CS2209A 2017

Applied Logic for Computer Science

Lecture 7

Propositional Logic:

Resolution

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False premises

- **Bertrand Russell**:
  "If 2+2=5, then I am the pope"

- Can you see how to prove this?

False premises

- **An argument can still be valid when some of its premises are false.**
  - Remember, false implies anything.
- Bertrand Russell: "If 2+2=5, then I am the pope"
  - Suppose 2+2=5
  - If 2+2=5, then 1=2 (subtract 3 from both sides).
  - So 1=2 (by modus ponens)
  - Me and the pope are two people.
  - Since 1=2, me and the pope are one person.
  - Therefore, I am the pope!

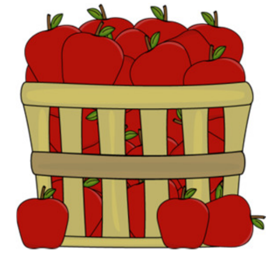
Natural deduction vs. Truth tables

- In this puzzle, it was faster to solve it using modus ponens (natural deduction method) than writing a truth table.
- But is it always better?
- The answer is...

Nobody knows!

- It is a very closely related to the question of how fast can one check if something is a tautology.
  - **And that's a million dollar question!**

The million dollar question



- In English, known as "**P vs. NP**" problem
  - P stands for "polynomial time computable".
  - NP is "polynomial time checkable"
    - non-deterministic polynomial-time computable
  - **Question: is everything efficiently checkable also efficiently computable?**
- In Russian, called "perebor" problem.
  - "perebor" translates as "exhaustive search".
  - Question: is it always possible to avoid looking through nearly all potential solutions to find an answer?
  - **Are there situations when exhaustive search is unavoidable?**

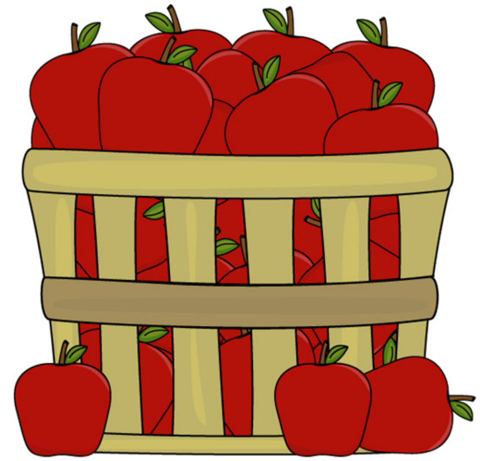
The million dollar question

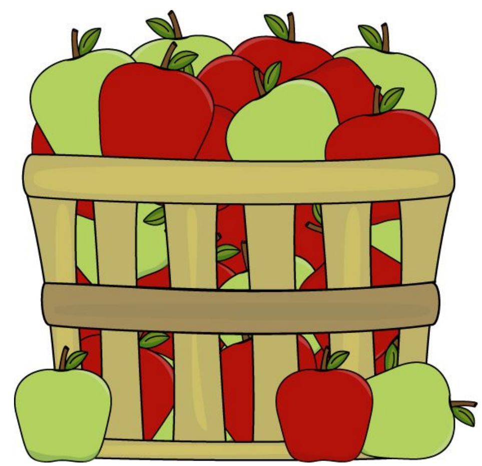
- **NP-completeness:** enough to answer for the problem of checking satisfiability (SAT)!
- A formula is like a basket of apples.

  formula is a tautology

  =

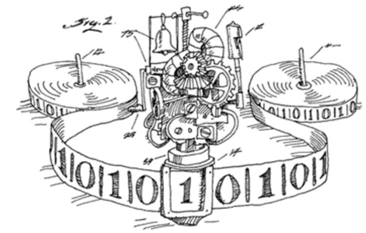
  "All apples in the basket are good".



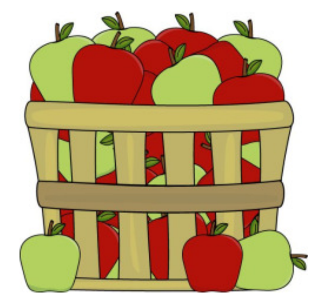


- Can you check that all apples are good without looking at every single one?
- Can you do it for every possible basket of apples?
  - Smell test?

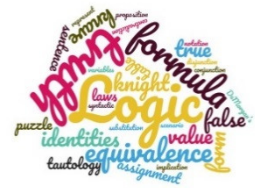
Automated provers



- How to make an **automated prover** which checks whether a formula is a tautology?
  - And so can check if an argument is valid, etc.
- **Truth tables**:
  - easy to program, but proofs are huge.
- **Natural deduction**:
  - proofs might be smaller than a truth table
    - Are they always? Good question...
  - even if there is a small proof, how can we find one quickly?
    - Nobody knows ...



