CS2209A 2017
Applied Logic for Computer Science

Lecture 8, 9

Propositional Logic:

Conjunctive Normal Form & Disjunctive Normal Form

Instructor: Yu Zhen Xie

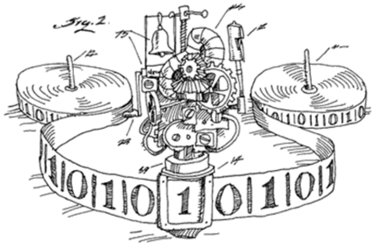
Conjunctive Normal Form (CNF)

- **Resolution works best when the formula is of the special form**: it is an $\wedge$ of $\vee$s of (possibly negated, $\neg$) variables (called **literals**).

- This form is called a **Conjunctive Normal Form**, or **CNF**.
  - $(y \vee \neg z) \wedge (\neg y) \wedge (y \vee z)$ is a CNF
  - $(x \vee \neg y \vee z)$ is a CNF. So is $(x \wedge \neg y \wedge z)$.
  - $(x \vee \neg y \wedge z)$ is not a CNF

- **An AND** $(\wedge)$ **of CNF formulas is a CNF formula**.
  - So if all premises are CNF and the negation of the conclusion is a CNF, then AND of premises AND NOT conclusion is a CNF.

CNF



- To convert a formula into a CNF.
  - Open up the implications to get ORs.
  - Get rid of double negations.
  - Convert $F \vee (G \wedge H)$ to $(F \vee G) \wedge (F \vee H)$
    //distributivity

- Example: $A \rightarrow (B \wedge C)$
  $\equiv \neg A \vee (B \wedge C)$
  $\equiv (\neg A \vee B) \wedge (\neg A \vee C)$

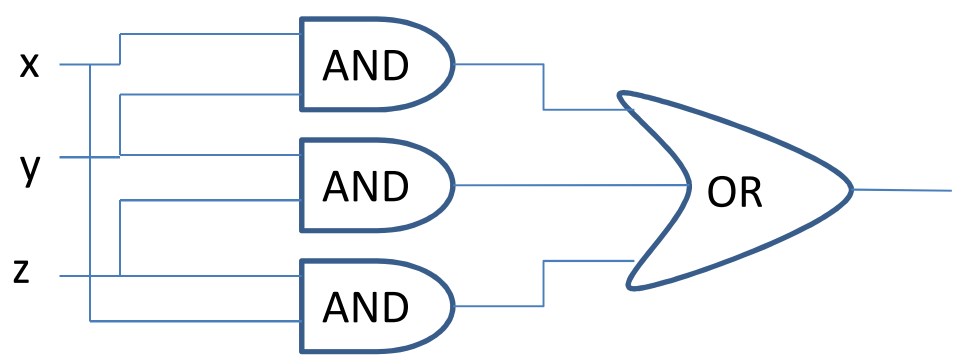
- In general, CNF can become quite big, especially when have $\leftrightarrow$. There are tricks to avoid that ...

Boolean functions and circuits

- What is the relation between **propositional logic** and **logic circuits**?
  - View a formula as computing a function (called a **Boolean function**),
    - inputs are values of variables,
    - output is either *true (1)* or *false (0)*.
  - For example, $Majority(x, y, z) = true$ when at least two out of x, y, z are true, and false otherwise.
  - Such a function is fully described by a truth table of its formula (or its circuit: circuits have truth tables too).

Boolean functions and circuits

- What is the relation between **propositional logic and logic circuits**?
  - So both formulas and circuits "compute" **Boolean functions** – that is, **truth tables**.
  - In a circuit, can "**reuse**" a piece in several places, so a circuit can be smaller than a formula.
    - Still, most circuits are big!
  - $Majority(x, y, z)$ is $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$



CNF and DNF

- Every truth table (Boolean function) can be written as either a conjunctive normal form (CNF) or disjunctive normal form (DNF)
- **CNF is an** $\wedge$ **of** $\vee$**s**, where $\vee$ is over variables or their negations (literals); an $\vee$ of literals is also called a **clause**.
- **DNF is an** $\vee$ **of** $\wedge$**s**; an $\wedge$ of literals is called a **term**.

Notations

- $\neg p, x, s$ are examples of literals
- Neither $\neg\neg p$ nor $(x \vee y)$ is a literal
- $(x \vee \neg y \vee z), (\neg p)$ are clause
- $(x \wedge \neg y \wedge z), (\neg p)$ are terms
- $(x \vee \neg z \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg y)$ is a CNF
- $(x \wedge z) \vee (\neg y \wedge z \wedge x) \vee (\neg x \wedge z)$ is a DNF
- $(x \wedge \neg(y \vee z) \vee u)$ is neither a CNF nor DNF, but is equivalent to DNF $((x \wedge \neg y \wedge \neg z) \vee u)$

Facts about CNF and DNF

- Any propositional formula is tautologically equivalent to some formula in disjunctive normal form.

