CS2209A 2017
Applied Logic for Computer Science

Lecture 10

Predicate Logic

- First-order formula

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Propositions with parameters

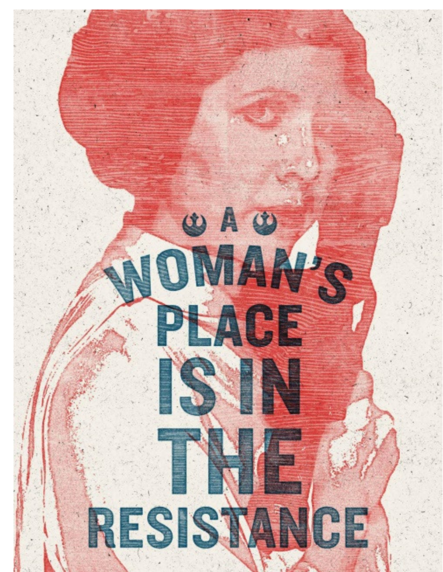
- "$x^2 \geq x$"
- Is it true?
  - Answer: depends.
    - When x is an integer, -1, 0, 1, 2, … ?
    - How about x is 0.5, 0.1, 1/3 …, real or rational numbers which is smaller than 1?
- How can we represent such type of knowledge and apply them in reasoning?

Puzzle

- Susan is 28 years old, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in anti-nuke demonstrations.

  Please rank the following possibilities by how likely they are. List them from least likely to most likely. Susan is:

1. a kindergarden teacher
2. works in a bookstore and takes yoga classes
3. an active feminist
4. a psychiatric social worker
5. a member of an outdoors club
6. a bank teller
7. an insurance salesperson
8. a bank teller and an active feminist



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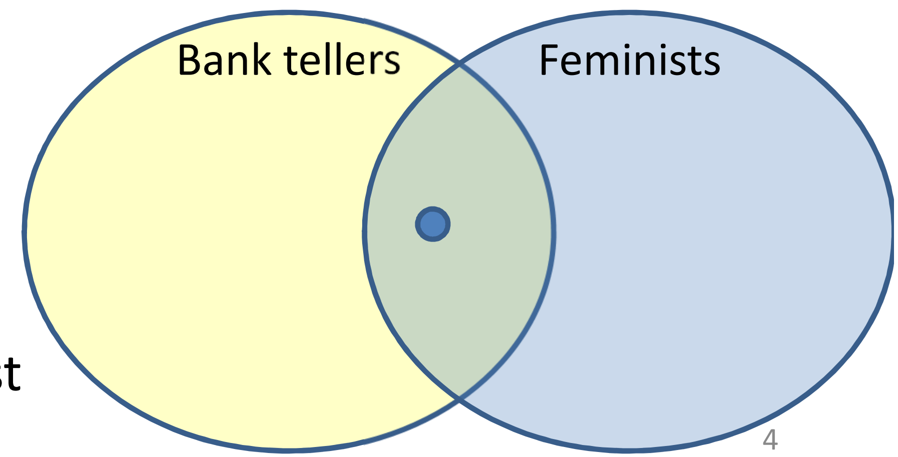


Scenarios and sets

- Want to reason about more general scenarios

- Rather than just **true/false**, vary over objects:
  - even numbers, integers, primes
  - people that are and are not bank tellers,
  - pairs of animals in the same ecosystem ...

- Want **multiple properties** of these objects:
  - an even number that is divisible by 4 and > 10,
  - a person that is also a bank teller ...

