Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

UWO – October 5, 2020

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Introduction

1. Sets are the basic building blocks for the types of objects considered in discrete mathematics and in mathematics, in general.
2. Set theory is an important branch of mathematics:
   a. Many different systems of axioms have been used to develop set theory.
   b. Here, we are not concerned with a formal set of axioms for set theory.
   c. Instead, we will use what is called naïve set theory.

Plan for Part I

1. Sets

1.1 Defining sets
1.2 Venn Diagram
1.3 Set Equality
1.4 Subsets
1.5 Venn Diagrams and Truth Sets
1.6 Set Cardinality
1.7 Power Sets
1.8 Cartesian Products

2. Set Operations

2.1 Boolean Algebra
2.2 Union
2.3 Intersection
2.4 Complement
2.5 Difference
2.6 The Cardinality of the Union of Two Sets
2.7 Set Identities
2.8 Generalized Unions and Intersections

Plan for Part I

Sets

1. A *set* is an unordered collection of "objects", e.g. intuitively described by some common property or properties (in *naïve set theory*):
   a. the students in this class,
   b. the chairs in this room.
2. The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
3. The notation $a \in A$ denotes that a is an element of set A.
4. If a is not a member of A, write $a \notin A$

Describing a set: the roster method

The roster method is defined as a way to show the members of a set by listing the members inside of brackets.

1. Example: $S = \{a, b, c, d\}$
2. The order of the elements in that list is not important.
   a. For instance, $\{a, b, c, d\} = \{b, c, a, d\}$
3. Each object in the universe is either a member or not.
4. Listing a member more than once does not change the set.
   a. For instance, $\{a, b, c, d\} = \{a, b, c, b, c, d\}$
5. *Ellipses* $(\ldots)$ may be used to describe a set without listing all of the members when the pattern is clear:
   a. For instance, $S = \{a, b, c, d, \ldots, z\}$.

The roster method: more examples

1. The set of all vowels in the English alphabet:

   $V = \{a,e,i,o,u\}$

2. The set of all odd positive integers less than 10:

   $O = \{1,3,5,7,9\}$

3. The set of all positive integers less than 100:

   $S = \{1,2,3,\ldots,99\}$

4. The set of all integers less than 0:

   $S = \{\ldots, -3,-2,-1\}$

Some important sets

$$
\begin{aligned}
\mathbb{N} &= \textit{natural numbers} && = \{0, 1, 2, 3, \ldots\} \\
\mathbb{Z} &= \textit{integers} && = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathbb{Z}^+ &= \textit{positive integers} && = \{1, 2, 3, \ldots\} \\
\mathbb{Q} &= \text{set of } \textit{rational numbers} \\
\mathbb{R} &= \text{set of } \textit{real numbers} \\
\mathbb{R}^+ &= \text{set of } \textit{positive real numbers} \\
\mathbb{C} &= \text{set of } \textit{complex numbers}.
\end{aligned}
$$

The **set-builder** notation

1. It is used to specify the property or properties that all members must satisfy. *Examples*:
   a. $S = \{x \mid x \text{ is a positive integer less than } 100\}$
   b. $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$
   c. $O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$
   d. the set of positive rational numbers:

$$\mathbb{Q}^+ = \{x \in \mathbb{R} \mid x = \tfrac{p}{q}, \text{ for some positive integers } p, q\}$$

2. A predicate may be used, as in $S = \{x \mid P(x)\}$
   a. *Example*: $S = \{x \mid Prime(x)\}$

The interval notation

1. It is used to specify a range of real numbers.
2. Consider two real numbers a, b with $a \leq b$. The following notations are commonly used:
   - $[a,b] = \{x \mid a \leq x \leq b\}$
   - $[a,b) = \{x \mid a \leq x < b\}$
   - $(a,b] = \{x \mid a < x \leq b\}$
   - $(a,b) = \{x \mid a < x < b\}$
   - *closed interval* [a,b]
   - *open interval* (a,b)

Truth sets of quantifiers

1. Given a predicate P and a domain D, we define the *truth set* of P to be the set of the elements in D for which $P(x)$ is true.
2. The truth set of $P(\mathrm{x})$ is denoted by:

$$\{x \in D \mid P(x)\}$$

3. **Example**: The truth set of $P(x)$ where the domain is the integers and $P(x)$ is "$|x| = 1$" is the set {-1,1}

Sets can be elements of sets

Examples:

1. $\{\{1,2,3\}, a, \{b, c\}\}$
2. $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$

Russell's paradox

1. Let S be the set of all sets which are not members of themselves.
2. A paradox results from trying to answer the question "Is S a member of itself?"

