Induction and Recursion

Chapter 5

UWO – November 15, 2020

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Climbing an infinite ladder

Suppose we have an infinite ladder:

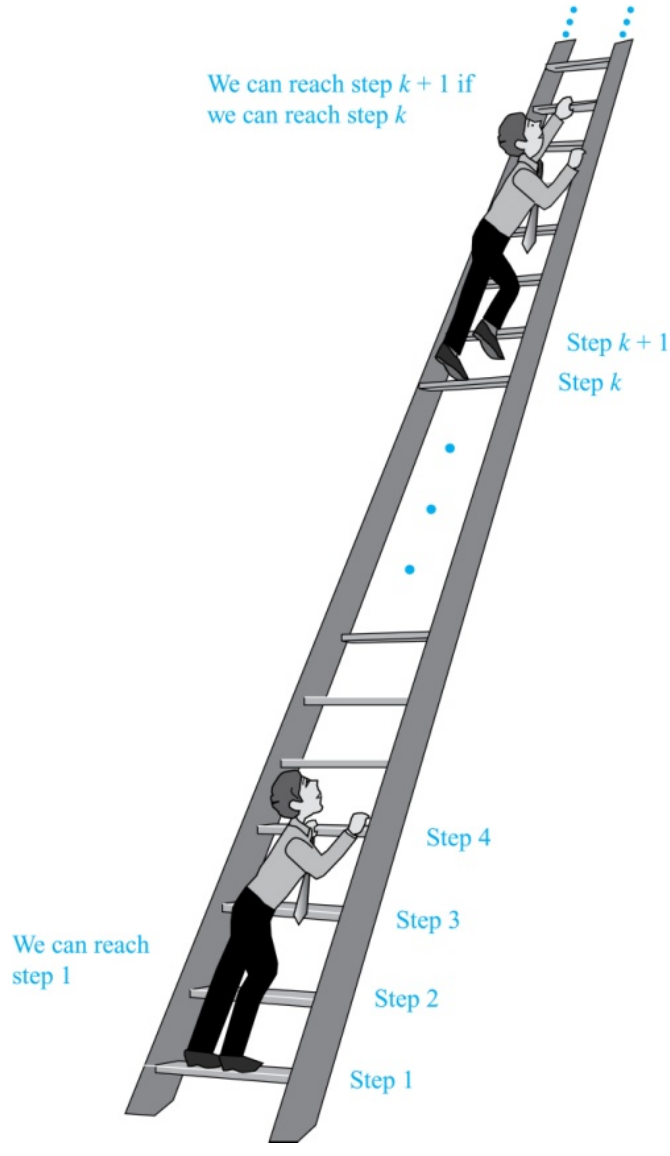
(1) We can
reach the first rung of the ladder.

(2) If we can reach
a particular rung of the ladder,
then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on.

We can apply (2) any number of times so that **we can reach any particular rung, no matter how high up**.

This example motivates the idea of proof by mathematical induction.



Principle of mathematical induction

To prove that a proposition $P(n)$ is true for all positive integers n, we complete these two steps:

1. *Basis Step*: Show that $P(1)$ is true.
2. *Inductive Step*: Show that the implication $P(k) \to P(k+1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, we show that $P(k+1)$ must be true.

Example (Climbing an infinite ladder)

1. BASIS STEP: By (1), we can reach Rung 1.
2. INDUCTIVE STEP: Assume the inductive hypothesis that we can reach Rung k. Then by (2), we can reach Rung $(k+1)$.

Hence, $P(k) \to P(k+1)$ is true for all positive integers k. We can reach every rung on the ladder. ■

Important points about using mathematical induction

1. Mathematical induction can be expressed as the rule of inference

$$(P(1) \wedge \forall k(P(k) \rightarrow P(k+1))) \quad \rightarrow \quad \forall n P(n)$$

   where the domain is the set of positive integers.
2. In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k+1)$ must also be true.
3. Proofs by mathematical induction do not always start at the integer 1. The basis step may begin at a starting point b where b is an integer. We will see examples of this soon.

Validity of mathematical induction

Mathematical induction is valid because of the well ordering property, which states that "*every nonempty subset of the set of positive integers has a least element.*"

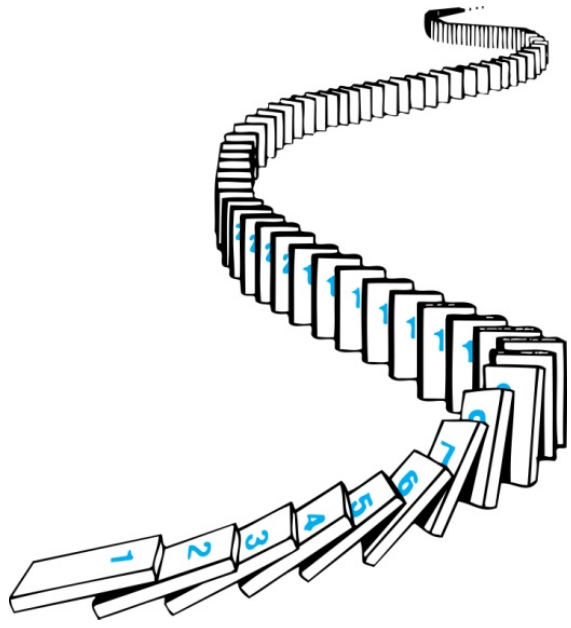
Here is the proof (**by contradiction**) that mathematical induction is valid:

1. Suppose that $P(1)$ holds and $P(k) \rightarrow P(k+1)$ is true for all positive integers k.
2. Assume there is at least one positive integer n for which P (n) is false. Then the set S of positive integers for which P (n) is false is nonempty.
3. By the well-ordering property, S has a least element, say m.
4. We know that m can not be 1 since $P(1)$ holds.
5. Since m is positive and greater than 1, $m-1$ must be a positive integer. Since $m-1 < m$, it is not in S, so $P(m-1)$ must be true.
6. But then, since the conditional $P(k) \rightarrow P(k+1)$ for every positive integer k holds, $P(m)$ must also be true. This contradicts $P(m)$ being false.
7. Hence, $P(n)$ must be true for every positive integer n.