Sets

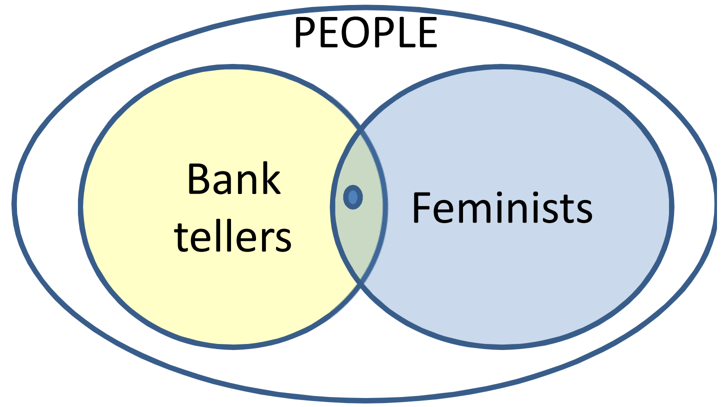
- A **set** is a collection of objects.
  - S_1={1, 2, 3}, S_2 = {Cathy, Alan, Keiko, Daniela}
  - S_3 = [-1, 2] (real numbers from -1 to 2, inclusive)
  - PEOPLE = {x | x is a person living on Earth now}
    - {x | such that x ... } is called **set builder notation**
  - S_4 = { (x,y) | x and y are people, and x is a parent of y}
  - BANKTELLERS = { x | x is a person who is a bank teller}
- The order of elements does not matter.
- There are no duplicates.

Special sets

- Notation for some **special sets** (much of which you are likely to have seen):
  - **Empty set** $\emptyset$
  - **Natural numbers** $\mathbb{N} = \{1, 2, 3, \dots\}$ (sometimes with 0)
  - **Integers** $\mathbb{Z} = \{\dots - 2, -1, 0, 1, 2, \dots\}$
  - **Rational numbers** $\mathbb{Q} = \left\{\frac{m}{n} \;\middle|\; m, n\ in\ \mathbb{Z}, n \neq 0\right\}$
  - **Real numbers** $\mathbb{R}$
  - **complex numbers** $\mathbb{C}$

Set elements

- $\boldsymbol{a} \in \boldsymbol{S}$ means that an element a is in a set S, and $\boldsymbol{a} \notin \boldsymbol{S}$ that a is not in S. That is, $\boldsymbol{a} \in \boldsymbol{S} \equiv \neg\,(\boldsymbol{a} \notin \boldsymbol{S})$
  - Susan ∈ PEOPLE, Susan ∉ BANKTELLERS
  - $0.23 \in [-1, 2]$. $3.14 \notin [-1, 2]$



- Also, write $\boldsymbol{x} \in \boldsymbol{S}$ for **a variable $\boldsymbol{x}$**.
  - BANKTELLERS = { $x \in PEOPLE$ | x is a bank teller}

- How do we generalize sentences like "x is a bank teller", where x is an element of some set?

Predicates

- A **predicate** $P(\mathrm{x}_1, \dots, x_n)$ is a "proposition with parameters", where values of the parameters $x_1, \dots, x_n$ come from some sets $S_1, \dots, S_n$, called their **domains** or **universes.**
  - $P(x)$ is true for some values of $x \in S$, and false for the rest.
    - Even(x) for $x \in \mathbb{Z}$, Feminist(y) for y∈ $PEOPLE$…
    - Here, domain of x is $\mathbb{Z}$, and domain of y is $PEOPLE$
    - Even(y) is not defined for y∈ $PEOPLE$, only for elements of $\mathbb{Z}$.
  - A predicate can have several variables:
    - $x > y$, for $x, y \in \mathbb{R}$
    - $\mathrm{Divides}(x, y)$, which is true for $x, y \in \mathbb{Z}$ such that x divides y.
- When all variables in a predicate are replaced with specific elements (**instantiated**), the predicate becomes a **proposition**.
  - "Even(3)" is false. "Feminist(Susan)" is true.

Predicates

- We can make formulas out of predicates the same way as we did for propositions using connectives, but now our formulas have **free variables**:
  - $Even(x) \vee Divides(3, x) \rightarrow \neg Prime(x)$
  - $Feminist(x) \wedge Bankteller(x)$
  - Now scenarios can correspond to values of x.
    - The first formula is false for x=2, because $Even(2) = true$, but $\neg Prime(2) = false$.
- This is called **predicate logic** (or **first-order logic**), as opposed to propositional logic we did so far.

