CS2209A 2017
Applied Logic for Computer Science

Lecture 21, 22

Recursive definition of sets and structural induction

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Tower of Hanoi game

 

- Rules of the game:
  - Start with all disks on the first peg.
  - At any step, can move a disk to another peg, as long as it is not placed on top of a smaller disk.
  - Goal: move the whole tower onto the second peg.

- Question: *how many steps are needed to move the tower of 8 disks? How about n disks?*

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- Let us call the number of moves needed to transfer n disks H(n).
  - Names of pegs do not matter: from any peg i to any peg $j \neq i$ would take the same number of steps.
- **Basis**: only one disk can be transferred in one step.
  - So H(1) = 1
- **Recursive** step:
  - suppose we have n-1 disks. To transfer them all to peg 2, need $H(n-1)$ number of steps.
  - To transfer the remaining disk to peg 3, 1 step.
  - To transfer n-1 disks from peg 2 to peg 3 need H(n-1) steps again.
  - So H(n) = 2H(n-1)+1 (recurrence).
- Closed form: H(n) = $2^n - 1$.

Recurrence relations

- **Recurrence**: an equation that defines an n^{th} element in a sequence in terms of one or more of previous terms.
  - H(n) = 2H(n-1)+1
  - F(n) = F(n-1)+F(n-2)
  - T(n) = aT(n-1)
- A **closed form** of a recurrence relation is an expression that defines an n^{th} element in a sequence in terms of n directly.
  - Often use recurrence relations and their closed forms to describe performance of (especially recursive) algorithms.

Recursive definitions of sets

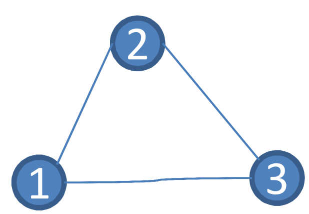
- So far, we talked about recursive definitions of **sequences**. We can also give recursive definitions of **sets**.
  - E.g: recursive definition of a set $S = \{0, 1\}^*$
    - **Basis**: empty string is in S.
    - **Recursive step**: if $w \in S$, then $w0 \in S$ and $w1 \in S$
      - Here, $w0$ means string w with 0 appended at the end; same for w1

Recursive definitions of sets

- Recursive definition of a set $S = \{0, 1\}^*$
  - Alternatively:
    - **Basis**: empty string, 0 and 1 are in S.
    - **Recursive step**: if s and t are in S, then $st \in S$
      - here, st is concatenation: symbols of s followed by symbols of t
      - If s = 101 and t= 0011, then st = 1010011
  - Additionally, need a **restriction condition**: the set S contains only elements produced from basis using recursive step rule.

Trees

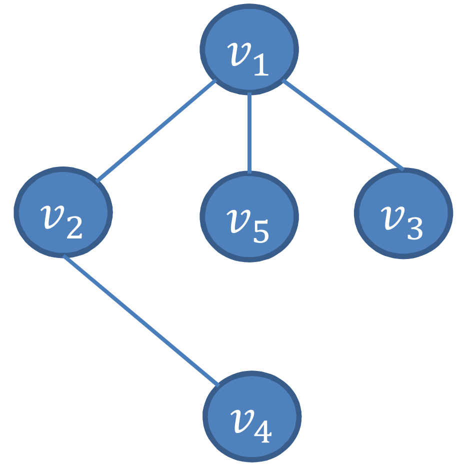
- In computer science, a **tree** is an undirected graph without cycles
  - **Undirected**: all edges go both ways, no arrows.
  - **Cycle**: sequence of edges going back to the same point.



Undirected cycle
(not a tree)

Trees

- **Recursive definition of trees**:
  - **Base**: A single vertex v is a tree.
  - **Recursion**:
    - Let T be a tree, and v a new vertex.
    - Then a new tree consist of T, v, and an edge (connection) between some vertex of T and v.
  - **Restriction**:
    - Anything that cannot be constructed with this rule from this base is not a tree.



