UWO CS2214

Tutorial #2

Problem 1 Suppose the variable x represents students, y represents courses, and $T(x, y)$ means "x is taking y". Translate the following statements into the language of predicate logic using these variables, predicate, and any needed quantifiers.

1. Every course is being taken by at least one student,
2. Some student is taking every course,
3. No student is taking all courses,
4. There is a course that all students are taking,
5. Every student is taking at least one course,
6. There is a course that no students are taking,
7. Some students are taking no courses,
8. No course is being taken by all students,
9. Some courses are being taken by no students,
10. No student is taking any course.

Solution 1

1. $(\forall C)(\exists S) \quad T(S, C)$,
2. $(\exists S)(\forall C) \quad T(S, C)$,
3. $(\forall S)(\exists C) \quad \neg T(S, C)$,
4. $(\exists C)(\forall S) \quad T(S, C)$,
5. $(\forall S)(\exists C) \quad T(S, C)$,
6. $(\exists C)(\forall S) \quad \neg T(S, C)$,
7. $(\exists S)(\forall C) \quad \neg T(S, C)$,
8. $(\forall C)(\exists S) \quad \neg T(S, C)$,

9. $(\exists C)(\forall S)\ \ \neg T(S, C)$,

10. $(\forall S)(\forall C)\ \ \neg T(S, C)$.

Problem 2 Prove that the following is true for all positive integers n: "n is even if and only if $3n^2 + 8$ is even".

Solution 2

- We first prove that if n is even then so is $3n^2+8$. Hence we assume that n is even and deduce that $3n^2 + 8$ is even as well. The assumption means that there exists an integer k such that $n = 2\,k$ holds. This leads to

  $$3\,n^2 + 8 \;=\; 3\,(2\,k)^2 + 8 \;=\; 12\,k^2 + 8 \;=\; 2\,(6\,k^2 + 4).$$

  Letting $k' = 6\,k^2 + 4$, we have established that there exists an integer k' such that $3\,n^2 + 8 = 2\,k'$, that is, the integer $3\,n^2 + 8$ is even.
- We then prove that if $3n^2 + 8$ is even then so is n. We prove the contrapositive. Hence we assume that n is odd and deduce that $3n^2+8$ is old as well. The assumption means that there exists an integer k such that $n = 2\,k + 1$ holds. This leads to

  $$3\,n^2+8 \;=\; 3\,(2\,k+1)^2+8 \;=\; 12\,k^2+12\,k+11 \;=\; 2\,(6\,k^2+6\,k+5)+1.$$

  Letting $k' = 6\,k^2 + 6\,k + 5$, we have established that there exists an integer k' such that $3\,n^2 + 8 = 2\,k' + 1$, that is, the integer $3\,n^2 + 8$ is odd.

Problem 3 Show that

$$((p \to q) \wedge (q \to r)) \to (p \to r) \quad \equiv \quad T$$

i.e. that the compound proposition on the left of $\equiv$ is a tautology. You should do it in two different ways using

(a) truth tables

(b) logical equivalences
HINT: first replace all "$\to$" using $a \to b \equiv \neg a \vee b$. Then, use Morgan's laws and other standard logical equivalence laws (slides 50-54 in propositional logic).

Solution 3 Our proposed solution combines both types of techniques. We have the following equivalences:

$$
\begin{array}{rc}
((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) & \iff \\
((\neg p \vee q) \wedge (\neg q \vee r)) \rightarrow (\neg p \vee r) & \iff \\
\neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r) & \iff \\
\neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r) & \iff \\
(p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r). &
\end{array}
$$

We shall prove now that the latter proposition (in the above chain of equivalences) is a true whatever are the truth values of p, q and r. We proceed by case inspection.

1. Consider the first parenthesized expression, namely $(p \wedge \neg q)$. Observe that if $p = \mathsf{true}$ and $q = \mathsf{false}$, then the whole proposition is true, whatever is r.
2. Assume now that either $p = \mathsf{false}$ or $q = \mathsf{true}$:
   - if $p = \mathsf{false}$, then $(q \wedge \neg r) \vee (\neg p \vee r)$ is true, whatever is r,
   - if $q = \mathsf{true}$, then $(q \wedge \neg r) \vee (\neg p \vee r)$ is true, whatever is r.

Finally, whatever are p, q, r, the formula is true. This concludes the proof.

Problem 4 Give a direct proof of the following: "If p is a prime number larger than 2 then p^2 is odd".

Clearly separate the hypotheses from the conclusion and provide detailed justification for your answer.

Solution 4 Assume that p is a prime number larger than 2. Let us prove that p^2 is odd. We saw in class that if an integer number is odd, then so is is square. The prime p (being greater than 2) is odd. Hence p^2 is odd.