Resolution rule



- Middle ground: use the **resolution rule**:
  - Basis for many practical provers (SAT solvers).
  - Used in verification, scheduling, etc...

$$
\begin{array}{l}
C \lor x \\
D \lor \neg x \\
\hline
\therefore C \lor D
\end{array}
$$

- $(C \lor x) \land (D \lor \neg x) \rightarrow (C \lor D)$

Resolution rule

$$\frac{\begin{array}{c} C \vee x \\ D \vee \neg x \end{array}}{\therefore C \vee D}$$

$$\frac{\begin{array}{c} y \vee \neg z \vee w \\ u \vee \neg w \end{array}}{\therefore y \vee \neg z \vee u}$$

$$\frac{\begin{array}{c} y \vee w \vee \neg z \\ \neg z \vee \neg w \end{array}}{\therefore y \vee \neg z}$$

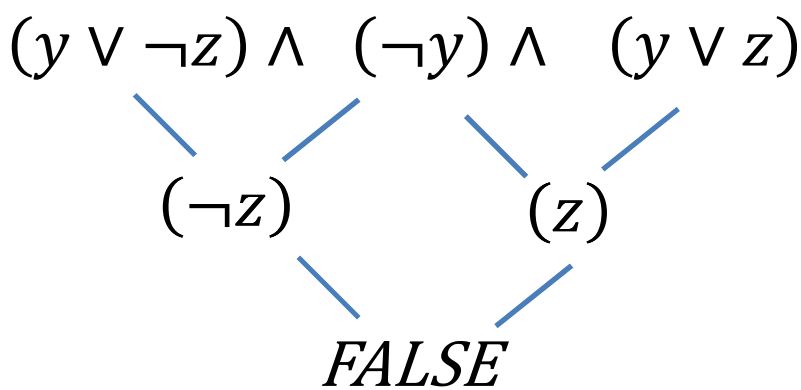
- Ignore order in an **OR** and remove duplicates.
- C and D are possibly empty
  - $\boldsymbol{x \wedge \neg x \equiv False}$ (same as saying it is a contradiction)

$$\frac{\begin{array}{c} x \\ \neg x \end{array}}{\therefore False}$$

Resolution proofs

- Rather than proving that F is a tautology, prove that $\neg \boldsymbol{F} \equiv \boldsymbol{FALSE}$. That is, a proof of F is a **refutation** of $\neg F$
  - To check that an argument is valid, refute AND of premises AND NOT conclusion.

- Last step of the resolution refutation of $\neg F$:
  - from x and $\neg x$ derive FALSE, for some variable x.
  - If you cannot derive anything new, then the formula is satisfiable.



Prove Modus Ponens by resolution

- If $\boldsymbol{p}$ then $\boldsymbol{q}$
- $\boldsymbol{p}$

$\overline{\qquad\qquad\qquad\qquad}\;\therefore$

$\boldsymbol{q}$

$$(\boldsymbol{p} \to \boldsymbol{q}) \land \boldsymbol{p} \to \boldsymbol{q}$$
$\boldsymbol{is\ a\ tautology}$

➢ Prove by resolution:

$$(p \to q) \land p \land (\neg q\ )$$
$is\ false$

$$(\neg \boldsymbol{p} \lor \boldsymbol{q}) \land \boldsymbol{p} \land \neg \boldsymbol{q}$$

$$(\neg p \lor q) \land\ p \land\ (\neg q)$$

$$q$$

$$FALSE$$

Prove Hypothetical Syllogism by resolution

- **If $\boldsymbol{p}$ then $\boldsymbol{q}$**
- **If $\boldsymbol{q}$ then $\boldsymbol{r}$**

$\therefore$ If $\boldsymbol{p}$ then $\boldsymbol{r}$

$$((p \to q) \land (q \to r)) \to (p \to r)$$

$is\ a\ tautology$

- Prove by resolution:

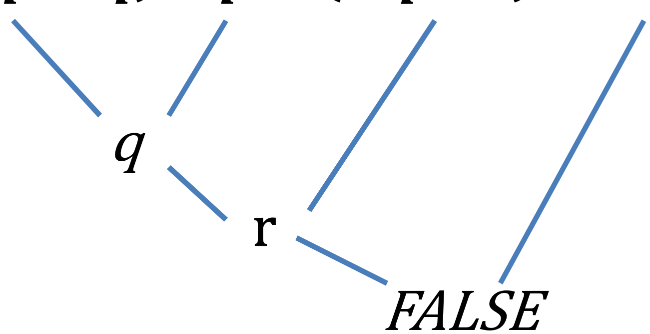
$$((p \to q) \land (q \to r)) \land (\neg(p \to r))$$