Arithmetic expressions

- Suppose you are writing a piece of code that takes an arithmetic expression and, say evaluates it.
  - "5*3-1", "40-(x+1)*7", etc

- *How to describe a valid arithmetic expression?*

Arithmetic expressions

- *How to describe a valid arithmetic expression?*
- Define a set of all valid arithmetic expressions **recursively**.
  - **Base**: A number or a variable is a valid arithmetic expression.
    - 5, 100, x, a
  - **Recursion**:
    - If A and B are valid arithmetic expressions, then so are (A), $A + B, A - B,\ A * B, A$ / B.
      - Constructing 40-(x+1)*7: first construct 40, x, 1, 7. Then (x+1). Then (x+1)*7, finally 40-(x+1)*7
      - Caveat: how do we know the order of evaluation? On that later.
  - **Restriction**: nothing else is a valid arithmetic expression.

Formulas

- What is a well-formed propositional logic formula?
  - $(p \vee \neg q) \wedge r \rightarrow (\neg p \rightarrow r)$

  - **Base**: a propositional variable p, q, r ...
    - Or a constant $TRUE, FALSE$
  - **Recursion**:
    - If F and G are propositional formulas, so are $(F),\ \neg F,\ F \wedge G, F \vee G, F \rightarrow G, F \leftrightarrow G$.
  - **And nothing else**.

Formulas

- What is a **well-formed predicate logic formula**?
  - $\exists x \in D\ \forall y \in \mathbb{Z}\ \ P\big((x, y) \vee Q(x, z)\big) \wedge x = y$
  - **Base**: a predicate with free variables
    - P(x), x=y, …
  - **Recursion**:
    - If F and G are predicate logic formulas, so are (F), $\neg F,\ F \wedge G, F \vee G, F \rightarrow G, F \leftrightarrow G$.
    - If F is a predicate logic formula with a free variable x, then $\exists x \in D\ F$ and $\forall x \in D\ F$ are predicate logic formulas.
  - **And nothing else**.
    - So $\exists x \in People\ \ Likes(x, y \wedge x),\ Likes(y \neq x)$ is not a well-formed predicate logic formula!

Grammars

- A context-free grammar consists of
  - A set V of **variables** (using capital letters)
    - Including a **start variable** S.
  - A set Σ of **terminals** (disjoint from V; alphabet)
  - A set R of **rules**, where each rule consists of a variable from V and a string of variables and terminals.
    - If $A \rightarrow w$ is a rule, we say variable A **yields** string w.
      - This is not the same "→ " as implication, a different use of the same symbol.
    - We use shortcut "|" when the same variable might yield several possible strings: $A \rightarrow w_1 | \, w_2 | \ldots | w_k$
    - Can use A again within the rule: **Recursion**!
      - Different occurrences of the same variable can be interpreted as different strings.
    - When left with just terminals, a string is **derived**.

Grammars

- A general recursive definition for these is called a **grammar**.
  - In particular, here we have "*context-free*" grammars, where symbols have the same meaning wherever they are.

- A **language generated by a grammar** consists of all strings of terminals that can be derived from the start variable by applying the rules.
  - All strings are derived by repeatedly applying the grammar rules to each variable until there are no variables left (just the terminals).

Examples of grammars

- Example: **language** {1, 00} consisting of two strings 1 and 00
  - $S \rightarrow 1 \mid 00$
    - Variables: S. Terminals: 1 and 00.

- Example: **strings** over {0, 1} with all 0s before all 1s.
  - $S \rightarrow 0S \mid S1 \mid \_$
    - Variables: S. Terminals: 0 and 1.

Examples of grammars

- Example: **propositional formulas**.
  1. $F \rightarrow \ F \vee F$
  2. $F \rightarrow F \wedge F$
  3. $F \rightarrow \ \neg F$
  4. $F \rightarrow \ (F)$
  5. $F \rightarrow p \mid q \mid r \mid TRUE \mid FALSE$
     - Here, the only variable is *F* (it is a start variable), and terminals are $\vee, \wedge, \neg, (,), p, q, r, TRUE, FALSE$
     - To obtain $(p \vee \neg q) \wedge r$, first apply rule 2, then rule 1, then rule 5 to get *p*, then rule 3, then rule 5 to get *q*, then rule 5 to get *r*.