Remembering how mathematical induction works

Consider an infinite
sequence of dominoes, labeled $1, 2, 3, \ldots$,
where each domino is standing.

1. Let $P(n)$ be the proposition that the $n-$ th domino is knocked over.
2. We know that the first domino is knocked down, i.e., $P(1)$ is true.
3. We also know that if whenever the k th domino is knocked over, it knocks over the $(k+1)$ ht domino, i.e, $P(k) \rightarrow P(k+1)$ is true for all positive integers k.
4. Hence, all dominos are knocked over.
5. $P(n)$ is true for all positive integers n.



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Proving a summation formula by mathematical induction

Example

Show that: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Note: Once we have this conjecture, mathematical induction can be used to prove it correct, that is, true for $P(k+1)$.

Solution:

1. BASIS STEP: $P(1)$ is true since $\frac{1(1+1)}{2} = 1$.
2. INDUCTIVE STEP: Assume true for $P(k)$.
   a. The inductive hypothesis is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
   b. Under this assumption,

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

that is, true for P(k+1)■

Conjecturing and proving correct a summation formula

Example

Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture.

Solution: We have:
$1 = 1, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, 1 + 3 + 5 + 7 + 9 = 25.$

1. We can conjecture that the sum of the first n positive odd integers is n^2,
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$
2. We will prove the conjecture is proved correct with **mathematical induction**:
   a. BASIS STEP: $P(1)$ is true since $1^2 = 1$.
   b. INDUCTIVE STEP: prove $P(k) \rightarrow P(k + 1)$ for every positive integer k. Assume the inductive hypothesis holds and then show that $P(k)$ holds has well.
   c. **Inductive Hypothesis** $P(k)$: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$
   d. So, assuming $P(k)$, it follows that:
   $$1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1)$$
   $$= k^2 + (2k + 1) \text{(by the inductive hypothesis)}$$
   $$= k^2 + 2k + 1$$
   $$= (k + 1)^2$$
   e. Hence, we have shown that $P(k + 1)$ **follows from** $P(k)$.
   f. Therefore the sum of the first n positive odd integers is n^2. ■

Proving inequalities

Example

Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution:

1. Let $P(n)$ be the proposition that $n < 2^n$.
2. BASIS STEP: $P(1)$ is true since $1 < 2 = 2^1$.
3. INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.
4. Must show that $P(k+1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:
   $k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$
5. Therefore $n < 2^n$ holds for all positive integers n. ■

Proving inequalities

Example

Use mathematical induction to prove that $2^n < n!$ for every integer $n \geq 4$.

Solution:

1. Let $P(n)$ be the proposition that $2^n < n!$
2. BASIS STEP: $P(4)$ is true since $2^4 = 16 < 24 = 4!$
3. INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. Must show that $P(k+1)$ holds:
   a. $2^{k+1} = 2 \cdot 2^k$
   b. $< 2 \cdot k!$ *by the inductive hypothesis* $P(k)$
   c. $< (k+1)k!$
   d. $= (k+1)!$
4. Therefore, $2^n < n!$ holds, for every integer $n \geq 4$. ■

Note that here the basis step is $P(4)$, since $P(0), P(1), P(2)$, and $P(3)$ are all false.

Proving divisibility results

Example

Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Solution:

1. Let $P(n)$ be the proposition that $n^3 - n$ is divisible by 3.
2. BASIS STEP: $P(1)$ is true since $1^3 - 1 = 0$, which is divisible by 3.
3. INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k^3 - k$ is divisible by 3, for an arbitrary positive integer k. To show that $P(k+1)$ follows:
   a. $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$
   b. $= (k^3 - k) + 3(k^2 + k)$
4. By the inductive hypothesis, the first term $k^3 - k$) is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. By part (i) of Theorem 1 (Sec.4.1), $(k+1)^3 - (k+1)$ is divisible by 3.
5. Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n. ■

Number of subsets of a finite set

Example

Use mathematical induction to show that if S is a finite set with n elements, where n is a non-negative integer, then S has 2^n subsets. That is, the cardinality of the *power set* for S is $| P(S) | = 2^k$

Solution:

1. Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.
2. Basis Step: $P(0)$ is true, because the empty set has only one subset (itself) and $1 = 2^0$.
3. Inductive Step: Assume $P(k)$ is true for an arbitrary non-negative integer k.