Related simple example:

a. Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell

(1872 - 1970)

Nobel Prize

Winner,

Cambridge, UK

1. To avoid this and other paradoxes, *sets* can be (formally) defined via appropriate axioms more carefully than just *an unordered collection of "objects"*
2. where objects are intuitively described by any given property *in naïve set theory*

The universal set U and the *empty set* $\varnothing$

1. The *universal set* is the set containing all the "objects" currently under consideration:
   a. often symbolized by U,
   b. sometimes implicitly stated,
   c. sometimes explicitly stated,
   d. its contents depend on the context.
2. The *empty set* is the set with no elements.
   a. symbolized by $\varnothing$, but $\{\}$ is also used.
   b. **Important**: the empty set is different from a set containing the empty set:

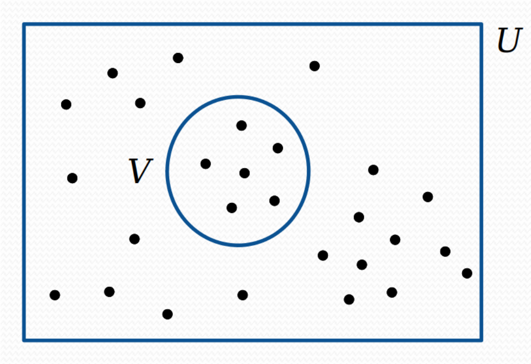
$$\varnothing \neq \{ \varnothing \}$$

Plan for Part I

Venn Diagram

John Venn (1834 - 1923) Cambridge, UK

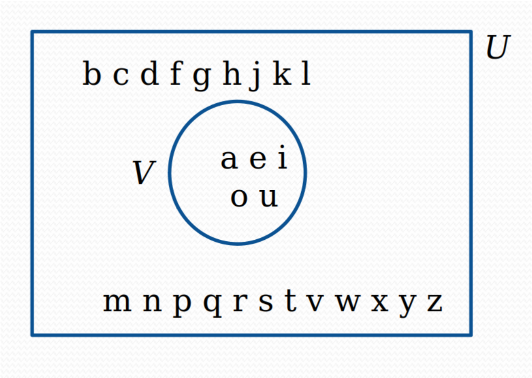
Sets and their elements can be represented via Venn diagrams



- □ – Universal set U
- • – elements
- ○ – Some set V

Venn Diagram

John Venn (1834 - 1923) Cambridge, UK



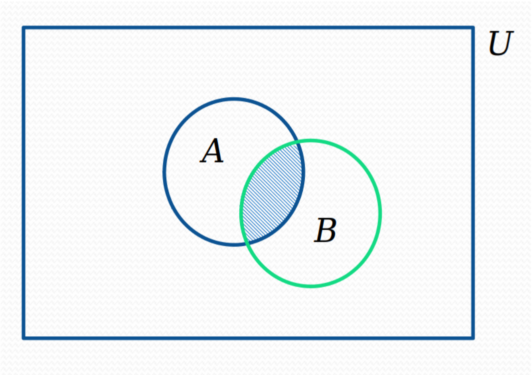
□ – Universal set U: all letters in the Latin alphabet

letters – elements

○ – Set V: all vowels

Venn Diagram

John Venn (1834 - 1923) Cambridge, UK



□ – Universal set U
○ – Set A
○ – Set B

1. Venn diagrams are often drawn to abstractly illustrate relations between multiple sets. Elements are implicit/omitted (shown as dots only when an explicit element is needed)
2. *Example*: shaded area illustrates a set of elements that are in both sets A and B (i.e. *intersection* of two sets, see later). E.g consider A={a,b,c,f,z} and B={c,d,e,f,x,y}.

Plan for Part I

Set equality

Definition: Two sets are *equal* if and only if they have the same elements.

