6, 7, 8)$ and its prefix sum $\vec{y} = (1, 3, 6, 10, 15, 21, 28, 36)$.
- Is there any value in adding, say, 4+5+6+7 on itw own?
- If we separately have 1+2+3, what can we do?
- Suppose we added 1+2, 3+4, etc. pairwise, what could we do?

Parallel scan: formal definitions

- Let S be a set, let $+ : S \times S \to S$ be an associative operation on S with 0 as identity. Let $A[1 \cdots n]$ be an array of n elements of S.
- Tthe *all-prefixes-sum* or *inclusive scan* of A computes the array B of n elements of S defined by

$$B[i] = \begin{cases} A[1] & \text{if} \quad i = 1 \\ B[i-1] + A[i] & \text{if} \quad 1 < i \leq n \end{cases}$$

- The *exclusive scan* of A computes the array B of n elements of S:

$$C[i] = \begin{cases} 0 & \text{if} \quad i = 1 \\ C[i-1] + A[i-1] & \text{if} \quad 1 < i \leq n \end{cases}$$

- An exclusive scan can be generated from an inclusive scan by shifting the resulting array right by one element and inserting the identity.
- Similarly, an inclusive scan can be generated from an exclusive scan.

Plan

Serial scan: pseudo-code

Here's a sequential algorithm for the inclusive scan.

```
function prefixSum(x)
   n = length(x)
   y = fill(x[1],n)
   for i=2:n
      y[i] = y[i-1] + x[i]
   end
   y
end
```

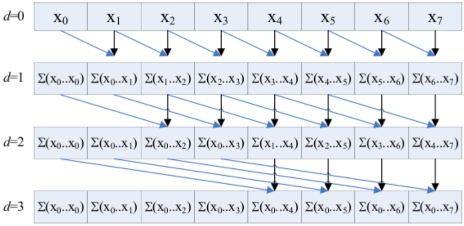
Comments

- Recall that this is similar to the *cumulated frequency computation* that is done in the prefix sum algorithm.
- Observe that this sequential algorithm performa $n - 1$ additions.

Naive parallelization (1/4)

Principles

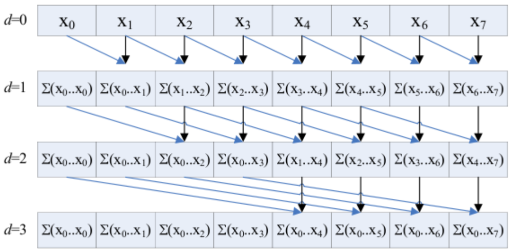
- Assume we have the input array has n entries and we have n workers at our disposal
- We aim at doing as much as possible per parallel step. For simplicity, we assume that n is a power of 2.
- Hence, during the first parallel step, each worker (except the first one) adds the value it owns to that of its left neighbour: this allows us to compute all sums of the forms $x_{k-1} + x_{k-2}$, for $2 \leq k \leq n$.
- For this to happen, we need to work OUT OF PLACE. More precisely, we need an auxiliary with n entries.



Naive parallelization (2/4)

Principles

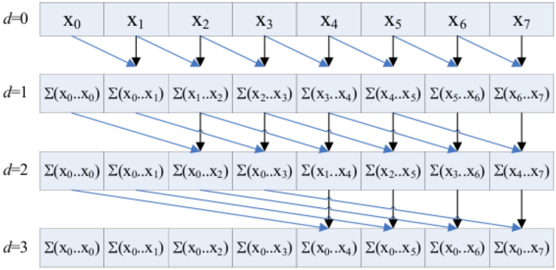
- Recall that the k-th slot, for $2 \leq k \leq n$, holds $x_{k-1} + x_{k-2}$.
- If $n = 4$, we can conclude by adding Slot 0 and Slot 2 on one hand and Slot 1 and Slot 3 on the other.
- More generally, we can perform a second parallel step by adding Slot k and Slot $k - 2$, for $3 \leq k \leq n$.



