Assignment #1
Due: Feb. 12, 2017, by 23:55
Submission: on the OWL web site of the course

Format of the submission. You must submit a **single** file which must be in **PDF** format. All other formats (text or Miscrosoft word format) will be **ignored** and considered as **null**. You are strongly encouraged to type your solutions using a text editor. To this end, we suggest the following options:

1. Miscrosoft word and convert your document to PDF
2. the typesetting system LaTeX; see `https://www.latex-project.org/` and `https://en.wikipedia.org/wiki/LaTeX#Example` to learn about LaTeX; see `https://www.tug.org/begin.html` to get started
3. using a software tool for typing mathematical symbols, for instance `http://math.typeit.org/`
4. using a Handwriting recognition system such as those equipping tablet PCs

Hand-writing and scanning your answers is allowed but not encouraged:

1. if you go this route please use a scanning printer and **do not take a picture of your answers with your phone**,
2. if the quality of the obtained PDF is too poor, your submission will be **ignored** and considered as **null**.

Problem 1 (Proving properties about the integers) *[15 marks]* Prove or disprove the following properties:

1. For every integer n we have $n \leq n^2$.
2. For every integer n, the integer $n^2 + n + 1$ is odd.

If the statement is true, your proof should be in the style of the proofs done in class, see Section 1.7 of the slides. If the statement is false, giving a counter-example is sufficient.

Solution 1

1. We proceed by cases, considering $n \leq 0$ and $1 \leq n$. If n is a non-positive integer, we have $n \leq 0$ and also $0 \leq n^2$, thus we have $n \leq n^2$. If n is a positive integer, we have $1 \leq n$ and $n \leq n$, which imply $n \leq n^2$.

2. We proceed by cases, considering n even abd n odd. If n is even, then there exists an integer k such that $n = 2\,k$ holds and we have $n^2 + n + 1 = 4\,k^2 + 2\,k + 1$, thus implying that $n^2 + n + 1$ is odd. If n is odd, then there exists an integer k such that $n = 2\,k + 1$ holds and we have $n^2 + n + 1 = 4\,k^2 + 6\,k + 3$, thus again implying that $n^2 + n + 1$ is odd.

Problem 2 (Proving properties about real numbers) *[15 marks]* Prove or disprove the following properties:

1. For every real number x, if $x \leq 0$ or $1 \leq x$ holds, then $x \leq x^2$ holds as well.
2. For all real number x we have $\lfloor 2x \rfloor = 2\lfloor x \rfloor$

If the statement is true, your proof should be in the style of the proofs done in class, see Section 1.7 of the slides. If the statement is false, giving a counter-example is sufficient.

Solution 2

1. One can prove the claim in the same way that we proved the first claim of Problem 1, thus by considering the two cases $x \leq 0$ and $1 \leq x$. Note that when $0 < x < 1$ holds we have $x^2 < x$; of course this configuration cannot happen in the integer case. An alternative proof of

$$((x \leq 0) \vee (1 \leq x)) \longrightarrow (x \leq x^2)$$

can be by solving the inequation

$$x \leq x^2$$

the solution set of which being precisely:

$$(x \leq 0) \vee (1 \leq x).$$

2. The claim is false. To show this, it is suffcient to exhibit a counter-example. Consider $x = 1.6$. We have $2\lfloor x \rfloor = 2 \times 1 = 2$ and $\lfloor 2x \rfloor = \lfloor 3.2 \rfloor = 3$. As an additional remark, consider a real number x of the form $n + \varepsilon$ where n is an integer and $\frac{1}{2} < \varepsilon < 1$ holds. Then we have

$$2\lfloor x \rfloor = 2n.$$

While we have

$$\lfloor 2x \rfloor = \lfloor 2(n + \varepsilon) \rfloor = \lfloor 2\,n + 1 + (2\,\varepsilon - 1) \rfloor = 2\,n + 1,$$

since $0 < 2\,\varepsilon - 1 < 1$ holds.

Problem 3 (Properties of preimage sets) *[20 marks]* Let f be a function from a set A to a set B. Let S and T be two subsets of B. Prove the following properties:

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$
2. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

Your proof should be in the style of the proofs done in class, see Section 1.7 of the slides.

Solution 3

1. By definition, the set $f^{-1}(S)$ is the set the elements $x \in A$ such that $f(x) \in S$ holds. Similarly:
   - the set $f^{-1}(T)$ is the set the elements $x \in A$ such that $f(x) \in T$ holds.
   - the set $f^{-1}(S \cup T)$ is the set the elements $x \in A$ such that $f(x) \in S \cup T$ holds.

   It follows that if an element $x \in A$ belongs to $f^{-1}(S \cup T)$, then either $f(x) \in S$ or $f(x) \in T$ holds, that is, either $x \in f^{-1}(S)$ or $x \in f(x) \in T$ holds. Thus, we have proved the following implication for all $x \in A$:

   $$x \in f^{-1}(S \cup T) \quad \longrightarrow \quad x \in\in f^{-1}(S) \cup f^{-1}(T),$$

   that is,

   $$f^{-1}(S \cup T) \;\subseteq\; f^{-1}(S) \cup f^{-1}(T).$$

   The inclusion

   $$f^{-1}(S) \cup f^{-1}(T) \;\subseteq\; f^{-1}(S \cup T)$$

   is proved in a similar way.
2. To prove the equality $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$ we could proceed as for $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$. For variety, we provide a more compact proof avoiding the proof of the two inclusions. Consider an arbitrary element x in A. Then, the following equivalences hold:

   $$\begin{array}{lcl} x \in f^{-1}(S \cap T) & \iff & f(x) \in S \cap T \\ & \iff & (f(x) \in S) \wedge (f(x) \in T) \\ & \iff & (x \in f^{-1}(S)) \wedge (x \in f^{-1}(T)) \\ & \iff & x \in f^{-1}(S) \cap f^{-1}(T). \end{array}$$

Problem 4 (Properties of functions) *[30 marks]* Which of the functions below is injective? surjective? When the function is bijective, determine its inverse. Justify your answers.

