UWO CS2214 Jan. 19, 2019

Assignment #1
Due: Jan. 29, 2017, by 23:55
Submission: on the OWL web site of the course

Format of the submission. You must submit a **single** file which must be in **PDF** format. All other formats (text or Miscrosoft word format) will be **ignored** and considered as **null**. You are strongly encouraged to type your solutions using a text editor. To this end, we suggest the following options:

1. Miscrosoft word and convert your document to PDF
2. the typesetting system LaTeX; see `https://www.latex-project.org/` and `https://en.wikipedia.org/wiki/LaTeX#Example` to learn about LaTeX; see `https://www.tug.org/begin.html` to get started
3. using a software tool for typing mathematical symbols, for instance `http://math.typeit.org/`
4. using a Handwriting recognition system such as those equipping tablet PCs

Hand-writing and scanning your answers is allowed but not encouraged:

1. if you go this route please use a scanning printer and **do not take a picture of your answers with your phone**,
2. if the quality of the obtained PDF is too poor, your submission will be **ignored** and considered as **null**.

Problem 1 (Undertsamding implication) *[20 marks] Let p, q be two Boolean variables. By definition, the implication $p \longrightarrow q$ is true if and only if p is false or q is true. Based on that, we have established the following practical tautologies:*

1. $(p \longrightarrow q) \iff (\neg q \longrightarrow \neg p)$
2. $(p \leftrightarrow q) \iff ((p \longrightarrow q) \wedge (q \longrightarrow p))$

Would these two tautologies still be true if we were changing the truth value of the implication $p \longrightarrow q$ to that of

1. $p \wedge q$?
2. $p \vee q$?

Justify your answer. Another way of phrasing the question would be the following. Would the above tautologies still be tautologies if we were

1. *replacing $\longrightarrow$ with $\wedge$?*
2. *replacing $\longrightarrow$ with $\vee$?*

Solution 1

1. Replacing $\longrightarrow$ with $\wedge$ in the first tautology yields

$$(p \wedge q) \iff (\neg q \wedge \neg p)$$

which is clearly false for $p = q =$ true. Replacing $\longrightarrow$ with $\wedge$ in the second tautology yields

$$(p \leftrightarrow q) \iff ((p \wedge q))$$

because of the commutativity of $\wedge$. This is clearly false for $p = q =$ false.

2. Replacing $\longrightarrow$ with $\vee$ in the first tautology yields

$$(p \vee q) \iff (\neg q \vee \neg p)$$

which is clearly false for $p = q =$ true. Replacing $\longrightarrow$ with $\vee$ in the second tautology yields

$$(p \leftrightarrow q) \iff ((p \vee q))$$

because of the commutativity of $\vee$. This is clearly false for $p = q =$ false.

Problem 2 (Proving theorems!) *[20 marks]* For each of the following statements, translate it into predicate logic and prove it, if the statement is true, or disprove it, otherwise:

1. for any two even integers, there exists a third integer (even or odd) the double of which is equal to the sum of the first two integers.
2. for any two odd integers, there exists a third integer (even or odd) the triple of which is equal to the sum of the first two integers.

Solution 2

1. Let x, y be any two even integers. There exist integers k, k' such that $x = 2\,k$ and $y = 2\,k'$. Therefore, we have:

$$x + y = 2(k + k').$$

Hence, there exists an integer $k'' = k + k$ such that $x + y = 2k''$ holds, which proves the claim.

2. The claim is false. Consider $x = y = 1$. Clearly $x + y = 2$ and $x + y$ is not the triple of an integer.

Problem 3 (Finding a treasure!) *[20 marks]* In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles. In the note he wrote that he had hidden a treasure somewhere on the property. He listed five true statements and challenged the reader to use them to figure out the location of the treasure.

1. If there is an old shipwreck near the beach, then the treasure is buried under a coconut palm tree.
2. There is a coconut palm tree growing either at the far end of the island or near the cave.
3. Either there is a shipwreck near the beach, or the treasure is hidden in a cave.
4. If there is a coconut palm tree at the far end of the island, then there is no shipwreck on the beach
5. There is no coconut palm tree near the cave.