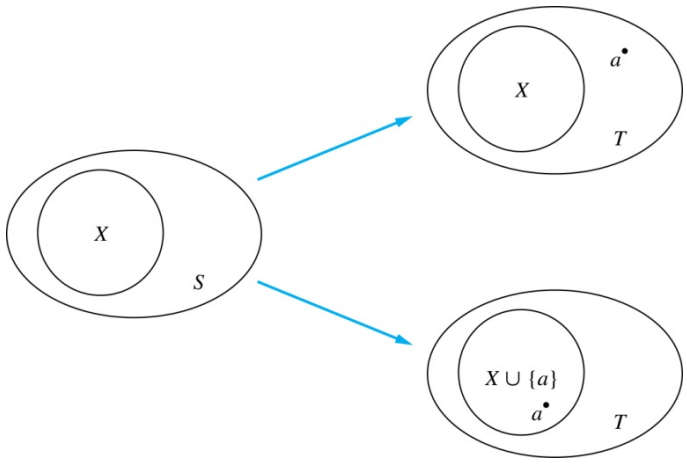
continued →

Number of subsets of a finite set

Inductive Hypothesis: For an arbitrary non-negative integer k, every set with k elements has 2^k subsets.

4. Let T be a set with $k+1$ elements. Choose any element $a \in T$. Then $T = S \cup \{a\}$, where $S = T - \{a\}$. Note that $| S | = k$.

5. For each subset X of S there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.



6. By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$. ■

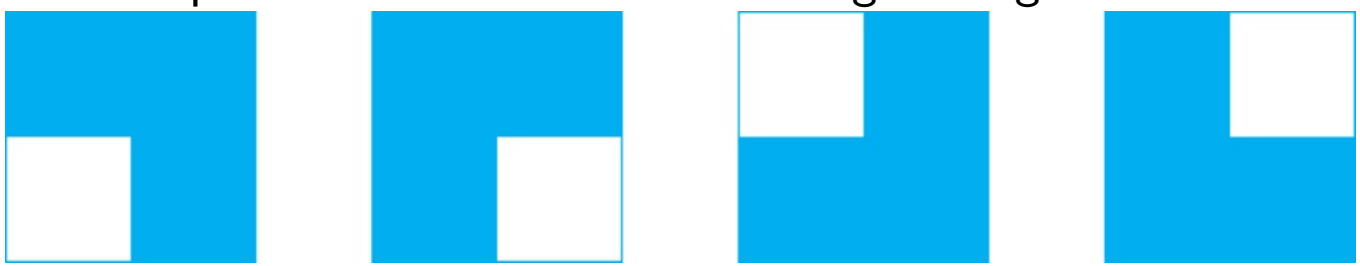
Tiling checkerboards

Example

Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right *triominoes*. A right triomino is an L-shaped tile which covers 3 squares at a time.

Solution:

1. Let $P(n)$ be the proposition that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using right triominoes. Use mathematical induction to prove that $P(n)$ is true for all positive integers n.
2. BASIS STEP: P(1) is true, because each of the four 2×2 checkerboards with one square removed can be tiled using one right triomino.



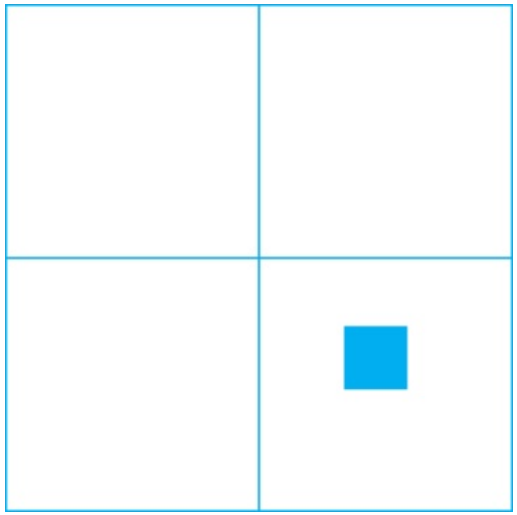
3. INDUCTIVE STEP: Assume that $P(k)$ is true for every $2^k \times 2^k$ checkerboard, for some positive integer k.

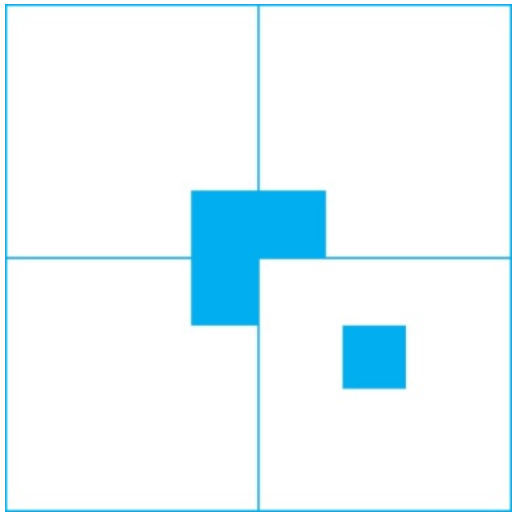
continued →

Tiling checkerboards

Inductive Hypothesis: Every $2^k \times 2^k$ checkerboard, for some positive integer k, with one square removed can be tiled using right triominoes.

4. Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. Split this checkerboard into four checkerboards of size $2^k \times 2^k$, by dividing it in half in both directions.





5. Remove a square from one of the four $2^k \times 2^k$ checkerboards. By the inductive hypothesis, this board can be tiled.
6. Also by the inductive hypothesis, the other three boards can be tiled with the square from the corner of the center of the original board removed. We can then cover the three adjacent squares with a triomino.
7. Hence, the entire $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be tiled using right triominoes. ■

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An incorrect "proof" by mathematical induction

Example

Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. Here is a "proof" that $P(n)$ is true for all positive integers $n \geq 2$.

1. BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point.
2. INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer $k \geq 2$, i.e., every set of k lines in the plane, no two of which are parallel, meet in a common point.
3. We must show that if $P(k)$ holds, then $P(k+1)$ holds, i.e., if every set of k lines in the plane, no two of which are parallel, $k \geq 2$, meet in a common point, then every set of k +1 lines in the plane, no two of which are parallel, meet in a common point.

An incorrect "proof" by mathematical induction

Inductive Hypothesis: Every set of k lines in the plane, where $k \geq 2$, no two of which are parallel, meet in a common point.

1. Consider a set of $k+1$ distinct lines in the plane, no two parallel. By the inductive hypothesis, the first k of these lines must meet in a common point p_1. By the inductive hypothesis, the last k of these lines meet in a common point p_2.
2. If p_1 and p_2 are different points, all lines containing both of them must be the same line since two points determine a line. This contradicts the assumption that the lines are distinct. Hence, point $p_1 = p_2$ lies on all $k+1$ distinct lines, and therefore $P(k+1)$ holds. Assuming that $k \geq 2$, distinct lines meet in a common point, then every $k+1$ lines meet in a common point.
3. There must be an error in this proof since the conclusion is absurd. But where is the error?
   Answer: $P(k) \rightarrow P(k+1)$ only holds for $k \geq 3$. It is not the case that $P(2)$ implies $P(3)$. The first two lines must meet in a common point p_1 and the second two must meet in a common point p_2. They do not have to be the same point since only the second line is common to both sets of lines.

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Guidelines: mathematical induction proofs

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

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Strong induction

Strong Induction:

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:

1. *Basis Step*: Verify that the proposition $P(1)$ is true.
2. *Inductive Step*: Show the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ holds for all positive integers k.

Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Strong induction and the infinite ladder

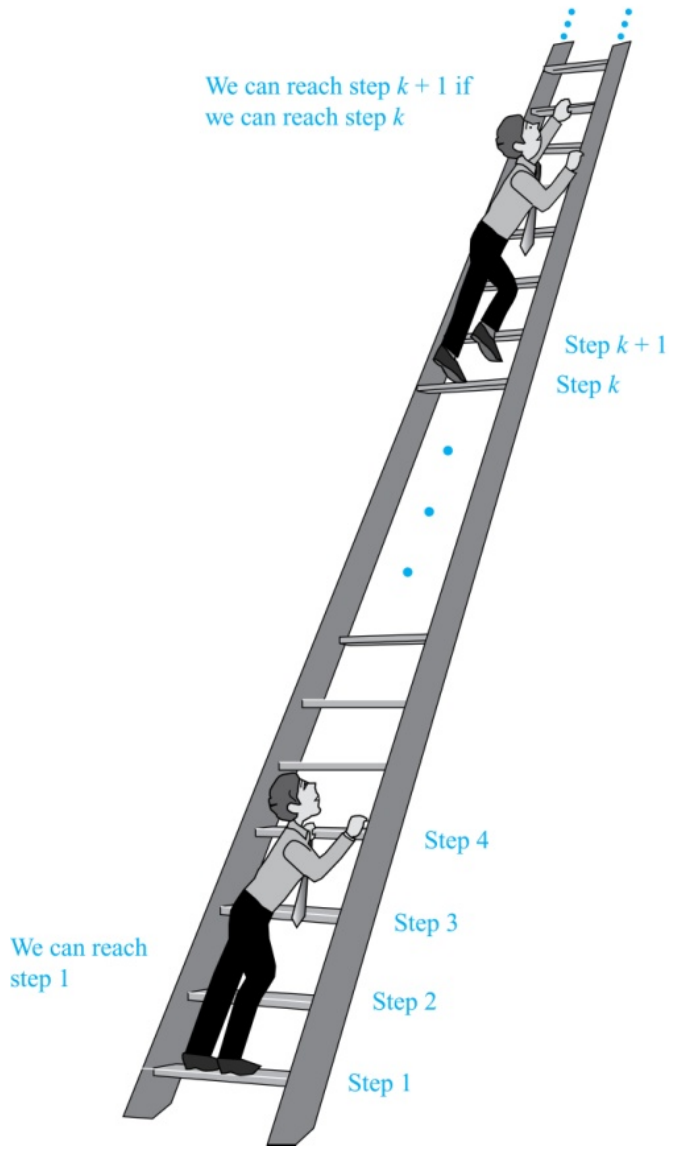
Strong induction
tells us that we can reach all rungs if:

1. We can
   reach the first rung of the ladder.
2. For every integer k, if we
   can reach the first k rungs, then
   we can reach the $(k+1)$-th rung.

To conclude that we can
reach every rung by strong induction:

1. BASIS STEP: $P(1)$ holds
2. INDUCTIVE STEP: Assume
   $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ holds for
   an arbitrary integer k, and show
   that $P(k+1)$ must also hold .

We will have then shown by strong induction that for every integer $n > 0$, the property $P(n)$ holds, that is, we can reach the n-th rung of the ladder.



Which form of induction should be used?