Quantifiers: universal (∀, for all)

- Theorems often look like this: "For all x , the following is true", and then a formula with x as a free variable.
  - For all $x \in \mathbb{Z}$, $Divides(6, x) \rightarrow Divides(3, x)$
  - For all $n \in \mathbb{N}$, $n > 4 \rightarrow 2^n > n^2$

- We write this in **predicate logic** using a **universal quantifier** (written as ∀):
  - $\forall x \in \mathbb{Z},\ Divides(6, x) \rightarrow Divides(3, x)$
  - $\forall n \in \mathbb{N},\ n > 4 \rightarrow 2^n > n^2$

Quantifiers: universal (∀)

- In general, for every formula F of predicate logic with a free variable x , we can write $\forall \boldsymbol{x} \in \boldsymbol{S},\ \boldsymbol{F}(\boldsymbol{x})$
  - The formula "$\forall x \in S,\ F(x)$" is true if and only if $F(a)$ is true for every $a \in S$.
  - That is, if $a_1, a_2, \dots, a_n, \dots$ is a list of all elements of S, then $\forall x \in S,\ F(x)$ is true if and only if $F(a_1) \wedge F(a_2) \wedge \cdots \wedge F(a_n) \wedge \cdots$ is true.
  - If there are no more free variables or quantifiers in F, then $\forall x \in S,\ F(x)$ is true if and only if $F(a_1) \wedge F(a_2) \wedge \cdots \wedge F(a_n) \wedge \cdots$ is a tautology.

Negating the universal

- What is the **negation** of "All"? When would a statement "$\forall x \in S,\ F(x)$" be false?
  - All girls hate math.
  - No!
    - All girls love math?
    - Some girls do not hate math!
  - Everybody in O'Brian family is tall
    - No, Jenny is O'Brian and she is quite short.
  - It is foggy all the time, every day in London
    - No, sometimes it is not foggy (like yesterday).

Quantifiers: existential (∃)

- To prove that something is not always true, we give a counter example. In predicate logic, use existential quantifier ∃.
- $\exists \boldsymbol{x} \in \boldsymbol{S}, \boldsymbol{F}(\boldsymbol{x})$ is true if and only if there exist some a ∈ S such that $F(a)$ is true (and we don't care for the rest). That is, when $\boldsymbol{F}(\boldsymbol{a_1}) \lor \boldsymbol{F}(\boldsymbol{a_2}) \lor \cdots \lor \boldsymbol{F}(\boldsymbol{a_n}) \lor$ … is true.
  - $\exists x \in \mathbb{N},\ Even(x) \land Prime(x)$.
- Negating the universal:
  $\neg\left(\forall \boldsymbol{x} \in \boldsymbol{S}, \boldsymbol{F}(\boldsymbol{x})\right) \equiv \exists \boldsymbol{x} \in \boldsymbol{S},\ \neg \boldsymbol{F}(\boldsymbol{x})$
- Once a variable is quantified, it is no longer free.
  - x is free in $Even(x) \land Prime(x)$,
  - But $\exists x \in \mathbb{N},\ Even(x) \land Prime(x)$ has no free variables.

Summary

- A formula $\forall x \in S,\ F(x)$, where $F(x)$ is a formula containing predicates, is true (on the domain of predicates) if it is true on every value of x from the domain. Here, $\forall$ is called a ***universal quantifier***, usually pronounced as “for all ...".
- A formula $\exists x \in S,\ F(x)$, where $F(x)$ is a formula containing predicates, is true (on the domain of predicates) if it is true on some value of x from the domain. Here, $\exists$ is called an ***existential quantifier***, usually pronounced as “exists ...".
- Universal and existential quantifiers are opposites of each other.
  - $\neg\left(\forall \boldsymbol{x} \in \boldsymbol{S}, \boldsymbol{F}(\boldsymbol{x})\right) \equiv \exists \boldsymbol{x} \in \boldsymbol{S},\ \neg \boldsymbol{F}(\boldsymbol{x})$
  - $\neg\left(\exists \boldsymbol{x} \in \boldsymbol{S}, \boldsymbol{F}(\boldsymbol{x})\right) \equiv \forall \boldsymbol{x} \in \boldsymbol{S},\ \neg \boldsymbol{F}(\boldsymbol{x})$