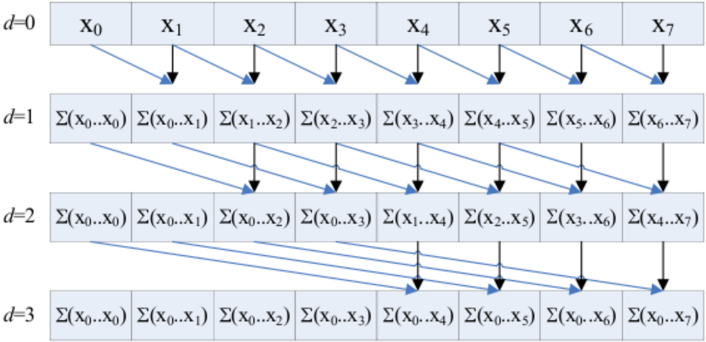
Naive parallelization (3/4)

Principles

- Now the k-th slot, for $4 \leq k \leq n$, holds $x_{k-1} + x_{k-2} + x_{k-3} + x_{k-4}$.
- If $n = 8$, we can conclude by adding Slot 5 and Slot 1, Slot 6 and Slot 2, Slot 7 and Slot 3, Slot 8 and Slot 4.
- More generally, we can perform a third parallel step by adding Slot k and Slot $k - 4$ for $5 \leq k \leq n$.



Naive parallelization (4/4)



Naive parallelization: pseudo-code (1/2)

Input: Elements located in $M[1], \ldots, M[n]$, where n is a power of 2.
Output: The n prefix sums located in $M[n+1], \ldots, M[2n]$.
Program:

```
Active Proocessors P[1], ...,P[n];
// id the active processor index
for d := 0 to (log(n) -1) do
if d is even then
  if id > 2^d then
      M[n + id] := M[id] + M[id - 2^d]
  else
      M[n + id] := M[id]
  end if
else
  if id > 2^d then
      M[id] := M[n + id] + M[n + id - 2^d]
  else
      M[id] := M[n + id]
  end if
end if
if d is odd then M[n + id] := M[id] end if
```

Naive parallelization: pseudo-code (2/2)

Pseudo-code

```
Active Proocessors P[1], ...,P[n]; // id the active processor index
for d := 0 to (log(n) -1) do
if d is even then
  if id > 2^d then
      M[n + id] := M[id] + M[id - 2^d]
  else
      M[n + id] := M[id]
  end if
else
  if id > 2^d then
      M[id] := M[n + id] + M[n + id - 2^d]
  else
      M[id] := M[n + id]
  end if
end if
if d is odd then M[n + id] := M[id] end if
```

Observations

- $M[n+1], \ldots, M[2n]$ are used to hold the intermediate results at Steps $d = 0, 2, 4, \ldots (\log(n) - 2)$.
- Note that at Step d, $(n - 2^d)$ processors are performing an addition.
- Moreover, at Step d, the distance between two operands in a sum is 2^d.

Naive parallelization: analysis

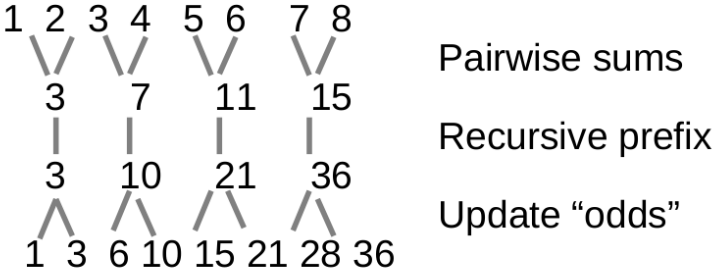
Recall

- $M[n+1], \ldots, M[2n]$ are used to hold the intermediate results at Steps $d = 0, 2, 4, \ldots (\log(n) - 2)$.
- Note that at Step d, $(n - 2^d)$ processors are performing an addition.
- Moreover, at Step d, the distance between two operands in a sum is 2^d.

Analysis

- It follows from the above that the naive parallel algorithm performs $\log(n)$ parallel steps
- Moreover, at each parallel step, at least $n/2$ additions are performed.
- Therefore, this algorithm performs at least $(n/2)\log(n)$ additions
- Thus, this algorithm is not work-efficient since the work of our serial algorithm is simply $n - 1$ additions.

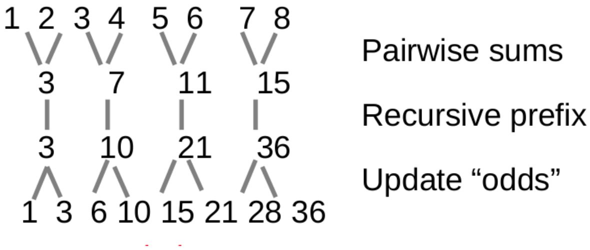
Parallel scan: a recursive work-efficient algorithm (1/2)



Algorithm

- Input: $x[1], x[2], \ldots, x[n]$ where n is a power of 2.
- Step 1: $(x[k], x[k-1]) = (x[k] + x[k-1], x[k]$ for all even k's.
- Step 2: Recursive call on $x[2], x[4], \ldots, x[n]$
- Step 3: $x[k-1] = x[k] - x[k-1]$ for all even k's.

