Problem 1 Let \(a, b, q_1, r_1, q_2, r_2\) be non-negative integer numbers such that \(b \neq 0\) and we have

\[
\begin{array}{c|c}
  a & b \\
  q_1 & r_1 \\
  q_2 & r_2 
\end{array}
\] (1)

Thus we have: \(a = bq_1 + r_1 = bq_2 + r_2\) as well as \(0 \leq r_1 < b\) and \(0 \leq r_2 < b\).

Prove that \(q_1 = q_2\) and \(r_1 = r_2\) necessarily both hold.

Solution 1 Let \(a = bq_1 + r_1 = bq_2 + r_2\), with \(0 \leq r_1 < b\) and \(0 \leq r_2 < b\), where \(a, b, q_1, r_1, q_2, r_2\) are non-negative integers. We wish to show that \(q_1 = q_2\) and \(r_1 = r_2\).

Assume that \(r_1 \neq r_2\). Then, without loss of generality, assume that \(r_2 > r_1\). We then have that

\[
bq_1 - bq_2 = r_2 - r_1 \\
\Rightarrow b(q_1 - q_2) = r_2 - r_1
\] (2)

Since \(0 \leq r_1 < b\) and \(0 \leq r_2 < b\), and \(r_2 > r_1\), it must be that

\[
0 < (r_2 - r_1) < b,
\] (3)

since the largest difference has \(r_2 = b - 1\) and \(r_1 = 0\), and \(r_1 \neq r_2\) by assumption (so \(r_2 - r_1 \neq 0\)). But equation (2) implies that \(b\) divides \(r_2 - r_1\), which cannot be given equation (3), because the multiples of \(b\) are \(0, \pm b, \pm 2b, \ldots\). This is a contradiction, and we conclude that \(r_1 = r_2\).

Since we have shown that \(r = r_1 = r_2\), it follows that

\[
bq_1 - bq_2 = r - r \\
\Rightarrow b(q_1 - q_2) = 0
\] (4)

But equation (4) implies either that \(b = 0\) or \(q_1 - q_2 = 0\). Since \(b \neq 0\) by the assumptions of the division theorem, we conclude that it must be that \(q_1 - q_2 = 0\), meaning that \(q_1 = q_2\), which is what we set out to prove. QED
Problem 2 In the previous exercise, if \( a, b, q_1, q_2 \), are non-negative integer numbers satisfying \( a = bq_1 + r_1 = bq_2 + r_2 \) while \( r_1, r_2 \) are integers satisfying \( -b < r_1 < b \) and \( -b < r_2 < b \). Do we still reach the same conclusion? Justify your answer.

Solution 2 No, we do not. Indeed, with \( a = 7 \) and \( b = 3 \), we then have two possible divisions:
\[
\begin{array}{c|c}
7 & 3 \\ \hline
1 & 2 \\
\end{array} \quad \text{and} \quad \begin{array}{c|c}
7 & 3 \\ \hline
-2 & 3 \\
\end{array}
\]

Problem 3 Consider the set of ordered pairs \((x, y)\) where \( x \) and \( y \) are real numbers. Such a pair can be seen as a point in the plane equipped with Cartesian coordinates \((x, y)\). For each of the following functions determine a \((2 \times 2)\)-matrix \( A \) so that the point of coordinates \((x, y)\) is sent to the point \((x', y')\) when we have
\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}
\] (5)
where
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\] (6)

1. \( F_1(x, y) = (x, y) \)
2. \( F_2(x, y) = (x, 0) \)
3. \( F_3(x, y) = (0, y) \)
4. \( F_4(x, y) = (y, x) \)

Solution 3

1. \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)
2. \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \)
3. \( A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)
4. \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)
**Problem 4** Following up on the previous problem, determine which of the above functions $F$ is injective? surjective?

**Solution 4**

1. $F_1$ is injective: Indeed, for all $(x_1, y_1)$ and $(x_2, y_2)$ if $F_1(x_1, y_1) = F_1(x_2, y_2)$ holds then we have $(x_1, y_1) = (x_2, y_2)$, which exactly means that $F_1$ is injective. $F_1$ is surjective: Indeed, every $(x', y')$ has a pre-image by $F_1$, namely itself, since $F_1(x', y') = (x', y')$ holds.

2. $F_2$ is not injective: Indeed, we have $F_2(0, 1) = (0, 0) = F_2(0, 2)$, thus two different points, namely $(0, 1)$ and $(0, 2)$ have the same image by $F_2$, namely $(0, 0)$. $F_2$ is not surjective: Indeed, $(1, 1)$ has no pre-image by $F_2$.

3. For similar reasons as those for $F_2$, $F_3$ is neither injective nor surjective.

4. $F_4$ is injective: Indeed, for all $(x_1, y_1)$ and $(x_2, y_2)$ if $F_4(x_1, y_1) = F_4(x_2, y_2)$ holds then we have $(y_1, x_1) = (y_2, x_2)$ that is, $y_1 = y_2$ and $x_1 = x_2$, thus $(x_1, y_1) = (x_2, y_2)$, which exactly means that $F_1$ is injective. $F_4$ is surjective: Indeed, every $(x', y')$ has a pre-image by $F_4$, namely $(y', x')$, since $F_1(y', x') = (x', y')$ holds.