First-order formula

- Variables that occur under quantifiers (that is, as $\forall x$ or $\exists x$) are called *bound variables*; if a variable is not bound, then it is called a *free variable*.
- A **predicate** is a first-order formula (possibly with free variables).
- A $\wedge$, $\vee$, $\neg$ of a first-order formula is a first-order formula.
- If a formula $F(x)$ has a free variable, then $\forall x\, F(x)$ and $\exists x\, F(x)$ are also first-order formulas.
- Note that this definition is very similar to the definition of propositional formulas except here there are **predicates** instead of propositions and there are **quantifiers**.
- The **truth value** of a first-order formula is defined when either free variables are given some specific values, or all variables are bound.

Quantifiers in English

- **Universal quantifier**: usually “every”, “all”, “each”, “any”.
  - Every day it is foggy.
    Each number is divisible by 1.

- **Existential quantifier**: “some”, “a”, “exists”
  - Some students got 100% on both quizzes.
  - There exists a prime number greater than 100.

- The word “any” can mean either!

Quantifiers in English: “any”

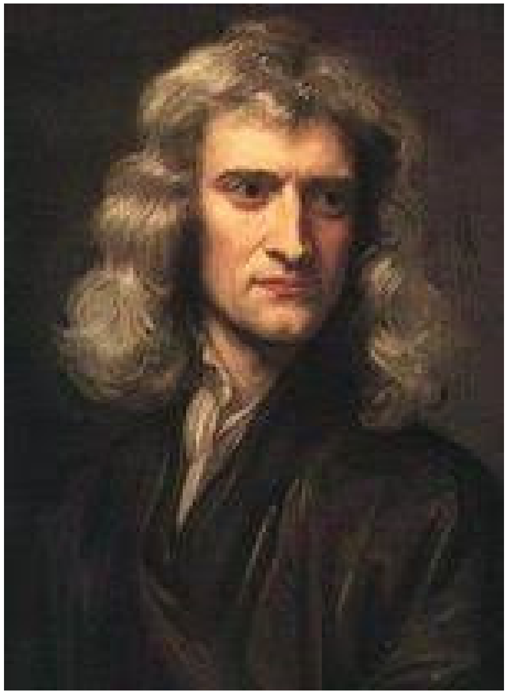
- “Any” can mean “every”:

  - Any student in our class knows logic 
  - Every student in our class knows logic. 

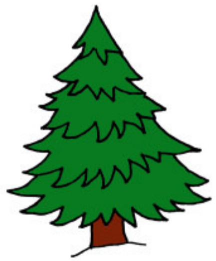
- But “any” can also mean “some”!

  - I will be happy if I do well on every quiz. 
  - I will be happy if I do well on any quiz. 

“What we know is a drop. What we do not know is an ocean.”



"NOT" makes life harder



- It is easy to visualize a tree, a number, or a person. It is harder to visualize a "not a tree", "not a number" or "not a person"
- So "NOT (ALL trees have leaves)" is harder to understand than "some trees have something other than leaves (e.g., needles).
- Here we really need to pay attention to the **domain of quantifiers**! **It stays the same when negating.**
  - Not all integers are even: $\neg\left(\forall \boldsymbol{x} \in \mathbb{Z}\ \boldsymbol{Even}(\boldsymbol{x})\right)$

$$\equiv$$

  - Some integers are not even: $\exists \boldsymbol{x} \in \mathbb{Z}\ \neg \boldsymbol{Even}(\boldsymbol{x})$