1. We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction.
2. In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent.
3. Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Complete proof of the fundamental theorem of arithmetic

Example

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution:

1. Let $P(n)$ be the proposition that n can be written as a product of primes.
2. BASIS STEP: $P(2)$ is true since 2 itself is prime.
3. INDUCTIVE STEP: The inductive hypothesis is $P(j)$ is true for all integers j with $2 \leq j \leq k$. To show that $P(k+1)$ must be true under this assumption, two cases need to be considered:
   a. If $k+1$ is prime, then $P(k+1)$ is true.
   b. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. By the inductive hypothesis a and b can be written as the product of primes and therefore $k+1$ can also be written as the product of those primes.
4. Hence, it has been shown that every integer greater than 1 can be written as the product of primes. ■

Proof using strong induction

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution:

1. Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.
2. BASIS STEP: $P(12), P(13), P(14)$, and $P(15)$ hold.
   a. $P(12)$ uses three 4-cent stamps.
   b. $P(13)$ uses two 4-cent stamps and one 5-cent stamp.
   c. $P(14)$ uses one 4-cent stamp and two 5-cent stamps.
   d. $P(15)$ uses three 5-cent stamps.
3. INDUCTIVE STEP: The inductive hypothesis states that $P(j)$ holds for $12 \leq j \leq k$, where $k \geq 15$. Assuming the inductive hypothesis, it can be shown that $P(k+1)$ holds.
4. Using the inductive hypothesis, $P(k-3)$ holds since $k-3 \geq 12$. To form postage of $k+1$ cents, add a 4-cent stamp to the postage for $k-3$ cents.
5. Hence, $P(n)$ holds for all $n \geq 12$. ■

Proof of the same example using mathematical induction

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let $P(n)$ be the proposition that postage of n cents can be formed using 4-cent and 5-cent stamps.

1. BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps. Thus, P(12) holds. We also checked P(13), P(14), P(15).
2. INDUCTIVE STEP: The inductive hypothesis $P(k)$ for any positive integer $k \geq 15$ is that postage of k cents can be formed using 4-cent and 5-cent stamps. To show $P(k+1)$ where $k \geq 15$, we consider **two cases**:
   a. If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of k +1 cents.
   b. Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of k +1 cents.
3. Hence, $P(n)$ holds for all $n \geq 12$. ■

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Well-ordering property

1. *Well-ordering property*: Every nonempty set of non-negative integers has a least element.
2. The well-ordering property is one of the axioms of the positive integers.
3. The well-ordering property can be generalized as follows.

Definition

A set is *well ordered* if every non-empty subset has a least element.

1. $\mathbb{N}$ is well ordered under $\leq$.
2. The set of finite strings over an alphabet using lexicographic ordering is well ordered.

Well-ordering property

Example

Use the well-ordering property to prove the *division algorithm* , which states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$, such that $a = dq + r$.

Solution:

1. Given a and $d > 0$, let S be the set of non-negative integers of the form $a - dq$ where q is an integer. The set is nonempty since $-dq$ can be made as large as needed.
2. By the well-ordering property, S has a least element $r = a - dq_0$. The integer r is non-negative.
3. It also must be the case that $r < d$. If $r \geq d$ would hold, then there would be a smaller non-negative element in S, namely,
$$r^* := a - d(q_0 + 1) = a - dq_0 - d = r - d \geq 0.$$
4. Therefore, there are integers q and r with $0 \leq r < d$. ■

(*uniqueness of q and r was proved in Tutorial 6*)

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Recursively defined functions

Definition

A *recursive* or *inductive* definition of a function consists of two steps.

1. BASIS STEP: Specify the value of the function at zero.
2. RECURSIVE STEP: Give a rule for finding function's value at an integer from its values at smaller integers.

NOTE: a function $f(n)$ is the same as a sequence $a_0, a_1, \ldots$ where $f(n) = a_n$. We previously used recurrence relations to define sequences. The above is essentially the same.

Recursively defined functions

Example

Suppose f is defined by: $f(0) = 3, \quad f(n+1) = 2f(n) + 3$
Find $f(1), f(2), f(3), f(4)$
Solution:

1. $f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9$
2. $f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21$
3. $f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45$
4. $f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93$

Example

Give a recursive definition of the factorial function $n!$:
Solution:

1. $f(0) = 1$
2. $f(n+1) = (n+1) \cdot f(n)$

Recursively defined functions

Example

Give a recursive definition of:

$$\sum_{k=0}^{n} a_k$$

Solution:

1. The first part of the definition is

$$\sum_{k=0}^{0} a_k = a_0$$

2. The second part is

$$\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}$$

Fibonacci Numbers

Fibonacci (1170 - 1250)

Example

The Fibonacci numbers are defined as follows:

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

Find f_2, f_3, f_4, f_5.

Solution:

1. $f_2 = f_1 + f_0 = 1 + 0 = 1$
2. $f_3 = f_2 + f_1 = 1 + 1 = 2$
3. $f_4 = f_3 + f_2 = 2 + 1 = 3$
4. $f_5 = f_4 + f_3 = 3 + 2 = 5$

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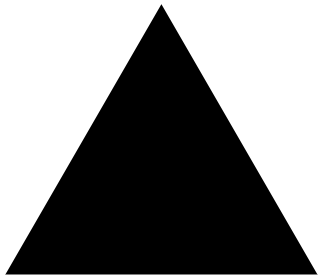
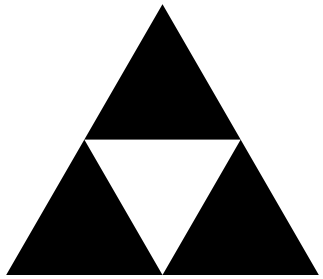
Sierpinski triangle

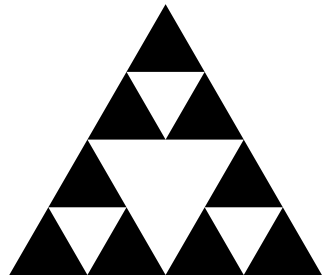
Sierpinski triangles are formed by starting with a triangle and then forming 3 triangles (black) within the original by connecting the midpoints of the sides of the original triangle.



