

Chapter 4

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4.1 Simplifying propositional formulas.

In the last class, we talked about logical equivalences, and listed a few most notable logical identities. Now we can apply these identities to simplify propositional formulas.

Example 1.

$$\begin{aligned} & (p \wedge q) \vee \neg(\neg p \vee \neg q) \\ \iff & (p \wedge q) \vee (\neg\neg p \wedge \neg\neg q) && \text{Apply DeMorgan's} \\ \iff & (p \wedge q) \vee (p \wedge q) && \text{Double Negation (twice)} \\ \iff & (p \wedge q) && \text{Idempotence} \end{aligned}$$

Notice that the logic identities are stated only for the logical connectives \wedge, \vee, \neg . In order to deal with \rightarrow and \iff we use their definitions: for example, $A \rightarrow B$ becomes $\neg A \vee B$.

Here we go over a few more examples similar to your assignment, simplifying formulas until they are as small as we can (reasonably easily) get using the rules from the last lecture. For the assignment question on negation, you would mainly apply DeMorgan and Double Negation to a negated formula, which is a much simpler procedure.

Example 2.

$$\begin{aligned} p &\leftrightarrow ((q \wedge \neg r) \rightarrow q) \\ \iff p &\leftrightarrow (\neg(q \wedge \neg r) \vee q) && \text{Definition of } \rightarrow \\ \iff p &\leftrightarrow ((\neg q \vee \neg\neg r) \vee q) && \text{DeMorgan} \\ \iff p &\leftrightarrow ((\neg\neg r \vee \neg q) \vee q) && \text{Commutativity} \\ \iff p &\leftrightarrow (\neg\neg r \vee (\neg q \vee q)) && \text{Associativity (dropping parentheses)} \\ \iff p &\leftrightarrow (\neg\neg r \vee T) && \text{Definition of T} \\ \iff p &\leftrightarrow T && \text{Identity} \\ \iff p &&& \text{Because } \leftrightarrow \text{ is an equivalence} \end{aligned}$$

The last step could be done more formally as follows:

$$\begin{aligned} p &\leftrightarrow T \\ \iff (p &\rightarrow T) \wedge (T \rightarrow p) && \text{Definition of } \leftrightarrow \\ \iff (\neg p &\vee T) \wedge (\neg T \vee p) && \text{Definition of } \rightarrow \\ \iff (\neg p &\vee T) \wedge (F \vee p) && \text{Definition of } F \\ \iff T &\wedge (F \vee p) && \text{Identity} \\ \iff (F &\vee p) && \text{Identity} \\ \iff p &&& \text{Identity} \end{aligned}$$

4.2 Conditional statements

Here is a puzzle that I was planning to give you in the last lecture.

Puzzle 3 (Wason Selection Task). You see 4 cards on the table; all cards have a letter on one side and a number on the other. On those four cards you see written “A”, “D”, “4” and “7”. Which cards do you need to turn over to verify the statement “If a card has a vowel on one side, then it has an even number on the other”?

Pretty much everybody immediately sees that a card with “A” needs to be turned over to verify that it has an even number on the other side. But for the other cards, many people would say “4”. However, turning “4” over does not give us much information about the statement: whether it has a vowel or a consonant, it still does not prove or disprove the statement: there are no restrictions on cards with consonants on them. To see which card to turn over, think of a *contrapositive* statement: “if a card does not have an even number on one side, then it does not have a vowel on the other”. So the card that has to be checked is the card with a “7”.

This puzzle gives an example of a conditional statement, that is, a statement of the form “if A then B ”, $A \rightarrow B$. Recall that we logically define $p \rightarrow q \iff (\neg p \vee q)$. Here is another example of a conditional statement. Note that its negation is *not* a conditional statement itself, but rather an “and”: an implication is false in only one situation, when A is true and B is false, and the negation of the implication states that it is indeed the case. Alternatively, you can verify it by applying DeMorgan’s law and double negations to the $\neg(\neg A \vee B)$ formula.

Example 3. If Jane is in London, then she is in England.
 Negation: Jane is in London, and she is not in England (e.g., London, Ontario).

We use the following terminology when talking about conditional statements:

- 1) *Contrapositive* of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. True whenever the original implication is.