Parallel scan: a recursive work-efficient algorithm (2/2)



Analysis

- Since the recursive call is applied to an array of size $n/2$, the total number of recursive calls is $\log(n)$.
- Before the recursive call, one performs $n/2$ additions
- After the recursive call, one performs $n/2$ subtractions
- Elementary calculations show that this recursive algorithm performs at most a total of $2n$ additions and subtractions
- Thus, this algorithm is work-efficient. In addition, it can run in $2\log(n)$ parallel steps.

Plan

Application to Fibonacci sequence computation

$$\boxed{F_{n+1} = F_n + F_{n-1}}$$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$

Can compute all F_n by matmul_prefix on

$$\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

Application to parallel addition (1/2)

Example		Notation
1 0 1 1 1	Carry	c_2 c_1 c_0
1 0 1 1 1	First Int	a_3 a_2 a_1 a_0
1 0 1 0 1	Second Int	b_3 b_2 b_1 b_0

Application to parallel addition (2/2)

Example							Notation			
1	0	1	1	1		Carry	c_2	c_1	c_0	
	1	0	1	1	1	First Int	a_3	a_2	a_1	a_0
	1	0	1	0	1	Second Int	a_3	b_2	b_1	b_0

$c_{-1} = 0$

(addition mod 2)

for $i = 0 : n-1$

$s_i = a_i + b_i + c_{i-1}$

$c_i = a_i b_i + c_{i-1}(a_i + b_i)$

end

$$\begin{bmatrix} c_i \\ 1 \end{bmatrix} = \begin{bmatrix} a_i + b_i & a_i b_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{i-1} \\ 1 \end{bmatrix}$$

Plan

Serial prefix sum: recall

```
function prefixSum(x)
   n = length(x)
   y = fill(x[1],n)
   for i=2:n
      y[i] = y[i-1] + x[i]
   end
   y
end

n = 10

x = [mod(rand(Int32),10) for i=1:n]

prefixSum(x)
```

Parallel prefix multiplication: live demo (1/4)

```
julia> n = 3000; k = 3;

julia> v=[randn(n,n) for i=1:2^k];

julia> w=copy(v);

julia> @time for i=2:2^k
         w[i]=w[i-1]*v[i];
       end

elapsed time: 32.458615523 seconds (516419092 bytes allocated)
```

Comments

- In the above we do a prefix multiplication with random matrices.
- We have $n = 2^k$.
- After randomly generating the matrices, we do the serial prefix mult.

Parallel prefix multiplication: live demo (2/4)

```
julia> l
4

julia> k
3

julia> p=workers()
4-element Array{Int64,1}:
 2
 3
 4
 5

julia> l=length(p)
4

julia> if l<2^k;
         addprocs(2^k-l+(l==1));
         p=workers();
        end
8-element Array{Int64,1}:
 2
 3
 4
 5
 6
 7
 8
 9
```

Comments

- We enforce 2^k worker processors.

Parallel prefix multiplication: live demo (3/4)

```
r=[@spawnat p[i] randn(n,n) for i=1:2^k ]
8-element Array{Any,1}:
 RemoteRef(2,1,1)
 RemoteRef(3,1,2)
 RemoteRef(4,1,3)
 RemoteRef(5,1,4)
 RemoteRef(6,1,5)
 RemoteRef(7,1,6)
 RemoteRef(8,1,7)
 RemoteRef(9,1,8)

julia> s=copy(r)
8-element Array{Any,1}:
 RemoteRef(2,1,1)
 RemoteRef(3,1,2)
 RemoteRef(4,1,3)
 RemoteRef(5,1,4)
 RemoteRef(6,1,5)
 RemoteRef(7,1,6)
 RemoteRef(8,1,7)
 RemoteRef(9,1,8)
```

Comments

- We create remote random matrices.

Parallel prefix multiplication: live demo (4/4)

```
 @time @sync begin
    for j=1:k
      for i in [2^j:2^j:2^k]
        s[i]=@spawnat p[i] fetch(s[i-2^(j-1)])*fetch(s[i]);
      end
    end
   for j=(k-1):-1:1
     for i in [3*2^(j-1):2^j:2^k]
       s[i]=@spawnat p[i] fetch(s[i-2^(j-1)])*fetch(s[i]);
     end
   end
end

elapsed time: 20.513351976 seconds (5045520 bytes allocated)
```

Comments

-
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-
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