1. If A and B are sets, then A and B are equal iff:

$$\forall x \ \ (x \in A \ \leftrightarrow \ x \in B)$$

2. We write $A = B$ if A and B are equal sets.

$$\{1, 3, 5\} = \{3, 5, 1\}$$
$$\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$$

Plan for Part I

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

1. The notation $A \subseteq B$ is used to indicate that A is a subset of the set B
2. $A \subseteq B$ holds if and only if $\forall x \ (x \in A \rightarrow x \in B)$ is true.
3. Observe that:
   a. Because $a \in \emptyset$ is always false, for every set S we have:
   $$\emptyset \subseteq S.$$
   b. Because $a \in S \rightarrow a \in S$, for every set S, we have:
   $$S \subseteq S.$$

Showing that a set is or is not a subset of another set

1. **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
2. **Showing that A is not a Subset of B**: To show that A is not a subset of B, that is $A \nsubseteq B$**,** find an element $x \in A$ with $x \notin B$**.** (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)
3. **Examples**:
   a. The set of all computer science majors at your school is a subset of all students at your school.
   b. The set of integers with squares less than 100 is not a subset of the set of all non-negative integers.

Another look at equality of sets

1. Recall that two sets A and B are *equal* (denoted by $A = B$) iff:

$$\forall x \ \ (x \in A \ \leftrightarrow \ x \in B)$$

2. That is, using logical equivalences we have that $A = B$ iff:

$$\forall x \quad ((x \in A \ \rightarrow \ x \in B) \quad \wedge \quad (x \in B \ \rightarrow \ x \in A))$$

3. This is also equivalent to:

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

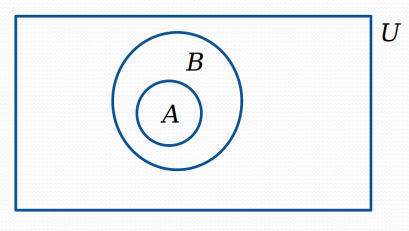
Proper subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then the following is true:

$$\forall x \quad (x \in A \rightarrow x \in B) \quad \wedge \quad \exists x \, (x \in B \wedge x \notin A)$$

Example: $A = \{c, f, z\}$ and $B = \{a, b, c, d, e, f, t, x, z\}$.

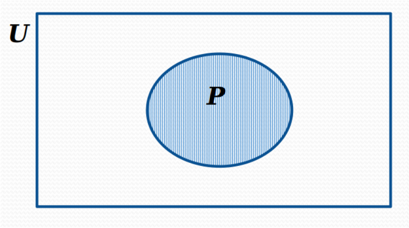
Venn Diagram for a proper subset $A \subset B$



Plan for Part I

Venn diagrams and truth sets

Consider any predicate P(x) for elements x in U and its truth set $P = \{x \mid P(x)\}$.



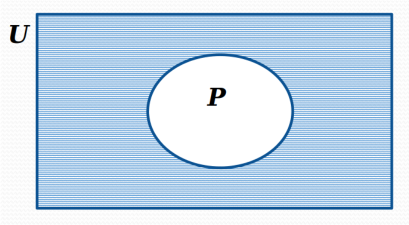
– truth set of $P = \{\ x \mid P(x)\ \}$

that is, all elements x where $P(x)$ is true

Note that: $x \in P \quad \equiv \quad P(x)$

Venn diagrams and logical connectives: negations

Consider any predicate $P(x)$ for elements x in U and its truth set $P = \{x \mid P(x)\}$.



– truth set of $\{x \mid \neg P(x)\}$

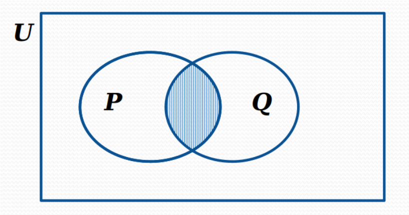
all elements x where $\neg P(x)$ is true, i.e. where $P(x)$ is false

Same as *complement* of set P (see section 2.4)

Note that: $x \notin P \equiv \neg P(x)$

Venn Diagrams and logical connectives: conjunctions

Consider two arbitrary predicates $P(x)$ and $Q(x)$ defined for elements x in U together with their corresponding truth sets $P = \{x \mid P(x)\}$ and $Q = \{x \mid Q(x)\}$.

