UWO CS2214

Tutorial #3

Problem 1 Professor Cuthbert Calculus has designed a machine which consists of three components A, B, C which are either running or stopped. The constraints on those components are the following:

1. if A is running, then at least one of the components B or C is stopped,
2. if B is stopped, then at least one of the components A or C is running,
3. if C is running, then B is running as well.

Can the machine of Professor Cuthbert Calculus be built, that is, is the conjunction of the above three statements satisfiable. Justify your answer.

Solution 1 Let us denote by A, B, C Boolean variables stating that the respective components A, B, C are running. Then the 3 constraints can be rephrased as follows in propositional logic:

1. $A \rightarrow (\neg B \vee \neg C)$,
2. $\neg B \rightarrow (A \vee C)$,
3. $C \rightarrow B$.

Because of the third constraint, namely $C \rightarrow B$, it is natural to test whether the conjunction of the three constraints is satisfiable with $B = C = \mathsf{true}$. Then, since $\neg B \vee \neg C = \mathsf{false}$, to satisfy the first constraint, we must have $A = \mathsf{false}$. With those values of the Boolean variables A, B, C, the second constraint is satisfied. Therefore, the the machine of Professor Cuthbert Calculus be built.

Problem 2 Prove that $\log_2(9)$ is irrational.

Solution 2 if $\log_2(9)$ were equal to $\frac{m}{n}$, with m, n positive integers, without common factors, then, by the definition of logarithms, we would have

$$2^{\frac{m}{n}} = 9.$$

Raising both sides to the n-th power, we obtain:

$$2^m = 9^n.$$

Since n and m are non-zero, the numbers 9^n and 2^m are greater or equal to 9 and 2, respectively. Moreover, the numbers 9^n and 2^m are odd and even, respectively. Since a number cannot be both even and odd, the numbers 9^n and 2^m cannot be equal and we have reached a contradiction. Therefore, the number $\log_2(9)$ is irrational.

Problem 3 Let p, q, r, s be Boolean variables. For each of the following propositions, determine whether it is satisfiable or not :

1. $(p\vee(q\wedge(q\vee s)) \;\wedge\; (\neg p\vee(\neg q\wedge(\neg q\vee r)) \;\wedge\; (p\vee s) \;\wedge\; (\neg p\vee r)$.
2. $(p\vee(q\wedge(q\vee s)) \;\wedge\; (\neg p\vee(\neg q\wedge(\neg q\vee r)) \;\wedge\; (p\vee\neg q) \;\wedge\; (\neg p\vee q)$

Solution 3

1. Using the absorption laws, the sub-expression $(q\wedge(q\vee s))$ can simply be rewritten as q and the sub-expression $(\neg q\wedge(\neg q\vee r))$ can simply be rewritten as $\neg q$. Therefore, the entire proposition becomes

$$(p\vee q) \;\wedge\; (\neg p\vee\neg q) \;\wedge\; (p\vee s) \;\wedge\; (\neg p\vee r).$$

   Let us look first at $(p\vee q) \wedge (\neg p\vee\neg q)$. Both $(p\vee q)$ and $(\neg p\vee\neg q)$ are true if and only if p and q have opposite truth values. (This can be verified with a truth table.) Assume we choose $p = \mathsf{true}$ and $q = \mathsf{false}$. Then $(p\vee s)$ is true whatever is the truth value of s, meanwhile satisfying $(\neg p\vee r)$ requires to set $r = \mathsf{true}$. Finally, we can conclude that the entire proposition is satisfied with $p = \mathsf{true}$, $q = \mathsf{false}$ and $r = \mathsf{true}$.
2. Here again, $(q\wedge(q\vee s))$ can simply be rewritten as q and $(\neg q\wedge(\neg q\vee r))$ can simply be rewritten as $\neg q$. And the entire proposition becomes

$$(p\vee q) \;\wedge\; (\neg p\vee\neg q) \;\wedge\; (p\vee\neg q) \;\wedge\; (\neg p\vee q).$$

   Remember that $(p\vee q) \wedge (\neg p\vee\neg q)$ means $p \leftrightarrow \neg q$, that is, p and q have **opposite** truth values. Similarly, the sub-expression $(p\vee\neg q) \;\wedge (\neg p\vee q)$ means that p and q have the same truth values, that is, $p \leftrightarrow q$. Therefore, the entire proposition becomes

$$(p \leftrightarrow \neg q) \;\wedge\; (p \leftrightarrow q),$$

   which is clearly false. Finally, we can conclude that the entire proposition cannot be satisfied.