The question being asked here is whether there is a treasure or not. And if yes, then where.

Solution 3 We consider the following 5 propositions:

- A: there is an old shipwreck near the beach,
- B: the treasure is buried under a coconut palm tree,
- C: there is a coconut palm tree growing at the far end of the island,
- D: there is a coconut palm tree growing near the cave,
- E: the treasure is hidden in a cave.

Translating the five facts into propositional formulas yields:

1. $A \longrightarrow B$,
2. $C \vee D$,
3. $A \vee E$,
4. $C \longrightarrow \neg A$,
5. $\neg D$.

The existence of the treasure (according to the pirate) is equivalent to

$$(A \longrightarrow B) \wedge (C \vee D) \wedge (A \vee E) \wedge (C \longrightarrow \neg A) \wedge (\neg D).$$

Hence, $\neg D$ must hold, we deduce $D =$ false. Since $C \vee D$ must hold, we deduce $C =$ true. Since $C \longrightarrow \neg A$ must hold, we deduce $A =$ false. Since $A \vee E$, we deduce $E =$ true. Since $A =$ false and $A \longrightarrow B$ both hold, we can either have $B =$ false or $B =$ true. But since $E =$ true anyway, B is certainly false.

Problem 4 (Deciding consistency) *[20 marks]* A set of propositions is *consistent* if there is an assignment of truth values to each of the propositional variables, that makes all propositions true. Is the following set of propositions consistent?

1. The system is in multiuser state if and only if it is operating normally.
2. If the system is operating normally, the kernel is functioning.
3. The kernel is not functioning or the system is in interrupt mode.
4. If the system is not in multiuser state, then it is in interrupt mode.
5. The system is in interrupt mode.

Solution 4 We consider the following 4 propositions:

- A: the system is in multiuser state,
- B: the system is operating normally,
- C: the kernel is functioning,
- D: the system is in interrupt mode.

Translating the five facts into propositional formulas yields:

1. $A \longleftrightarrow B$,
2. $B \longrightarrow C$,
3. $\neg C \vee D$,
4. $\neg A \longrightarrow D$,
5. D.

The consistency of the system is equivalent to

$$(A \longleftrightarrow B) \wedge (B \longrightarrow C) \wedge (\neg C \vee D) \wedge (\neg A \longrightarrow D) \wedge (D). \tag{1}$$

Hence D must hold, we deduce $D =$ true. Since $\neg A \longrightarrow D$ must hold and we have $D =$ true, we can choose $A =$ false. Since $A \longleftrightarrow B$ must hold and we have $A =$ false, we deduce $B =$ false. Note that, with $B =$ false, the constraint $B \longrightarrow C$ is satisfied whatever is C. Since $\neg C \vee D$ must hold and we have $D =$ true, we can choose $C =$ false. In summary, we can find values of A, B, C, D such that the Formula (1) is satisfied.

Problem 5 (Deciding satisfiability) *[20 marks]* Let p, q, r be three Boolean variables. For each of the following propositional formulas determine whether it is satisfiable or not.

1. $p \wedge (q \vee \neg p) \wedge (\neg q \vee \neg r)$
2. $p \wedge (q \vee \neg p) \wedge (\neg q \vee \neg p)$

Solution 5

1. satisfiable for $p = \mathsf{true}$, $q = \mathsf{true}$, $r = \mathsf{false}$. Indeed:
    - the first clause, namely p, implies that p must be true,
    - then, the second clause, namely $(q \vee \neg p)$, implies that q must be true,

- then, the third clause, namely $(\neg q \lor \neg r)$ implies that r must be false.

2. unsatisfiable. Indeed:
    - the first clause, namely p, implies that p must be true,
    - then, the second clause, namely $(q \lor \neg p)$, implies that q must be true,
    - then, the third clause, namely $(\neg q \lor \neg p)$ is false,
    - thus the whole formula cannot be satisfied.