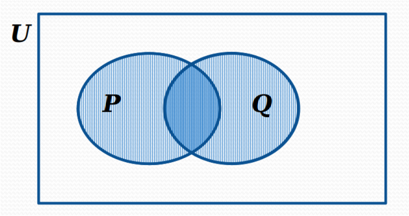


– truth set of $\{x \mid P(x) \wedge Q(x)\}$ that is, all elements x where both $P(x)$ and $Q(x)$ is true

Same as *intersection* of sets P and Q (see section 2.3)

Venn Diagrams and logical connectives: disjunctions

Consider two arbitrary predicates $P(x)$ and $Q(x)$ defined for elements x in U together with their corresponding truth sets $P = \{x \mid P(x)\}$ and $Q = \{x \mid Q(x)\}$.

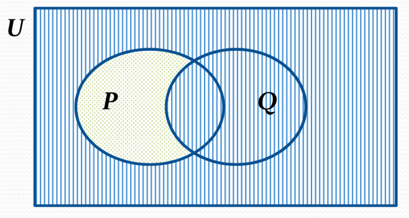


– truth set $\{x \mid P(x) \vee Q(x)\}$ that is, all elements x where $P(x)$ or $Q(x)$ is true

Same as *union* of sets P and Q (see section 2.2)

Venn diagrams and logical connectives: implications

Consider two arbitrary predicates $P(x)$ and $Q(x)$ defined for elements x in U together with their corresponding truth sets $P = \{x \mid P(x)\}$ and $Q = \{x \mid Q(x)\}$.



– truth set $\{x \mid P(x) \rightarrow Q(x)\}$

(all x where implication $P(x) \rightarrow Q(x)$ is true)

– set where implication $P(x) \rightarrow Q(x)$ is false:

$$\{x \mid \neg(\neg P(x) \lor Q(x))\} = \{x \mid P(x) \land \neg Q(x)\}$$
$$= \{x \mid x \in P \land x \notin Q\}$$

1. Remember:

$$p \rightarrow q \equiv \neg p \lor q$$

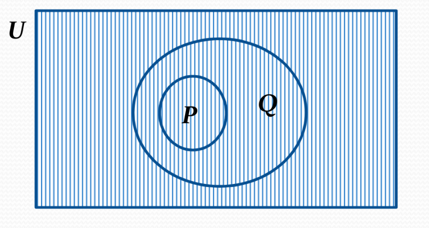
2. Thus, we have:

$$\{x \mid P(x) \rightarrow Q(x)\} = \{x \mid \neg P(x) \lor Q(x)\} = \{x \mid x \notin P \lor x \in Q\}$$

Venn diagram and implication: special case

1. Assume that implication $P(x) \to Q(x)$ is true for all x.
2. That is, assume $\{x \mid P(x) \to Q(x)\} \equiv U$ is true.
3. Note that, from the definition of subsets:

$$\forall x (P(x) \to Q(x)) \equiv \forall x (x \in P \to x \in Q) \equiv P \subseteq Q$$



– truth set $\{x \mid P(x) \to Q(x)\}$

(all x where implication $P(x) \to Q(x)$ is true)

– set where implication $P(x) \to Q(x)$ is false: empty in this case: $\{x \mid x \in P \wedge x \notin Q\} = \varnothing$

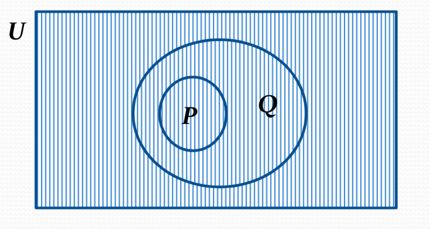
Venn diagram for $P \subseteq Q$ often shows P as a proper subset of Q, thus assuming $P \neq Q$

$$\begin{aligned} \forall x (P(x) \to Q(x)) &\equiv \forall x (\neg P(x) \vee Q(x)) \\ &\equiv \neg \exists x (P(x) \wedge \neg Q(x)) \\ &\equiv \neg \exists x (x \in P \wedge x \notin Q) \end{aligned}$$