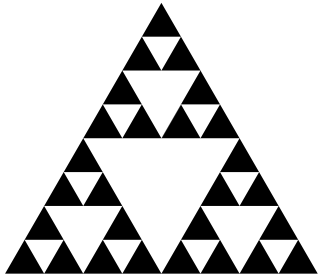
Iteration 0



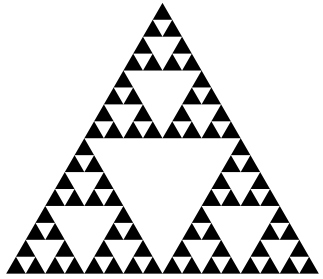
Iteration 1



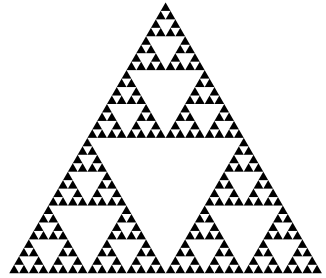
Iteration 2



Iteration 3



Iteration 4



Iteration 5

Recursively defined sets and structures

1. *Recursive definitions* of sets have two parts:
   a. The *basis step* specifies an initial collection of elements.
   b. The *recursive step* gives the rules for forming new elements in the set from those already known to be in the set.
2. Sometimes the recursive definition has an *exclusion rule*, which specifies that the set contains nothing other than those elements specified in the basis step and generated by applications of the rules in the recursive step.

   We will always assume that the exclusion rule holds, even if it is not explicitly mentioned.
3. We will later develop a form of induction, called *structural induction*, to prove results about recursively defined sets.

Recursively defined sets and structures

Example

Subset of Integers S :

1. BASIS STEP: $3 \in S$.
2. RECURSIVE STEP: If $x \in S$ and $y \in S$, then $x + y$ is in S.
3. Initially 3 is in S, then $3 + 3 = 6$, then $3 + 6 = 9$, etc.

Example

The natural numbers $\mathbb{N}$.

1. BASIS STEP: $0 \in \mathbb{N}$.
2. RECURSIVE STEP: If n is in $\mathbb{N}$, then $n + 1$ is in $\mathbb{N}$.
3. Initially 0 is in S, then $0 + 1 = 1$, then $1 + 1 = 2$, etc.

Strings

Definition

The set Σ^* of *strings* over the alphabet Σ:

1. BASIS STEP: $\lambda \in \Sigma^*$ (λ is the empty string)
2. RECURSIVE STEP: If $w \in \Sigma^*$ and $x \in \Sigma$ then $wx \in \Sigma^*$.

Example

If $\Sigma = \{0, 1\}$, the strings in in Σ^* are the set of all bit strings:
$\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots$ etc.

Example

If $\Sigma = \{a, b\}$, show that aab is in Σ^*.

1. Since $\lambda \in \Sigma^*$ and $a \in \Sigma$, $a \in \Sigma^*$.
2. Since $a \in \Sigma^*$ and $a \in \Sigma$, $aa \in \Sigma^*$.
3. Since $aa \in \Sigma^*$ and $b \in \Sigma$, $aab \in \Sigma^*$.

Length of a string

Example

Give a recursive definition of function $\ell : \Sigma^* \to \mathbb{Z}^+$ specifying length of any given string in Σ^*.

Solution: The length of a string can be recursively defined by:

1. $\ell(\lambda) = 0$;
2. $\ell(wx) = \ell(w) + 1$ if $w \in \Sigma^*$ and $x \in \Sigma$.

String concatenation

Definition

Two strings can be combined via the operation of *concatenation*. Let Σ be a set of symbols and Σ^* be the set of strings formed from the symbols in Σ. We can recursively define the concatenation operator " $\cdot$ " mapping two strings to a string, thus defining concatenation as function from $\Sigma^* \times \Sigma^*$ to Σ^*.

1. BASIS STEP: If $w \in \Sigma^*$ then $w \cdot \lambda = w$
2. RECURSIVE STEP: If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$ then $w_1 \cdot (w_2 x) = (w_1 \cdot w_2)x$

1. Often the concatenation $v \cdot u$ of two strings u and v is written as vu.
2. If $v =$ "*abra* " and $u =$ "*cadabra* " , the concatenation is $vu =$ "*abracadabra*".

Rooted trees

Definition

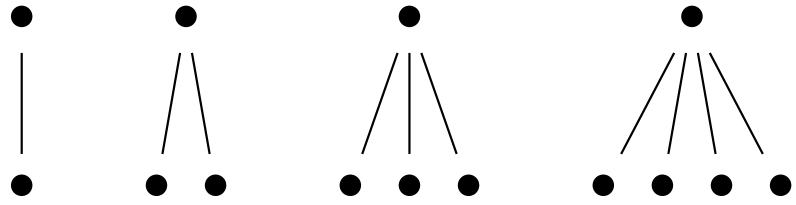
The set of *rooted trees*, where a rooted tree consists of a set of vertices containing a distinguished vertex called the *root*, and edges connecting these vertices, can be defined recursively by these steps:

1. BASIS STEP: A single vertex r is a rooted tree.
2. RECURSIVE STEP: Suppose that $T_1, T_2, \ldots, T_n$ are disjoint rooted trees with roots $r_1, r_2, \ldots, r_n$ respectively. Then the structure formed by starting with a root r (which is not in any of the rooted trees $T_1, T_2, \ldots, T_n$) and adding an edge from r to each of the vertices $r_1, r_2, \ldots, r_n$ is also a rooted tree.