Proof. Recall that $(p \rightarrow q) \iff (\neg p \vee q)$. Now,

$$\begin{aligned} & \neg q \rightarrow \neg p \\ \iff & (\neg \neg q \vee \neg p) && \text{Definition of } \rightarrow \\ \iff & (q \vee \neg p) && \text{Double negation} \\ \iff & (\neg p \vee q) && \text{Commutativity} \iff (p \rightarrow q) && \text{Definition of } \rightarrow \end{aligned}$$

Thus, a contrapositive of an if-then statement is logically equivalent to the original statement. □

In the cards example, the contrapositive is “if a card does not have an even number on one side, then it does not have a vowel on the other side”. Thus, cards with odd number facing up need to be checked.

- 2) *Converse* $q \rightarrow p$ and *inverse* $\neg p \rightarrow \neg q$. Contrapositives of each other, can have a different truth value from $p \rightarrow q$.

So a converse in the cards example would be “if a card has an even number on one side, then it has a vowel on the other”. You can see that this does not have the same truth value as the original statement: the first one is true when the card with 4 has a consonant (say, B) on the other side, but the converse would be falsified by this scenario.

- 3) *Sufficient* condition: p is sufficient for q if $p \rightarrow q$. The “if” part of if-then.
Necessary condition: q is necessary for p if $\neg q \rightarrow \neg p$, that is, $p \rightarrow q$. The “then” part of “if-then”.

So if we know that the puzzle statement is true for the 4 given cards, then it is sufficient to know that a card has a vowel (say A) on one side to conclude that the other side

has an even number. And, if a card has say A on one side, it is necessary for it to have an even number on the other for the whole “if card has a vowel on one side then it has an even number on the other” to be true.

4) If and only if (p iff q , $p \leftrightarrow q$) means $(p \rightarrow q) \wedge (q \rightarrow p)$.

In many daily situations people use “if.. then..” construct implying a biconditional (if and only if), and sometimes even the “only if” direction. For example, a parent telling a kid “if you eat your veggies then you’ll get to eat your desert” probably means the converse – if the kid doesn’t eat her veggies, then she’d be punished by not getting her desert. In that case, the context would allow us to interpret the sentence the way it was intended (although a logic-savvy kid would catch her dad on that). But in sciences, medicine, law where it is very important to avoid ambiguity; there, “if.. then” statements should be interpreted according to the rules of logic.

In a book “Logic made easy” by Deborah Bennet (where she also talks about the Wason Selection test and the puzzle I will give at the end of the class) she mentions a common mistake doctors made when interpreting the meaning of sensitivity of a test (probability of a correct positive answer). The sensitivity is often stated as in “if a person is sick, then the probability of the test being positive is 90%”. Now suppose somebody tested positive. What is the probability that they are indeed sick? It is tempting to say “90%”; however this number is irrelevant here: it could have been a test that often returns positive even for healthy people.

For example, consider airport security. Suppose somebody says that if a person is a terrorist carrying a weapon, then the metal detector rings with probability 90%. But this is definitely not the same as saying that every time the detector rings, with 90% chance it is a terrorist: much more likely a person forgot to take keys or coins out of their pocket.

You can see how this can come up in e.g. American court system a lot: saying “if a subject committed a crime, then his fingerprint test will come up positive with 99% probability, and it did come up positive” will make jurors think that the accused really committed the crime. If they know logic, though, they will ask: “but what is the probability that a test will be positive for an innocent person”?

Let me finish with a version of another puzzle from the same book.

Puzzle 4 (Colours and shapes). Suppose you meet a parent with a little baby girl; the parent tells you that she is attracted to some colours and some shapes, and goes for a toy which has either the colour or the shape she likes (or both, of course). In her play area, you see several plush toys: a blue square, a blue circle, a yellow square and a yellow circle. You see the baby reaching for a blue circle. What, if anything, can you infer about her liking or not liking other three toys?

Chapter 5

5.1 Proof techniques

Example 1. Consider the sentence “if n is divisible by 4, then n is divisible by 2” (we will use the notation $n|4$ to mean n is divisible by 4). This is an if-then statement. Its contrapositive is “if $n \nmid 2$ then $n \nmid 4$. That is, if n is an odd number then it is definitely not divisible by 4. So $n|4$ is sufficient for $n|2$ (if n is divisible by 4, it is sufficient for n to be divisible by 2). On the other hand, $n|2$ is necessary for $n|4$.