Mixing quantifiers

- We can make statements of predicate logic **mixing existential and universal quantifiers**.
- *Order of variables under the same quantifier does not matter. Under different ones does.*
  - **Predicate: Loves(x,y). Domain: people.**

  - Everybody loves somebody: $\forall x\ \exists y$ Loves(x,y)
    - Normal people

  - Somebody loves everybody: $\exists x\ \forall y$ Loves(x,y)
    - Mother Teresa
  - Everybody is loved by somebody: $\forall x\ \exists y$ Loves(y,x)
    - Their mother

  - Somebody is loved by everybody: $\exists x\ \forall y$ Loves(y,x)
    - Elvis Presley
  - Everybody is loved by everybody: $\forall x\ \forall y$ Loves(x,y)
    - Domain is a good family

Negating mixed quantifiers

- Now, a "**not**" in front of such a sentence means all ∀ and ∃ are interchanged, and the inner part becomes negated.
  - Everybody loves somebody: $\forall x\ \exists y\ Loves(x,y)$

    - Somebody does not love anybody: $\exists\ x\ \forall\ y\ \neg Loves(x,y)$
    - Can also say "Somebody loves nobody" in English.
    - Not the same as "somebody does not love everybody": here, "somebody does not (love everybody)" meaning $\exists x\ \neg\left(\forall y\ Loves(x,y)\right) \equiv\ \exists x\ \exists y\ \neg Loves(x,y)$
    - But the formula $\exists x\ \exists y\ \neg Loves(x,y)$ is the negation of $\forall x\ \forall y\ Loves(x,y)$

Negating mixed quantifiers

- Everybody loves somebody: $\forall x \exists y \, Loves(x,y)$
  - Somebody does not love anybody $\exists X \forall Y \neg Loves(x,y)$

- Somebody loves everybody: $\exists x \forall y \, Loves(x,y)$
  - Everyone doesn't like somebody $\forall x \exists y \neg Loves(x,y)$

- Everybody is loved by somebody: $\forall x \exists y \, Loves(y,x)$
  - Somebody is not loved by anybody $\exists x \forall y \neg Loves(y,x)$

- Somebody is loved by everybody: $\exists x \forall y \, Loves(y,x)$
  - For everyone, somebody does not love them $\forall x \exists y \neg Loves(y,x)$

- Everybody is loved by everybody: $\forall x \forall y \, Loves(y,x)$
  - Somebody does not love someone $\exists x \exists y \neg Loves(y,x)$

Scope of quantifiers

- Like in programming, a **scope** of a quantified variable continues until a new variable with the same name is introduced.
  - $\forall \boldsymbol{x}\left(\exists \boldsymbol{y}\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})\right) \wedge \left(\exists \boldsymbol{y}\ \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y})\right)$
    - For everybody there is somebody who loves them and somebody who hates them.
  - Not the same as $\forall \boldsymbol{x}\left(\exists \boldsymbol{y}\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y})\right)$
    - For everybody there is somebody who both loves and hates them.

- Better to avoid using same names for different variables since it is confusing.
  - $\forall \boldsymbol{x}\left(\exists \boldsymbol{y}\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})\right) \wedge \left(\exists \boldsymbol{y}\ \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{y})\right)$

    $\equiv$
  - $\forall \boldsymbol{x}\left(\exists \boldsymbol{y}\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y})\right) \wedge \left(\exists \boldsymbol{z}\ \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{z})\right)$

    $\equiv$
  - $\forall \boldsymbol{x}\ \exists \boldsymbol{y}\ \exists \boldsymbol{z}\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{z})$

    $\equiv$
  - $\forall \boldsymbol{x}\ \exists \boldsymbol{z}\ \exists \boldsymbol{y}\ \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{z})$

Equivalence for predicate logic

- Two predicate logic formulas are **equivalent** if they have the same truth value for every setting of free variables, **no matter what the predicates are**.
  - $(\exists y\, P(x,y)) \wedge (\exists y\, Q(x,y))$
  - $\equiv (\exists y\, P(x,y)) \wedge (\exists z\, Q(x,z))$
  - $\equiv \exists y\, \exists z\, P(x,y) \wedge Q(x,z)$
  - $\equiv \exists z\, \exists y\, P(x,y) \wedge Q(x,z)$
    Why?
  - But $\exists x\, \forall y\, P(x,y,z)$ is not equivalent to $\forall y\, \exists x\, P(x,y,z)$ Why?