Venn diagram and implication: special case

1. Assume that implication $P(x) \rightarrow Q(x)$ is true for all x.
2. That is, assume $\{x \mid P(x) \rightarrow Q(x)\} \equiv U$ is true.
3. Note that, from the definition of subsets:

$$\forall x(P(x) \rightarrow Q(x)) \quad \equiv \quad \forall x(x \in P \rightarrow x \in Q) \quad \equiv \quad P \subseteq Q$$



 – truth set $\{x \mid P(x) \rightarrow Q(x)\}$

(all x where implication $P(x) \rightarrow Q(x)$ is true)

 – set where implication $P(x) \rightarrow Q(x)$ is false: empty in this case: $\{x \mid x \in P \wedge x \notin Q\} = \varnothing$

Venn diagram for $P \subseteq Q$ often shows P as a proper subset of Q, thus assuming $P \neq Q$

1. This gives intuitive interpretation for logical "implications":
2. Proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$ is equivalent to proving the subset relationship for the truth sets $P \subseteq Q$.

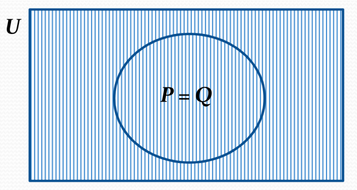
Venn diagram and logical connectives: biconditional

1. Similarly one can show that

$$\forall x P(x) \leftrightarrow Q(x) \equiv P = Q$$

2. that is,

$$\forall x (P(x) \to Q(x) \land Q(x) \to P(x)) \equiv P \subseteq Q \land Q \subseteq P.$$



 – truth set of $\{x \mid P(x) \leftrightarrow Q(x)\}$

assuming $\forall x (P(x) \leftrightarrow Q(x))$ is true

1. This gives intuitive interpretation for "biconditional":
2. Proving theorems of the form $\forall x (P(x) \leftrightarrow Q(x))$ is equivalent to proving the subset relationship for the truth sets $P = Q$.

Plan for Part I

Set cardinality

Definition: If there are exactly n (distinct) elements in S where n is a non-negative integer, we say that S is *finite*. Otherwise, the set S is said *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\varnothing| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\varnothing\}| = 1$
5. The set of integers is infinite.

Plan for Part I

Power sets

Definition: The set of all subsets of a set A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example: If $A = \{a,b\}$ then:

$$\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$$

1. If a set has n elements, then the cardinality of the power set is 2^n.
2. In Chapters 5 and 6, we will discuss different ways to show this.

Plan for Part I

Tuples

1. The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
2. Two n-tuples are equal if and only if their corresponding elements are equal.
3. 2-tuples are called *ordered pairs*, e.g. (a_1, a_2)
4. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product

René

Descartes

(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

Example:

$$A = \{a, b\}, B = \{1, 2, 3\}$$

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. Relations will be covered in depth in Chapter 9.

Cartesian product

Definition: The Cartesian products of the sets $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Question: What is $A \times B \times$ C where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$?

Solution: $A \times B \times$ C $=$ {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

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Plan for Part I

Boolean Algebra

1. Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in CS2209.
2. The operators in set theory are analogous to corresponding operators in propositional calculus.
3. As always there must be a universal set U.
4. All sets $A, B, \ldots$ shown in the next slides are assumed to be subsets of U.

Plan for Part I

Union

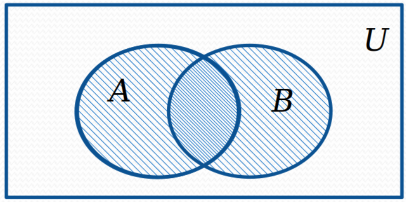
1. **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

2. **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Union is analogous to disjunction, see earlier slides.

Plan for Part I

Intersection

1. **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is:

$$\{x \mid x \in A \wedge x \in B\}$$

2. If the intersection is empty, then A and B are said to be *disjoint*.

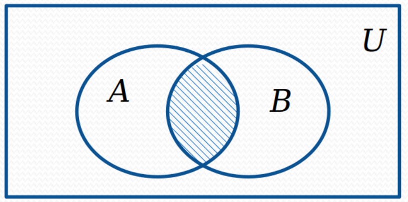
1. **Example**: What is $\{1,2,3\} \cap \{3,4,5\}$?