- 1) *Direct proof:* show that if p is true directly.
- 2) *Proof by contrapositive:* instead of $p \rightarrow q$ prove $\neg q \rightarrow \neg p$.

Lemma 1. *If n^2 is even, then n is even.*

Proof. We will show this by showing that if n is odd, then n^2 is odd. If n is odd, then $n = 2k + 1$ for some k . Then $(2k + 1)^2 = 2(2k^2 + 2k) + 1$, which is an odd number. This proves that if n is odd, then n^2 is odd, thus proving the contrapositive of if n^2 is even then n is even, and so proving the statement “if n^2 is even then n is even” itself. \square

- 3) *Proof by contradiction:* to show that p is true, show that $\neg p \rightarrow F$.
It is easy to show that $(\neg p \rightarrow F)$ is logically equivalent to p : just note that $(\neg\neg p \vee F) \iff (p \vee F) \iff p$ by applying the definition of implication followed by the double negation law followed by the identity law.

Theorem 1. *$\sqrt{2}$ is irrational.*

Proof. Recall that a number is called *rational* if it can be represented as an (irreducible) fraction of two integers. Assume, for the sake of contradiction, that $\sqrt{2}$ is rational: that is, there are integers m and n which do not have any common divisors > 1 such that $\sqrt{2} = m/n$.

- If $\sqrt{2} = m/n$ then $(\sqrt{2})^2 = m^2/n^2$.
- From here, $2n^2 = m^2$, which means that m^2 is even.

- By the lemma above, then m is even, so $m = 2k$ for some k .
- Then $m^2 = 4k^2$. So $2n^2 = 4k^2$, and, dividing by 2, $n^2 = 2k^2$. So n^2 is even.
- Using the lemma again, conclude that n is even.
- So both n and m are even, but we assumed that m and n do not have a non-trivial common divisor. This is a contradiction.

□

4) *Proof by cases:* to show that p is true, prove $(q \rightarrow p) \wedge (\neg q \rightarrow p)$.

Lemma 2. For any natural number n , $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$

Here, $\lfloor k \rfloor$ is a *floor* of a (real) number, defined to be the largest integer smaller than or equal to k . For example, $\lfloor 5/2 \rfloor = 2$, and $\lfloor 4/2 \rfloor = 2$. Similarly, a *ceiling* of a number is the smallest integer larger than or equal to that number. So $\lceil 5/2 \rceil = 3$ and $\lceil 4/2 \rceil = 2$. For an integer, both its floor and its ceiling are equal to that integer itself; for a non-integer, the floor is its rounded-down value and the ceiling is rounded up.

Proof. Case 1: n is even. Then $\lfloor (n+1)/2 \rfloor = n/2 = \lceil n/2 \rceil$.

Case 2: n is odd. Then $\lfloor (n+1)/2 \rfloor = (n+1)/2 = \lceil n/2 \rceil$. □

Puzzle 4. A from the island of knights and knaves said: “If I am a knight, then I’ll eat my hat!”. Prove that A will eat his hat.

Chapter 6

6.1 More about implications

In the previous lecture we had the following knights-and-knaves puzzle:

A said: “If I am a knight I’ll eat my hat!”. Show that A will eat his hat.

Note that this statement is an implication. Lets set p : “A is a knight” and q : “A will eat his hat”. Then what A said is $p \rightarrow q$. Now, consider the truth table for the statement saying that what A said is truth if and only if A is a knight (that is, $(p \rightarrow q) \leftrightarrow p$.)

p	q	$p \rightarrow q$	$(p \rightarrow q) \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

So the only situation which is possible (that is, A is a knight and he told the truth or A is a knave and he lied) is when A is a knight and he told the truth. And since what he said is true, and the left hand side of the implication (that is, p) is true, q also has to be true. So A will eat his hat.

Let us look what happens when A is a knave. Then what he said must be false. But the only time $p \rightarrow q$ is false is when p is true and q is false (that is, $\neg(p \rightarrow q) \iff (p \wedge \neg q)$, you can check by the definition of implication and the DeMorgan’s law that this holds). So here is where the contradiction comes: the only time the implication could be false (that is, uttered by a knave) is when its left-hand-side is true (that is, A is a knight).