Problem 4 For any real number x, the *absolute value* of x, denoted by $|x|$, is defined as follows:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Prove that for all real numbers a, b, the following properties hold:

1. $|a + b| \leq |a| + |b|$ (called the *triangular inequality*),
2. $|a - b| \geq \big| |a| - |b| \big|$ (called the *reverse triangular inequality*),
3. if b is non-negative then we have: $|a| \leq b \iff -b \leq a \leq b$.

Solution 4

1. Let a, b be two real numbers. We consider 4 cases
   - **Case 1:** $a \geq 0$ and $b \geq 0$. Then $a + b \geq 0$ and we have:
     $$|a + b| = a + b = |a| + |b|.$$
   - **Case 2:** $a < 0$ and $b \geq 0$. In this case $a + b = -|a| + b$ and thus the sign of $a + b$ depends on whether $|a| \leq b$ or $|a| > b$ holds. If $|a| \leq b$ holds, then $a + b \geq 0$ and we have:
     $$|a + b| = -|a| + b \leq |a| + b = |a| + |b|.$$
     If $|a| > b$ holds, then $a + b < 0$ and we have:
     $$|a + b| = |a| - b \leq |a| + b = |a| + |b|.$$
   - **Case 3:** $a \geq 0$ and $b < 0$. This case is simply deduced from the previous one by exchanging the role of a and b.
   - **Case 4:** $a < 0$ and $b < 0$. Then $a + b < 0$ and we have:
     $$|a + b| = -(a + b) = -|a| - |b| \leq |a| + |b|.$$

   QED. It should be noted, as pointed by one student in class, that other formulas about absolute values can be used to avoid the case discussion. These formulas are
   $$\sqrt{a^2} = |a| \quad \text{and} \quad |a \times b| = |a| \times |b|.$$
   Since $ab \leq |a \times b|$ and $|a \times b| = |a| \times |b|$ both hold, we deduce:
   $$2ab \leq 2|a| \times |b|,$$
   and thus
   $$a^2 + 2ab + b^2 \leq |a|^2 + 2|a| \times |b| + |b|^2,$$
   leading to
   $$(a + b)^2 \leq (|a| + |b|)^2.$$
   Taking the square-root of each side yields:
   $$\sqrt{(a + b)^2} \leq \sqrt{(|a| + |b|)^2},$$
   that is,
   $$|a + b| \leq ||a| + |b|| = |a| + |b|.$$
2. Let a, b be two real numbers. One could proceed again by case inspection, discussing whether $a - b$ is non-negative or not, and discussing whether $|a| - |b|$ is non-negative or not. But there is a faster way, by applying the triangular inequality twice:

- From $a = (a - b) + b$, we deduce

$$|a| \leq |a - b| + |b|,$$

  and thus

$$|a| - |b| \leq |a - b|.$$

- From $-b = (a - b) + (-a)$ and $|a| = |-a|$ and $|b| = |-b|$, we deduce

$$|b| \leq |a - b| + |a|,$$

  and thus

$$|b| - |a| \leq |a - b|.$$

From $|a| - |b| \leq |a - b|$ and $|b| - |a| \leq |a - b|$, we deduce

$$\big|\, |a| - |b| \,\big| \leq |a - b|$$

Indeed, $\big|\, |a| - |b| \,\big|$ is equal to either $|a| - |b|$ or $|b| - |a|$. QED.

3. Let a, b be two real numbers. We have the following equivalences:

$$\begin{array}{lll} |a| \leq b & \iff & (a \geq 0 \wedge |a| \leq b) \;\vee\; (a < 0 \wedge |a| \leq b) \\ & \iff & (a \geq 0 \wedge a \leq b) \;\vee\; (a < 0 \wedge -a \leq b) \\ & \iff & (a \geq 0 \wedge a \leq b) \;\vee\; (a < 0 \wedge -b \leq a) \\ & \iff & (a \geq 0 \wedge -b \leq a \leq b) \;\vee\; (a < 0 \wedge -b \leq a \leq b) \\ & \iff & -b \leq a \leq b \end{array}$$

Indeed, for the second last equivalence, we can replace $(a \geq 0 \wedge a \leq b)$ with $(a \geq 0 \wedge -b \leq a \leq b)$ since we know that $b \geq 0$ holds anyway. Similarly, we can replace $(a < 0 \wedge -b \leq a)$ with $(a < 0 \wedge -b \leq a \leq b)$ for the same reason. QED. Of course, we can also prove the property

$$|a| \leq b \quad \iff \quad -b \leq a \leq b$$

by case inspection, discussing $a \geq 0$ or $a < 0$.