- Any propositional formula is tautologically equivalent to some formula in conjunctive normal form.

Why CNF and DNF?

- Convenient normal forms
- **Resolution** works best for formulas in CNF
- Useful for constructing formulas given a **truth table**
  - **DNF**: take a disjunction (that is, $\vee$) of all **satisfying** truth assignments
  - **CNF:** take a conjunction ($\wedge$) of **negations** of **falsifying** truth assignments

From truth table to DNF and CNF

- A **minterm** is a conjunction of literals in which each variable is represented exactly once
  - If a Boolean function (truth table) has the variables (p, q, r) then $p \wedge \neg q \wedge r$ is a minterm but $p \wedge \neg q$ is not.
- Each minterm is true for exactly one assignment.
  - $p \wedge \neg q \wedge r$ is true if p is true (1) , q is false (0) and r is true (1).
  - Any deviation from this assignment would make this particular minterm false.
- A disjunction of minterms is true only if at least one of its constituents minterms is true.

From truth table to DNF

- If a function, e.g. **F**, is given by a truth table, we know exactly for which assignments it is true.
- Consequently, we can select the minterms that make the function true and form the disjunction of these minterms.
- **F** is true for three assignments:
  - p, q, r are all true, $(p \wedge q \wedge r)$
  - p, ¬q, r are all true, $(p \wedge \neg q \wedge r)$
  - ¬p, ¬q, r are all true, $(\neg p \wedge \neg q \wedge r)$
- **DNF of F**: $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

p	q	r	F
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

From truth table to CNF

- **Complementation** can be used to obtain **conjunctive normal forms** from truth tables.
- If A is a formula containing only the connectives $\neg$, $\vee$ and $\wedge$, then its complement is formed by
  - replacing all $\vee$ by $\wedge$
  - replacing all $\wedge$ by $\vee$
  - replacing all atoms by their complements.
    - The complement of q is $\neg q$
    - The complement of $\neg q$ is q
- Example: Find the complement of the formula
    - $(p \wedge q) \vee \neg r$

      $(\neg p \vee \neg q) \wedge r$

From truth table to CNF

- Solution: $\neg$**G** is **true** for the following assignments.

  $p = 1;\ \ q = 0;\ \ r = 1$

  $p = 1;\ \ q = 0;\ \ r = 0$

  $p = 0;\ \ q = 0;\ \ r = 1$

- The **DNF** of $\neg$**G** is therefore:

$$(\boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r}) \vee (\boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}) \vee (\neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r})$$

- The formula has the complement:

$$(\neg \boldsymbol{p} \vee \boldsymbol{q} \vee \neg \boldsymbol{r}) \wedge (\neg \boldsymbol{p} \vee \boldsymbol{q} \vee \boldsymbol{r}) \wedge (\boldsymbol{p} \vee \boldsymbol{q} \vee \neg \boldsymbol{r})$$

  - It is the desired CNF of **G**

p	q	r	G
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	1

Canonical CNF and DNF

- So for every formula, there is a unique **canonical CNF** (and a truth table, and a Boolean function).
- And for every possible truth table (a Boolean function), there is a formula (the canonical CNF).
- Recall, to make a **canonical DNF** from a truth table:
  - Take all satisfying assignments.
  - Write each as an AND of literals, as before.
  - Then take an OR of these ANDs.

Complete set of connectives

- CNFs only have $\neg, \vee, \wedge$, yet any formula can be converted into a CNF
  - Any truth table can be coded as a CNF
- Call a set of connectives which can be used to express any formula **a complete set** of connectives.
  - In fact, $\neg, \vee$ is already complete. So is $\neg, \wedge$.
    - By DeMorgan, $(A \vee B) \equiv \neg(\neg A \wedge \neg B)$ No need for $\vee$!
  - But $\wedge, \vee$ is not: cannot do $\neg$ with just $\wedge, \vee$.
    - Because when both inputs have the same value, both $\wedge, \vee$ leave them unchanged.

Complete set of connectives

- How many connectives is enough?
  - Just one: **NAND** (NotAND), also called the Sheffer stroke, written as |
  - $\neg A \equiv A \mid A$
  - $A \vee B \equiv \neg(\neg A \wedge \neg B)$
    $\equiv (\neg A \mid \neg B)$
    $\equiv ((A \mid A) \mid (B \mid B))$
  - In practice, most often stick to ∧,∨, ¬

A	B	A | B
True	*True*	*False*
True	*False*	*True*
False	*True*	*True*
False	*False*	*True*

Puzzle

- Susan is 28 years old, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with issues of discrimination and social justice and also participated in anti-nuke demonstrations.

  Please rank the following possibilities by how likely they are. List them from least likely to most likely. Susan is:

1. a kindergarden teacher
2. works in a bookstore and takes yoga classes
3. an active feminist
4. a psychiatric social worker
5. a member of an outdoors club
6. a bank teller
7. an insurance salesperson
8. a bank teller and an active feminist