   Solution: $\{3\}$

1. **Example:** What is $\{1,2,3\} \cap \{4,5,6\}$?

   Solution: $\varnothing$

Venn Diagram for $A \cap B$



Intersection is analogous to conjunction, see earlier slides.

Plan for Part I

Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\overline{A}$ is the set:

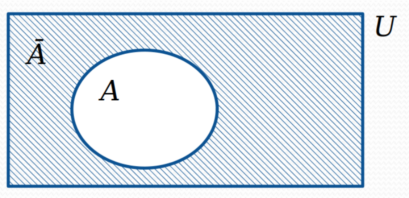
$$\overline{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$?

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for complement



Complement is analogous to negation, see earlier.

Plan for Part I

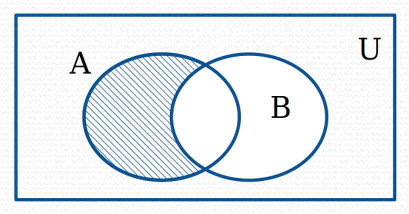
Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Note: $\overline{A} = U$ - A

Venn Diagram for $A - B$

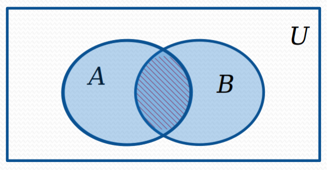


Plan for Part I

The cardinality of the union of two sets

1. Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Venn Diagram for $A, B, A \cap B, A \cup B$

1. **Example**:
   a. Let A be the math majors in your class and B be the CS majors in your class.
   b. To count the number of students in your class who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
   c. We will return to this principle in Chapter 6 and Chapter 8, where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Review questions

Example: Given $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$ solve the following:

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\overline{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\overline{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Plan for Part I

Set identities

1. Identity laws

$$A \cup \varnothing = A \qquad\qquad A \cap U = A$$

2. Domination laws

$$A \cup U = U \qquad\qquad A \cap \varnothing = \varnothing$$

3. Idempotent laws

$$A \cup A = A \qquad\qquad A \cap A = A$$

4. Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide ↪

Set identities

1. Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

2. Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide ↪

Set identities

1. De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

2. Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

3. Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Proving set identities

1. Different ways to prove set identities:
   a. Prove that each set (i.e. each side of the identity) is a subset of the other.
   b. Use set builder notation and propositional logic.
   c. Membership tables

(to be explained)

Proof of second De Morgan law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1. $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$
2. $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide $\hookrightarrow$

Proof of second De Morgan law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	definition of complement
$\neg((x \in A) \wedge (x \in B))$	definition of intersection
$\neg(x \in A) \vee \neg(x \in B)$	De Morgan's 1^{st} Law
$(x \notin A) \vee (x \notin B)$	definition of negation
$(x \in \overline{A}) \vee (x \in \overline{B})$	definition of complement
$x \in \overline{A} \cup \overline{B}$	definition of union

Continued on next slide ↪

Proof of second De Morgan law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	definition of union
$(x \notin A) \vee (x \notin B)$	definition of complement
$\neg(x \in A) \vee \neg(x \in B)$	definition of negation
$\neg((x \in A) \wedge (x \in B))$	De Morgan's 1^{st} Law
$\neg x \in (A \cap B)$	definition of intersection
$x \in \overline{A \cap B}$	definition of complement

■

Set-builder notation: second De Morgan law

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg x \in (A \cap B)\} && \text{by definition of `not in'} \\
&= \{x \mid \neg((x \in A) \wedge (x \in B))\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by De Morgan's } 1^{st} \text{ Law} \\
&= \{x \mid (x \notin A) \vee (x \notin B)\} && \text{by definition of `not'} \\
&= \{x \mid (x \in \overline{A}) \vee (x \in \overline{B})\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by definition of notation}
\end{aligned}
$$

■

Membership table

Example: Construct a membership table to show that the distributive law holds:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Plan for Part I

Generalized unions and intersections

1. Let A_1, A_2 ,. . . , A_n be an indexed collection of sets.
   We define:
   $$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$
   $$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$
   These are well defined, since union and intersection are associative.
2. *Example*: for $(i = 1, 2, \ldots)$ let $A_i = \{i, i+1, i+2, \ldots\}$. Then,
   $$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$
   $$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$