One of the main things to remember about the implication is that **falsity can imply anything!**. That is, if pigs can fly, then $2+2=5$, and also if pigs can fly then $2+2=4$. Both of these are true implications, provided that pigs cannot fly. No matter what kind of statement is q , the implication $F \rightarrow q$ is always true.

A brilliant example that shows that falsity can imply anything via a valid argument was presented by the famous logician Bertrand Russell (author of the “Russell’s paradox” that

we will see later in the course.)

Example 1. Bertrand Russell: “If $2+2=5$, then I am the Pope”.¹

Proof. If $2+2=5$ then $1=2$ by subtracting 3 from both sides.

Bertrand Russell and Pope are two people.

Since $1=2$, Bertrand Russell and Pope are one person.

□

Note that the steps of this proof are perfectly fine logically. Every line follows from the previous line correctly. The strange conclusion follows instead from $2+2=5$ being a false statement.

6.2 Valid and invalid arguments

Bertrand Russell’s proof of “if $2+2=5$, then I am the Pope” is an example of a valid argument (that is, every step logically followed from the previous). In this subsection we will discuss what constitutes a valid argument, and later discuss methods for proving that arguments are valid.

We will start with a (propositional form of) Aristotle’s classic example of a valid argument (his original version had a quantifier; we will get to that in a few lectures).

If Socrates is a man, then Socrates is mortal
Socrates is a man
∴ Socrates is mortal

Terminology: the final statement is called *conclusion*, the rest are *premises*. The symbol ∴ reads as *therefore*. An argument is *valid* if no matter what statements are substituted into variables, if all premises are true then the conclusion is true. That is, if there are formulas P_1, \dots, P_k that are all the premises (one per line, as above), and a formula C is the conclusion, then an argument is valid iff $P_1 \wedge P_2 \wedge \dots \wedge P_k \rightarrow C$ is a tautology.

The valid form of argument is called *rules of inference*. The most known is called *Modus Ponens* (“method of affirming”). Its contrapositive is called *Modus Tollens* (“method of denying”):

¹According to other sources, the statement Bertrand Russell was proving was “if $1+1=1$, then I am the Pope”. In this case, the first line can be omitted.

Modus Ponens

If p then q
 p
 $\therefore q$

Modus Tollens

If p then q
 $\neg q$
 $\therefore \neg p$

This is another way to describe “proof by contrapositive”. Similarly we can write the proof by cases, by contradiction, by transitivity and so on. They can be derived from the original logic identities. For example, modus ponens becomes $((p \rightarrow q) \wedge p) \rightarrow q$.

Example 2. The general form of the proof by cases above can be written as follows:

$p \vee q$	“ n is even or n is odd”
$p \rightarrow r$	“if n is even then $\lfloor (n + 1)/2 \rfloor = \lceil n/2 \rceil$ “
$q \rightarrow r$	“if n is odd then $\lfloor (n + 1)/2 \rfloor = \lceil n/2 \rceil$ “
$\therefore r$	“therefore $\lfloor (n + 1)/2 \rfloor = \lceil n/2 \rceil$ “

To show that an argument is invalid give a truth table where all premises are true and the conclusion is false. That is, show that a propositional formula of the form $\wedge(\text{of premises}) \rightarrow \text{conclusion}$ is not a tautology.

Example 3. For example, although modus ponens is valid, the following is not a valid inference:

$p \rightarrow q$
 q **INVALID ARGUMENT!**
 $\therefore p$

We can write it as $((p \rightarrow q) \wedge q) \rightarrow p$ and show that this is not a tautology using a truth table.

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$(p \rightarrow q) \wedge q \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

The truth table showed us a situation when both premises $(p \rightarrow q)$ and q are true, but the conclusion p is false. Therefore, $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology and thus the argument based on it is not a valid argument.

However, note that if any of the premises are false, a valid argument can produce a most weird conclusion: remember that if p is false in $p \rightarrow q$, then $p \rightarrow q$ is true for any value of q . Thus, if a premise is false, a false conclusion can be reached (even though an argument is valid); a true conclusion can be reached as well.