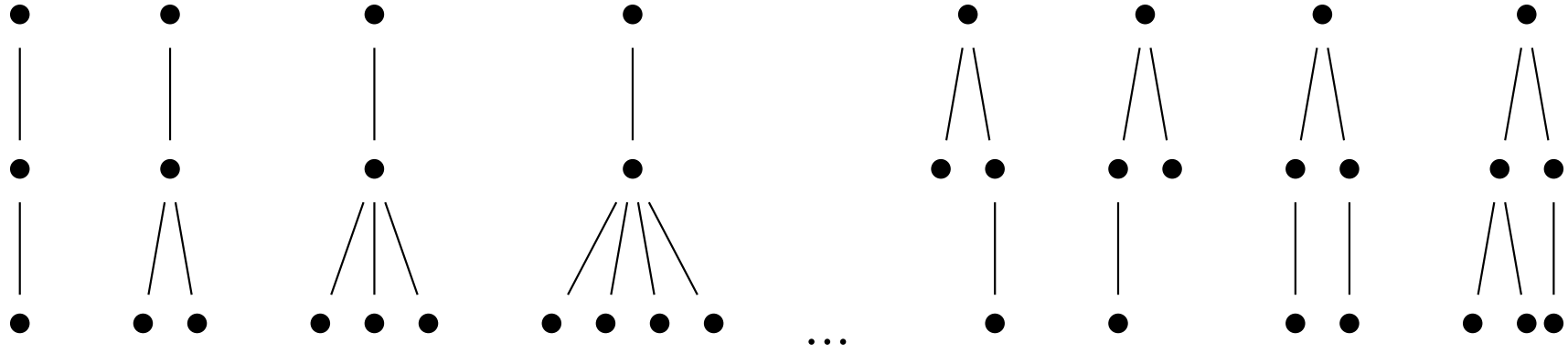
Building up rooted trees

Basis Step

Step 1



Step 2



Next we look at a special type of tree, the full binary tree.

Full binary trees

Definition

The set of *full binary trees* can be defined recursively by these steps.

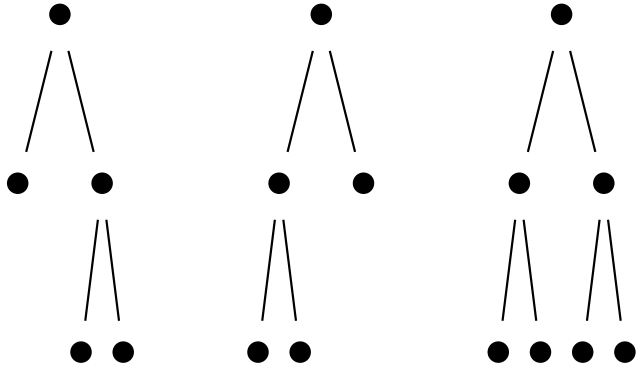
1. BASIS STEP: There is a full binary tree consisting of only a single vertex r.
2. RECURSIVE STEP: If T_1 and T_2 are disjoint full binary trees, there is a full binary tree (denoted by $T_1 \cdot T_2$) consisting of a root r together with edges connecting the root to each of the roots of the left subtree T_1 and the right subtree T_2.

Building up full binary trees

Basis Step

Step 1

Step 2



Induction and recursively defined sets

Example

Show that the set S (defined by specifying that $3 \in S$ and that if $x \in S$ and $y \in S$, then $x + y$ is in S) is the set of all positive integers that are multiples of 3.

Solution:

1. Let A be the set of all positive integers divisible by 3.
2. To prove that $A = S$, show that A is a subset of S and S is a subset of A.
3. A $\subset$ S:
   a. Let $P(n)$ be the statement that $3n$ belongs to S.
   b. BASIS STEP: $3 \cdot 1 = 3 \in S$, by the first part of recursive definition.
   c. INDUCTIVE STEP: Assume $P(k)$ is true. By the second part of the recursive definition, if $3k \in S$, then since $3 \in S$, $3k + 3 = 3(k + 1) \in S$. Hence, $P(k + 1)$ is true.
4. S $\subset$ A:
   BASIS STEP: $3 \in S$ by the first part of recursive definition, and $3 = 3 \cdot 1$.
   INDUCTIVE STEP: The second part of the recursive definition adds $x + y$ to S, if both x and y are in S. If x and y are both in A, then both x and y are divisible by 3. It follows that $x + y$ is divisible by 3.

Part (4) is known as structural induction.

Plan for Chapter 5

Structural induction

Definition

To prove a property of the elements of a recursively defined set, we use *structural induction*.

1. BASIS STEP: Show that the result holds for all elements specified in the basis step of the recursive definition.
2. RECURSIVE STEP: Show that if the statement is true for each of the (old) elements used to construct new elements in the recursive step of the definition, then the result holds for these new elements.
3. The validity of structural induction can be shown to follow from the principle of mathematical induction.

Full binary trees

Definition

The *height* h(T) of a full binary tree T is defined recursively as follows:

1. BASIS STEP: The height of a full binary tree T consisting of only a root r is $h(T) = 0$.
2. RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height $h(T) = 1 + max(h(T_1), h(T_2))$.