1. $f_1 : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ n & \longmapsto & 2019n+1 \end{array}$

2. $f_2 : \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ n & \longmapsto & \lfloor n/2 \rfloor + \lceil n/2 \rceil \end{array}$

3. $f_3 : \begin{array}{ccc} [1,2) & \to & [0,1) \\ x & \longmapsto & x - \lfloor x \rfloor \end{array}$

4. $f_4 : \begin{array}{ccc} [1,2) & \to & [0,1) \\ x & \longmapsto & (f_3(x))^2 \end{array}$

Solution 4

1. f_1 is injective. Indeed, if two integers n_1 and n_2 have the same image by f_1, then we have $2019n_1 + 1 = 2019n_2 + 1$, which implies $2019n_1 = 2019n_2$ and thus $n_1 = n_2$. However, f_1 is not surjective. Indeed, the integer $m = 0$ has no pre-images via f_1 since $2019n + 1 = 0$ yields $n = -\frac{1}{2019}$ which is not an integer.
2. Let us first understand what f_2 computes. It is natural to distinguish two cases: n even and n odd. If n is even, then there exists an integer k such that we have $n = 2k$. In this case, we have

$$\lfloor n/2 \rfloor + \lceil n/2 \rceil = \lfloor (2k)/2 \rfloor + \lceil (2k)/2 \rceil = \lfloor k \rfloor + \lceil k \rceil = k + k = n.$$

If n is odd, then there exists an integer k such that we have $n = 2k+1$. In this case, we have

$$\lfloor n/2 \rfloor + \lceil n/2 \rceil = \lfloor (2k+1)/2 \rfloor + \lceil (2k+1)/2 \rceil = \lfloor k+1/2 \rfloor + \lceil k+1/2 \rceil = k+k+1 = n.$$

Therefore, for all integer n, we have $f_2(n) = n$. It follows that if two integers n_1 and n_2 have the same image by f_2, then we have $n_1 = n_2$, that is, f_2 is injective. Similarly, every integer m has a pre-image by f_2, namely itself, thus f_2 is surjective. Consequently, f_2 is bijective and f_2 is its own inverse function.
3. Let us first understand what f_3 computes. Observe that for all $x \in [1,2)$, we have $1 \leq x < 2$, thus $\lfloor x \rfloor = 1$, hence $f_3(x) = x - 1$. It follows that if two real numbers x_1 and x_2 have the same image by f_3, then we have $x_1 - 1 = x_2 - 1$, thus $x_1 = x_2$, that is, f_3 is injective. Similarly, every real number $y \in [0,1)$ has a pre-image by f_3 in $[1,2)$, namely $y + 1$, thus f_3 is surjective. Consequently, f_3 is bijective and its inverse function is: $f_3^{-1} : \begin{array}{ccc} [0,1) & \to & [1,2) \\ y & \longmapsto & y+1. \end{array}$

4. Let us first understand what f_4 computes. From the previous question, we have: $f_4 : \begin{array}{ccc} [1,2) & \rightarrow & [0,1) \\ x & \longmapsto & (x-1)^2 \end{array}$ The function f_4 is injective. Indeed, if x_1 and x_2 are real numbers in the interval $[1, 2)$ with the same image by f_4 then we have $(x_1 - 1)^2 = (x_2 - 1)^2$, which implies $(x_1 - 1 - x_2 + 1)(x_1 - 1 + x_2 - 1) = 0$, that is, $x_1 = x_2$ or $x_1 = -x_2$. Since $x_1 = -x_2$ cannot hold for x_1 and x_2 in $[1, 2)$, we deduce $x_1 = x_2$, that is f_4 is injective. The function f_4 is surjective. Indeed, if $y \in [0, 1)$ then $x = \sqrt{y} + 1$ is a pre-image of y in $[1, 2)$. Consequently, f_4 is bijective and its inverse function is: $f_4^{-1} : \begin{array}{ccc} [0,1) & \rightarrow & [1,2) \\ y & \longmapsto & \sqrt{y}+1. \end{array}$

Problem 5 (Properties of functions) *[20 marks]* Let f be a surjective function from a set A to a set B and g be a function from B to a set C. Prove or disprove the following properties:

1. if g is surjective then so is gof.
2. if f and g are both injective, then so is gof.

If the statement is true, your proof should be in the style of the proofs done in class, see Section 1.7 of the slides. If the statement is false, giving a counter-example is sufficient.

Solution 5

1. Assume that g is surjective and let us prove that gof is surjective as well. That is, let us prove that all $z \in C$ there exists $x \in A$ such that $g(f(x)) = z$. Let $z \in C$. Since g is surjective there exists $y \in B$ such that $g(y) = z$. Since f is surjective there exists $x \in A$ such that $f(x) = y$. Hence, there exists $x \in A$ such that we have $gof(x) = z$. Therefore, we have proved that gof is surjective.
2. Assume that f and g are both injective and let us prove that gof is injective as well. Let x_1 and x_2 be in A such that $gof(x_1) = gof(x_2)$ holds. Thus, we have $g(f(x_1)) = g(f(x_2))$. Since g is injective, we deduce $f(x_1) = f(x_2)$. Since f is injective, we deduce $x_1 = x_2$. Therefore, we have proved that gof is injective.