Examples of grammars

- Example: **arithmetic expressions**
  - $EXPR \rightarrow EXPR + EXPR \mid EXPR - EXPR \mid EXPR * EXPR \mid EXPR \,/\, EXPR \mid (EXPR) \mid NUMBER \mid$-NUMBER
  - NUMBER $\rightarrow 0DIGITS \mid \ldots \mid 9DIGITS$
  - $DIGITS \rightarrow$ _$\mid NUMBER$
    - Here, _ stands for empty string.
      Variables: EXPR, NUMBER, DIGITS (S is starting).
      Terminals: +,-,*, /, 0,…,9.
    - We used separate NUMBER to avoid multiple “-”.
    - And separate DIGITS to have an empty string to finish writing a number, but to avoid an empty number.

Encoding order of precedence

- Easier to specify in which order to process parts of the formula.
  - Better grammar for arithmetic expressions (for simplicity, with x,y,z instead of numbers):
    1. $EXPR \rightarrow EXPR + TERM\ |EXPR - TERM|\ TERM$
    2. $TERM \rightarrow TERM\ * FACTOR \mid TERM\ / FACTOR \mid FACTOR$
    3. $FACTOR\ \rightarrow (EXPR)$ | x | y | z
  - Here, variables are EXPR, TERM and FACTOR (with EXPR a starting variable).
  - Now can encode precedence.
    - And put parentheses more sensibly.

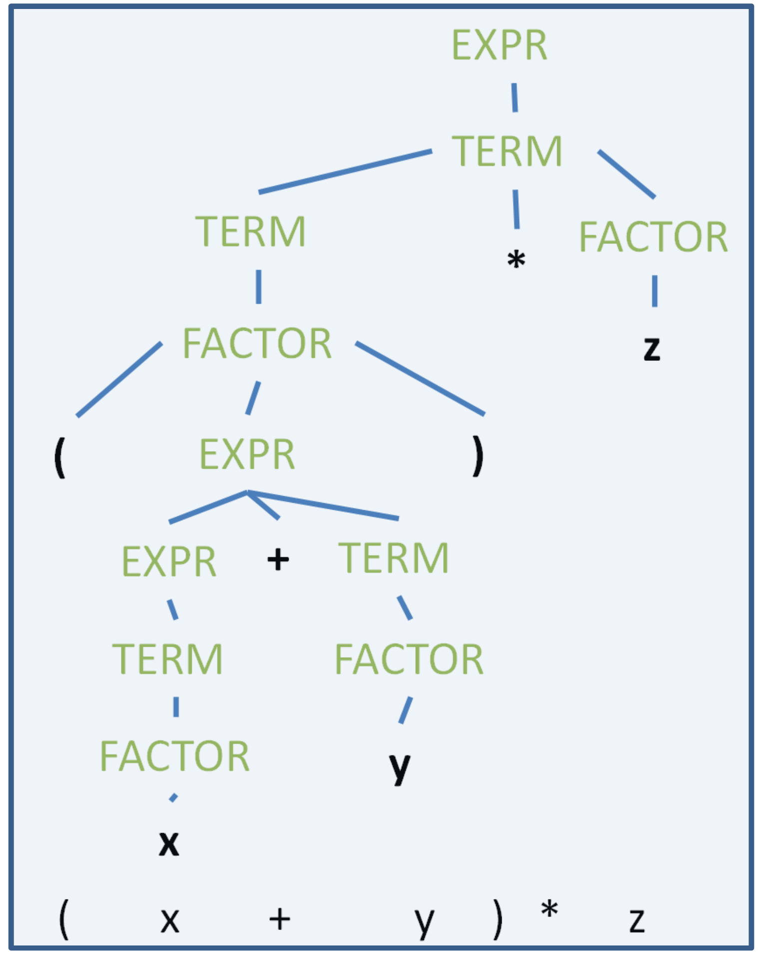
Parse trees.

Visualization of derivations:

parse trees.