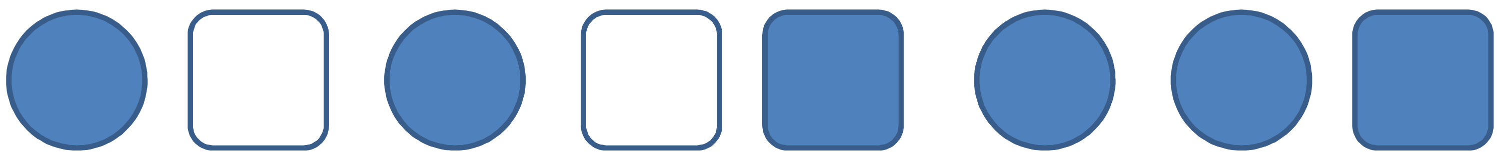
Prenex normal form

- When all quantified variables have different names, can move all quantifiers to the front of the formula, and get an equivalent formula: this is called **prenex normal form**.
  - $\forall x \exists y \exists z \, P(x,y) \land Q(x,z)$ is in prenex normal form
  - $\forall x \left(\exists y \, P(x,y)\right) \land \left(\exists z \, Q(x,z)\right)$ is not in prenex normal form.

- Be careful with implications: when in doubt, open into $\neg A \lor B$. Move all negations inside.
  - $\forall x \ \left(\left(\exists y \, P(x,y)\right) \to Q(x)\right)$ actually has two universal quantifiers!
  - Its equivalence in prenex normal form is $\forall x \, \forall y \left(\neg P(x,y) \lor Q(x)\right)$

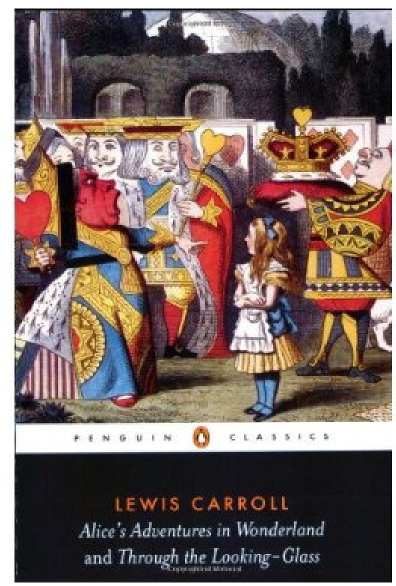
Quantifiers and conditionals

- Which statements are true?
  - All squares are white. All white shapes are squares
  - All circles are blue. All blue shapes are circles.



  - All lemurs live in the trees. All animals living in the trees are lemurs.
  - $\forall \boldsymbol{x} \in \boldsymbol{S},\ \boldsymbol{P}(\boldsymbol{x}) \rightarrow \boldsymbol{Q}(\boldsymbol{x})$
    - For all objects, if it is white, then it is a square.
    - If an object is white, then it is a square.
    - If an animal is a lemur, then it lives in the trees.

- Then you should say what you mean,’ the March Hare went on.
- `I do,’ Alice hastily replied; `at least–at least I mean what I say–that’s the same thing, you know.’
- `Not the same thing a bit!’ said the Hatter. `You might just as well say that “I see what I eat” is the same thing as “I eat what I see”!’
- `You might just as well say,’ added the March Hare, `that “I like what I get” is the same thing as “I get what I like”!’
- `You might just as well say,’ added the Dormouse, who seemed to be talking in his sleep, `that “I breathe when I sleep” is the same thing as “I sleep when I breathe”!’



“Alice’s Adventures in Wonderland” by Lewis Carroll