$is\ false$

$$(\neg \boldsymbol{p} \lor \boldsymbol{q}) \land (\neg \boldsymbol{q} \lor \boldsymbol{r}) \land \neg(\neg \boldsymbol{p} \lor \boldsymbol{r})$$

$$\equiv (\neg \boldsymbol{p} \lor \boldsymbol{q}) \land (\neg \boldsymbol{q} \lor \boldsymbol{r}) \land \boldsymbol{p} \land \neg \boldsymbol{r}$$ //De Morgan, double negation

$$(\neg \boldsymbol{p} \lor \boldsymbol{q}) \land \boldsymbol{p} \land (\neg \boldsymbol{q} \lor \boldsymbol{r}) \land \neg \boldsymbol{r}$$



Natural deduction

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

1. If A then not B
2. If C then B
3. A
4. C or D
5. If E then F
6. Not B
7. Not C
8. D

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- Therefore, the treasure is not in the kitchen.

Treasure hunt: resolution

1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the font yard is an elm, then the treasure is in the kitchen.
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
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1. $A \rightarrow \neg B$
2. $C \rightarrow B$
3. A
4. C ∨ D
5. E → F

1. $\neg A \vee \neg B$
2. $\neg C \vee B$
3. A
4. C ∨ D
5. $\neg E \vee$ F

Conclusion: D

Check validity of the argument using resolution

$(\neg A \vee \neg B) \wedge (\neg C \vee B) \wedge (A) \wedge (C \vee D) \wedge (\neg E \vee F) \wedge \neg D$

$(\neg A \vee \neg C)$

(C)

$(\neg C)$

$FALSE$

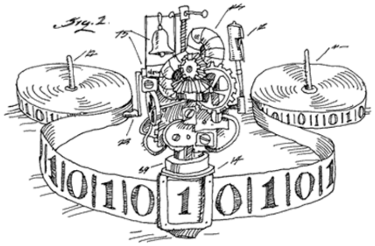
Conjunctive Normal Form (CNF)

- **Resolution works best when the formula is of the special form**: it is an AND of ORs of (possibly negated) variables (called **literals**).

- This form is called a **Conjunctive Normal Form**, or **CNF**.
  - $(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$ is a CNF
  - $(x \vee \neg y \vee z)$ is a CNF. So is $(x \wedge \neg y \wedge z)$.
  - $(x \vee \neg y \wedge z)$ is not a CNF

- **An AND of CNF formulas is a CNF formula**.
  - So if all premises are CNF and the negation of the conclusion is a CNF, then AND of premises AND NOT conclusion is a CNF.

CNF



- To convert a formula into a CNF.
  - Open up the implications to get ORs.
  - Get rid of double negations.
  - Convert $F \vee (G \wedge H)$ to $(F \vee G) \wedge (F \vee H)$ //distributivity

- Example: $A \rightarrow (B \wedge C)$
  $\equiv \neg A \vee (B \wedge C)$
  $\equiv (\neg A \vee B) \wedge (\neg A \vee C)$

- In general, CNF can become quite big, especially when have $\leftrightarrow$. There are tricks to avoid that ...

Puzzle

- Suppose that nobody in our class carries more than 10 pens.
- There are 70 students in our class.

- Prove that there are at least 2 students in our class who carry the same number of pens.
  - In fact, there are at least 7 who do.

Pigeonhole Principle

- The Pigeonhole Principle:
  - If there are n pigeons
  - And n-1 pigeonholes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

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- The Pigeonhole Principle:
  - If there are n pigeons and n-1 pigeonholes
  - Then if every pigeon is in a pigeonhole
  - At least two pigeons sit in the same hole

- Applying to our problem:
  - n-1 = 11 possible numbers of pens (from 0 to 10)
  - Even with n=12 people, there would be 2 who have the same number.
  - If there were less than 7, say 6 for each scenario, total would be 66.
  - Note that it does not tell us which number or who these people are!

Resolution and Pigeons

- It is not that hard to write the Pigeonhole Principle as a tautology
- But we can prove that resolution has trouble with this kind of reasoning
  - the smallest resolution proof of this tautology is **exponential** size!
- By contrast, natural deduction (and you!) can figure it out fairly quickly
  - though it is not straightforward.
- The problem is that resolution **cannot count**.
  - But ability to count makes things harder...