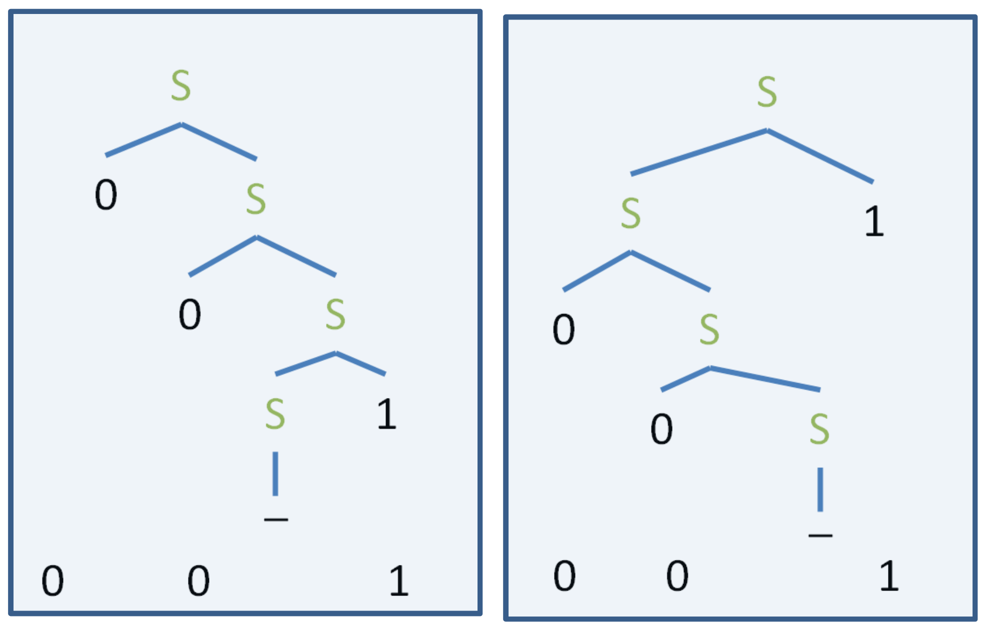
1. $EXPR \rightarrow EXPR + TERM \mid EXPR - TERM \mid TERM$
2. $TERM \rightarrow TERM * FACTOR \mid TERM \; / \; FACTOR \mid FACTOR$
3. $FACTOR \rightarrow (EXPR) \mid \mathrm{x} \mid \mathrm{y} \mid \mathrm{z}$

- String (x+y)*z



Parse trees.

- Visualization of derivations: **parse trees**.
  - Simpler example:
    - $S \rightarrow 0S \mid S1 \mid \_$
    - String 001



Puzzle

- Do the following two English sentences have the same parse trees?
  - Time flies like an arrow. 
  - Fruit flies like an apple. 

Structural induction

- Let $S \subseteq U$ be a **recursively defined set**, and F(x) is a property (of $x \in U$).
- Then
  - if all x in the base of S have the property,
  - and applying the recursion rules preserves the property,
  - then all elements in S have the property.

Multiples of 3

- Let’s define a set S of numbers as follows.
  - **Base**: $3 \in S$
  - **Recursion**: if $x, y \in S$, then $x + y \in S$
- **Claim: all numbers in S are divisible by 3**
  - That is, $\forall x \in S\ \exists\, z \in \mathbb{N}\ x = 3z$.

Multiples of 3

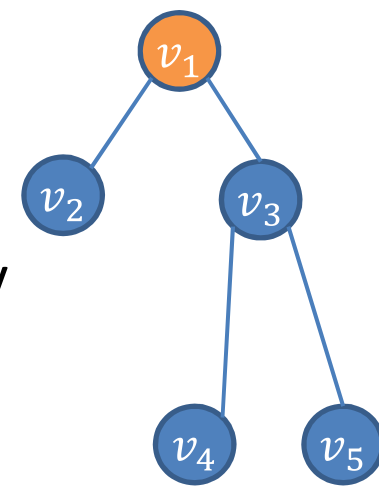
- Proof (by **structural induction**).
  - **Base case**: 3 is divisible by 3 (y=1).
  - **Recursion**: Let $x, y \in S$. Then $\exists z, u \in \mathbb{N}\; x = 3z \wedge y = 3u$.
    - Then $x + y = 3z + 3u = 3(z + u)$.
    - Therefore, $x + y$ is divisible by 3.
  - As there are **no other elements** in S except for those constructed from 3 by the recursion rule, all elements in S are divisible by 3.