The number of vertices $n(T)$ of a full binary tree T satisfies the following recursive formula:

1. **BASIS STEP**: The number of vertices of a full binary tree T consisting of only a root r is $n(T) = 1$.
2. **RECURSIVE STEP**: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has the number of vertices $n(T) = 1 + n(T_1) + n(T_2)$.

Structural induction and binary trees

Theorem

If T is a full binary tree, then $n(T) \leq 2^{h(T)+1} - 1$.

Proof.

Use structural induction.

1. BASIS STEP: The result holds for a full binary tree consisting only of a root, $n(T) = 1$ and $h(T) = 0$. Hence, $n(T) = 1 \leq 2^{0+1} - 1 = 1$.
2. RECURSIVE STEP: Assume $n(T_1) \leq 2^{h(T1)+1} - 1$ and also $n(T_2) \leq 2^{h(T2)+1} - 1$ whenever T_1 and T_2 are full binary trees.

$$
\begin{aligned}
n(T) &= 1 + n(T_1) + n(T_2) && \textit{by recursive formula of } n(T) \\
&\leq 1 + (2^{h(T1)+1} - 1) + (2^{h(T2)+1} - 1) && \textit{by inductive hypothesis} \\
&\leq 2 \cdot max(2^{h(T1)+1}, 2^{h(T2)+1}) - 1 \\
&= 2 \cdot 2^{max(h(T1),h(T2))+1} - 1 && \textit{since } max(2^x, 2^y) = 2^{max(x,y)} \\
&= 2 \cdot 2^{h(T)} - 1 && \textit{by recursive definition of } h(T) \\
&= 2^{h(T)+1} - 1
\end{aligned}
$$

■

Plan for Chapter 5

1. Mathematical Induction

1.1 Mathematical Induction
1.2 Examples of Proof by Mathematical Induction
1.3 Mistaken Proofs by Mathematical Induction
1.4 Guidelines for Proofs by Mathematical Induction

2. Strong Induction and Well-Ordering

2.1 Strong Induction
2.2 Well-Ordering Property

3. Recursive Definitions and Structural Induction

3.1 Recursively Defined Functions
3.2 Recursively Defined Sets and Structures
3.3 Structural Induction

4. Recursive Algorithms

4.1 Recursive Algorithms
4.2 Proving Correctness of Recursive Algorithms

Plan for Chapter 5

Recursive algorithms

Definition

An algorithm is called *recursive* if it solves a problem by reducing it to an instance of the same problem with smaller input size.

1. As for any algorithm, proving a recursive algorithm amounts to proving that this algorithm is correct (that is, satisfies its specifications) and proving that this algorithm terminates (that is, any call to that algorithm executes in finitely many steps).
2. For the algorithm to terminate, the instance of the problem must eventually be reduced to some initial case for which the solution is known.

Recursive factorial algorithm

Example

Give a recursive algorithm for computing $n!$, where n is a non-negative integer.

Solution: Use the recursive definition of the factorial function.

Algorithm 1 factorial (n)

Require: $n \in \mathbb{Z}^+$
Ensure: $n!$, the factorial of n.

```
1: if n = 0 then
2:     return 1
3: else
4:     return n · factorial(n − 1)
5: end if
```

Recursive exponentiation algorithm

Example

Give a recursive algorithm for computing a^n , where a is a nonzero real number and n is a non-negative integer.

Solution: Use the recursive definition of a^n.

Algorithm 2 power (a, n)

Require: $a \in \mathbb{R}, n \in \mathbb{Z}^+, \quad a \neq 0$
Ensure: a^n, the power of a to n.

1: **if** $n = 0$ **then**
2: **return** 1
3: **else**
4: **return** $a \cdot power(a, n-1)$
5: **end if**

Recursive GCD algorithm

Example

Give a recursive algorithm for computing the greatest common divisor of two non-negative integers a and b with $a < b$. **Solution**: Use the reduction $\gcd(a, b) = \gcd(a,\ a \bmod b)$ and the condition $\gcd(0, b) = b$ when $b > 0$.

Algorithm 3 $\gcd(a, b)$

Require: $a, b \in \mathbb{Z}^+, \quad a < b$

Ensure: $\gcd(a, b)$, the GCD of a and b.

```
1: if a = 0 then
2:     return b
3: else
4:     return gcd(a, a mod b)
5: end if
```

Plan for Chapter 5

Proving recursive algorithms correct

Both mathematical and strong induction are useful techniques to show that recursive algorithms always produce the correct output.

Example

Algorithm 2 power (a, n)

Require: $a \in \mathbb{R}, n \in \mathbb{Z}^+, \quad a \neq 0$
Ensure: a^n, the power of a to n.

```
1: if n = 0 then
2:     return 1
3: else
4:     return a · power(a, n − 1)
5: end if
```

Prove that the algorithm for computing the powers of real numbers is correct.
Solution: Use mathematical induction on the exponent n.

1. BASIS STEP: $a^0 = 1$ for every nonzero real number a, and $power(a, 0\ ) = 1$.
2. INDUCTIVE STEP: The inductive hypothesis is that $power(a, k) = a^k$, for all $a \neq 0$.
3. Assuming the inductive hypothesis, the algorithm correctly computes a^{k+1}, since $power(a, k + 1) = a \cdot power(a, k) = a \cdot a^k = a^{k+1}$. ■