Binary trees

- **Rooted trees** are trees with a special vertex designated as a root.
  - Rooted trees are **binary** if every vertex has **at most three edges**: one going towards the root, and two going away from the root. **Full** if every vertex has either 2 or 0 edges going away from the root.

Binary trees

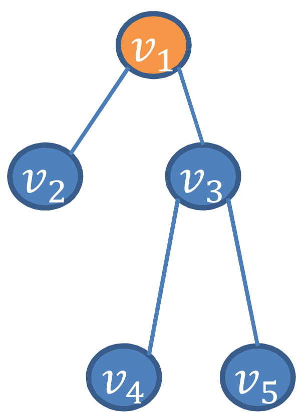
- **Recursive** definition of **full binary trees**:
  - **Base**: A single vertex v is a full binary tree with that vertex as a root.
  - **Recursion**:
    - Let T_1, T_2 be full binary trees with roots r_1, r_2, respectively. Let v be a new vertex.
    - A new full binary tree with root v is formed by connecting r_1 and r_2 to v.
  - **Restriction**:
    - Anything that cannot be constructed with this rule from this base is not a full binary tree.



Height of a full binary tree

- The **height** of a rooted tree, $h(T)$, is the maximum number of edges to get from any vertex to the root.
  - Height of a tree with a single vertex is 0.

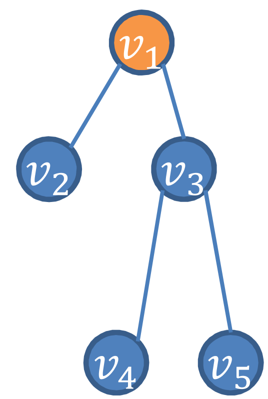
- Claim: Let $n(T)$ be the number of vertices in a full binary tree T. Then $n(T) \leq 2^{h(T)+1} - 1$



Height 2

Height of a full binary tree

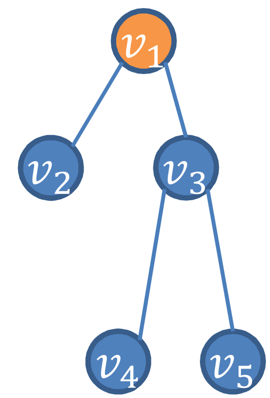
- Proof (by **structural induction**)
  - **Base case**: a tree with a single vertex has $n(T) = 1$ and $h(T) = 0$. So $2^{h(T)+1} - 1 = 1 \geq 1$
  - **Recursion**: Suppose T was built by attaching T_1, T_2 to a new root vertex v.
    - Number of vertices in T is $\mathrm{n(T)} = n(T_1) + n(T_2) + 1$
    - Every vertex in T_1 or T_2 now has one extra step to get to the new root in T. So $h(T) = 1 + \max(h(T_1), h(T_2))$
    - By the induction hypothesis, $n(T_1) \leq 2^{h(T_1)+1} - 1$ and $n(T_2) \leq 2^{h(T_2)+1} - 1$
    - $$\begin{aligned} \mathrm{n(T)} &= n(T_1) + n(T_2) + 1 \\ &\leq 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \\ &\leq 2 \cdot \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 \\ &\leq 2 \cdot 2^{\max\left(h(T_1), h(T_2)\right)+1} - 1 \\ &= 2 \cdot 2^{h(T)} - 1 = 2^{h(T)+1} - 1 \end{aligned}$$
  - Therefore, the number of vertices of any binary tree T is less than $2^{h(T)+1} - 1$



Height 2

Height of a full binary tree

- Claim: Let $n(T)$ be the number of vertices in a full binary tree T. Then $n(T) \leq 2^{h(T)+1} - 1$
- Alternatively, height of a binary tree is at least $\log_2 n(T)$
  - If you have a recursive program that calls itself twice (e.g, within if … then … else …)
  - Then if this code executes n times (maybe on n different cases)
  - Then the program runs in time at least $\log_2 n$, even when cases are checked in parallel.



Height 2